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# Some formulas for the central trinomial and Motzkin numbers

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**Abstract**

We prove two new formulas for the central trinomial coefficients and the Motzkin numbers.

## 1 Introduction

Let $c_n$ denote the $n$th *central trinomial coefficient*, defined as the coefficient of $x^n$ in the expansion of $(1+x+x^2)^n$, or more combinatorially as the number of planar paths starting at $(0,0)$ and ending at $(n,0)$, whose allowed steps are $(1,0),(1,1),(1,-1)$. Let $m_n$ denote the $n$th *Motzkin number*, defined as the number of such planar paths which do not descend below the $x$-axis. The first few $c_n$'s are $1,3,7,19,51,\ldots$, and the first few $m_n$'s are $1,2,4,9,21,\ldots$. We prove

**Theorem 1**

$$m_n = \sum_{k=\lceil (n+2)/3 \rceil}^{\lfloor (n+2)/2 \rfloor} \frac{(3k-2)!}{(2k-1)!(n+2-2k)!(3k-n-2)!} \tag{1}$$

$$c_n = (-1)^{n+1} + 2n \sum_{k=\lceil n/3 \rceil}^{\lfloor n/2 \rfloor} \frac{(3k-1)!}{(2k)!(n-2k)!(3k-n)!} \tag{2}$$

It is interesting to compare these formulas with some of the other known formulas [6] for $m_n$ and $c_n$:

$$m_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(k+1)!(n-2k)!}$$

$$m_n = \sum_{k=0}^{n} \frac{(-1)^{n+k}\; n!\,(2k+2)!}{k!\,((k+1)!)^2\,(k+2)(n-k)!}$$

$$c_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(k!)^2(n-2k)!}$$

$$c_n = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{3^k(2n-2k)!}{k!(n-k)!(n-2k)!}$$

Formulas such as (1) and (2) can be proven automatically by computer, using the methods and software of Petkovšek, Wilf and Zeilberger [5]. We offer an independent, non-automatic proof that involves a certain symmetry idea which might lead to the discovery of other such identities. Two simpler auxiliary identities used in the proof are also automatically verifiable and shall not be proved.

## 2 Proof of the main result

**Proof of (1).** Our proof uses a variant of the generating function [6] for the numbers $m_n$, namely

$$f(x) = \frac{1 - x + \sqrt{1 + 2x - 3x^2}}{2} = 1 - x^2 + \sum_{n=3}^{\infty} (-1)^{n+1} m_{n-2} x^n$$

Then $f$ satisfies $f(0) = 1, f(1) = 0$ and is decreasing on $[0, 1]$. Another property of $f$ that will be essential in the proof is that it satisfies the functional equation

$$f(x)^2 - f(x)^3 = x^2 - x^3, \qquad 0 \le x \le 1, \tag{3}$$

as can easily be verified. A simple corollary of this is that $f(f(x)) = x$ for $x \in [0, 1]$.

Next, define

$$g(x) = \sum_{k=1}^{\infty} \frac{2(3k-2)!}{(2k)!(k-1)!}(x^2 - x^3)^k$$

Since on $[0, 1]$, the maximal value attained by $x^2 - x^3$ is $4/27$ (at $x = 2/3$), by Stirling's formula the series is seen to converge everywhere on $[0, 1]$, to a function $g(x)$ which is real-analytic except at $x = 2/3$. We now expand $g(x)$ in powers of $1 - x$; all rearrangement operations are permitted by absolute convergence:

$$g(x) = \sum_{k=1}^{\infty} \frac{2(3k-2)!}{(2k)!(k-1)!} x^{2k}(1-x)^k =$$

$$= \sum_{k=1}^{\infty} \frac{2(3k-2)!}{(2k)!(k-1)!}(1-x)^k \sum_{j=0}^{k} \binom{2k}{j} (-1)^j (1-x)^j =$$

$$= \sum_{n=1}^{\infty} \left( \sum_{k=\lceil n/3 \rceil}^{n} \binom{2k}{n-k} (-1)^{n+k} \frac{2(3k-2)!}{(2k)!(k-1)!} \right) (1-x)^n = 1 - x,$$

where the last equality follows from the automatically verifiable [5] identity

$$\sum_{k=\lceil n/3\rceil}^{n} \frac{(-1)^k(3k-2)!}{(k-1)!(n-k)!(3k-n)!} = 0, \qquad n > 1.$$

We have shown that $g(x) = 1 - x$ near $x = 1$. But since $g(x)$ is defined as a function of $x^2 - x^3$, by (3) it follows that $g(f(x)) = g(x)$, and therefore near $x = 0$ we have

$$g(x) = g(f(x)) = 1 - f(x) = x^2 + \sum_{n=3}^{\infty} (-1)^n m_{n-2} x^n.$$

Now to prove (1), we expand $g(x)$ into powers of $x$, again using easily justifiable rearrangement operations

$$g(x) = \sum_{k=1}^{\infty} \frac{2(3k-2)!}{(2k)!(k-1)!} x^{2k}(1-x)^k =$$

$$= \sum_{k=1}^{\infty} \frac{2(3k-2)!}{(2k)!(k-1)!} x^{2k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j x^j =$$

$$= \sum_{n=2}^{\infty} \left( (-1)^n \sum_{k=\lceil n/3\rceil}^{\lfloor n/2\rfloor} \frac{(3k-2)!}{(2k-1)!(n-2k)!(3k-n)!} \right) x^n.$$

Equating coefficients in the last two formulas gives (1). ∎

**Proof of (2).** We use a similar idea, this time using instead of the function $f(x)$ the function $-\log f(x)$, which generates a sequence related to $c_n$. Since the generating function for $c_n$ is well known [6] to be $1/\sqrt{1-2x-3x^2}$, it is easy to verify that

$$\frac{f'(x)}{f(x)} = \sum_{n=0}^{\infty} \frac{(-1)^n c_{n+1} - 1}{2} x^n$$

and therefore

$$-\log f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n c_n + 1}{2n} x^n.$$

Now define the function

$$h(x) = \sum_{k=1}^{\infty} \frac{(3k-1)!}{k!(2k)!} (x^2 - x^3)^k$$

which again converges for all $x \in [0, 1]$ to a function which is analytic except at $x = 2/3$. Expanding $h(x)$ into powers of $1 - x$ gives

$$h(x) = \sum_{k=1}^{\infty} \frac{(3k-1)!}{k!(2k)!} (1-x)^k \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (1-x)^j =$$

$$= \sum_{n=1}^{\infty} \left( \sum_{k=\lceil n/3 \rceil}^{n} \binom{2k}{n-k} (-1)^{n-k} \frac{(3k-1)!}{k!(2k)!} \right) (1-x)^n =$$

$$= \sum_{n=1}^{\infty} \frac{(1-x)^n}{n} = -\log x,$$

again making use of a verifiable identity [5], namely that

$$(-1)^n \sum_{k=\lceil n/3 \rceil}^{n} \frac{(-1)^k (3k-1)!}{k!(n-k)!(3k-n)!} = \frac{1}{n}, \qquad n \geq 1. \tag{4}$$

So $h(x) = -\log x$ near $x = 1$, and therefore because of the symmetry property (3) we have that $h(x) = -\log f(x)$ near $x = 0$. Expanding $h(x)$ in powers of $x$ near $x = 0$ gives

$$-\log f(x) = h(x) = \sum_{k=1}^{\infty} \frac{(3k-1)!}{k!(2k)!} x^{2k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j x^j =$$

$$= \sum_{n=2}^{\infty} \left( (-1)^n \sum_{k=\lceil n/3 \rceil}^{\lfloor n/2 \rfloor} \frac{(3k-1)!}{(2k)!(n-2k)!(3k-n)!} \right) x^n$$

Equating coefficients with our previous expansion of $h(x)$ gives (2). ∎

**Remarks.**

1. One obvious question on seeing formulas (1) and (2) is, Can they be explained combinatorially? That is, do there exist bijections between sets known to be enumerated by the numbers $m_n$ and $c_n$, and sets whose cardinality is seen to be the right-hand sides of (1) and (2)? Such explanations elude us currently.

2. Identity (4) is a special case of a more general identity [4, Eq. (6)] that was discovered by Thomas Liggett.

3. See [1, 2, 3, 6] for some other formulas involving the central trinomial coefficients and the Motzkin numbers, and for more information on the properties, and the many different combinatorial interpretations, of these sequences.

# 3 Acknowledgments

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(Concerned with sequences A001006 and A002426.)

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