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Matrix Transformations of Integer Sequences

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Abstract: The integer sequences with first term 1 comprise a group $\mathcal{G}$ under convolution, namely, the Appell group, and the lower triangular infinite integer matrices with all diagonal entries 1 comprise a group $\mathbb{G}$ under matrix multiplication. If $A \in \mathcal{G}$ and $M \in \mathbb{G}$, then $MA \in \mathcal{G}$. The groups $\mathcal{G}$ and $\mathbb{G}$ and various subgroups are discussed. These include the group $\mathbb{G}^{(1)}$ of matrices whose columns are identical except for initial zeros, and also the group $\mathbb{G}^{(2)}$ of matrices in which the odd-numbered columns are identical except for initial zeros and the same is true for even-numbered columns. Conditions are determined for the product of two matrices in $\mathbb{G}^{(m)}$ to be in $\mathbb{G}^{(1)}$. Conditions are also determined for two matrices in $\mathbb{G}^{(2)}$ to commute.

1 Introduction

Let $\mathcal{G}$ be the set of integer sequences $(a_1, a_2, a_3, \ldots)$ for which $a_1 = 1$. The notations $A = (a_1, a_2, a_3, \ldots)$, $B = (b_1, b_2, b_3, \ldots)$, $C = (c_1, c_2, c_3, \ldots)$ will always refer to elements of $\mathcal{G}$. The finite sequence $(a_1, a_2, a_3, \ldots, a_n)$ will be denoted by A_n, and likewise for B_n and C_n. Let $\star$ denote convolution; i.e., if $C = A \star B$, then

$$c_n = \sum_{k=1}^{n} a_k b_{n-k+1},$$

which we shall sometimes write as $A_n \circledast B_n$, so that $A \star B$ is the sequence having $A_n \circledast B_n$ as nth term. Formally,

$$\sum_{k=1}^{\infty} c_k x^{k-1} = \left(\sum_{k=1}^{\infty} a_k x^{k-1} \right) \left(\sum_{k=1}^{\infty} b_k x^{k-1} \right).$$

In particular, if $c_1 = 1$ and $c_k = 0$ for $k \geq 2$, then the sequence B has generating function $1/(a_1 + a_2 x + a_3 x^2 + \cdots)$, and A and B are a pair of convolutory inverses.

Let $\mathcal{G}_n$ denote the group of finite sequences A_n under $\star$; the identity is $I_n = (1, 0, 0, \ldots, 0)$, and A_n^{-1} is the sequence B_n given inductively by $b_1 = 1$ and

$$b_n = -\sum_{k=1}^{n-1} a_{n-k+1} b_k. \tag{1}$$

for $n \geq 2$. The algebraic system $(\mathcal{G}, \star)$ is a commutative group known as the Appell subgroup of the Riordan group. Its elements, the Appell sequences, are special cases of the Sheffler sequences, which play a leading role in the umbral calculus [2, Chapter 4]; however, the umbral developments are not used in this paper. In $\mathcal{G}$, the identity and A^{-1} are the limits of I_n and A_n^{-1}. (Here, limits are of the combinatorial kind: suppose $j_1, j_2, j_3, \ldots$ is an unbounded nondecreasing sequence of positive integers and $\{a_{i,j}\}$ is a sequence of sequences such for each i,

$$(a_{k,1}, a_{k,2}, a_{k,3}, \ldots, a_{k,j_i}) = (a_{i,1}, a_{i,2}, a_{i,3}, \ldots, a_{i,j_i})$$

for every $k > i$. Then

$$\lim_{i \to \infty} (a_{i,1}, a_{i,2}, a_{i,3}, \ldots)$$

is defined as the sequence $(a_1, a_2, a_3, \ldots)$ such that for every n there exists i_0 such that if $i > i_0$, then

$$(a_1, a_2, a_3, \ldots, a_n) = (a_{i,1}, a_{i,2}, a_{i,3}, \ldots, a_{i,n}).)$$

The study of the group $(\mathcal{G}, \star)$, we shall soon see, is essentially that of a certain group of matrices. However, we shall consider first a more general group of matrices.

For any positive integer n, let $\mathbb{G}_n$ be the set of lower triangular $n \times n$ integer matrices with all diagonal entries 1, and let $\cdot$ denote matrix multiplication. Then $(\mathbb{G}_n, \cdot)$ is a noncommutative group. Now let $\mathbb{G}$ denote the set of lower triangular infinite integer matrices with all diagonal entries 1. In such a matrix, every column, excluding the zeros above the diagonal, is an element of $\mathcal{G}$, and $(\mathbb{G}, \cdot)$ is a noncommutative group. Properties of matrices in $\mathbb{G}$ arise via limits of those of matrices in $\mathbb{G}_n$. For example, if $M = (m_{ij}) \in \mathbb{G}$, then the matrix $M_n := (m_{ij})$, where $1 \leq i \leq n$ and $1 \leq j \leq n$, is an element of $\mathbb{G}_n$, and

$$M^{-1} = \lim_{n \to \infty} M_n^{-1}.$$

It is easy to check that if $A \in \mathcal{G}$ and $M \in \mathbb{G}$, then $M \cdot A \in \mathcal{G}$; here A is regarded as an infinite column vector.

Among subgroups of $\mathbb{G}$ is the Riordan group (in the case that the coefficients are all integers) introduced in [3]. Although the Riordan group will not be further discussed in this paper, the reader may wish to consult the references listed at A053121 (the Catalan triangle) in [4].

Suppose $T = (t_1, t_2, t_3, \ldots) \in \mathcal{G}$. Let $\mathbb{T}$ be the matrix in $\mathbb{G}$ whose ith row is

$$t_i, t_{i-1}, \ldots, t_1, 0, 0 \ldots,$$

so that the first column of $\mathbb{T}$ is T, and each subsequent column contains T as a subsequence. Let $\mathbb{G}^{(1)}$ be the set of all such matrices $\mathbb{T}$. If $\mathbb{T}$ and $\mathbb{U}$ in $\mathbb{G}^{(1)}$ have first columns T and U, respectively, then the first column of $\mathbb{T} \cdot \mathbb{U}$ is the sequence $T \star U$, and $\mathbb{T} \cdot \mathbb{U} \in \mathbb{G}^{(1)}$. Clearly, $(\mathbb{G}^{(1)}, \cdot)$ is isomorphic to $(\mathcal{G}, \star)$. Matrices in $\mathbb{G}^{(1)}$ will be called *sequential matrices.*

One more property of the group $\mathbb{G}$, with easy and omitted proof, will be useful: if $M = (m_{ij}) \in \mathbb{G}$ and $f(M) := ((-1)^{i+j} m_{ij})$, then

$$(f(M))^{-1} = f(M^{-1}). \tag{2}$$

2 The Appell group $(\mathcal{G}, \star)$

The first theorem in this section concerns the convolutory inverse of a linear recurrence sequence of order $m \geq 2$.

Theorem 1. *Suppose $m \geq 2$, and $a_1 = 1, a_2, \ldots, a_m$ are initial values of an mth order recurrence sequence given by*

$$a_n = u_1 a_{n-1} + u_2 a_{n-2} + \cdots + u_m a_{n-m} + r_{n-m} \tag{3}$$

for $n \geq m+1$, where $u_1, u_2, \ldots, u_m$ and $r_1, r_2, r_3, \ldots$ are integers and $u_m \neq 0$. Then the convolutory inverse, B, of A, is a sequence

$$(1, b_2, \ldots, b_m, b_{m+1}, b_{m+2}, \ldots)$$

for which the subsequence $(b_{m+2}, b_{m+3}, \ldots)$ satisfies

$$b_n = \sum_{k=1}^{m-1} b_{n-k} c_k - B_{n-m} \circledast R_{n-m},$$

where

$$c_k = -a_{k+1} + \sum_{j=1}^{k} u_j a_{k+1-j}$$

for $n \geq m+2$.

Proof: By (1), $b_1 = a_1 = 1$. Also, $b_2 = -a_2$, and

$$b_n = -a_n b_1 - a_{n-1} b_2 - \cdots - a_2 b_{n-1}$$

for $n \geq 3$. For the rest of this proof, assume that $n \geq m+2$, and for later convenience, let

$$s_n = -a_n b_1 - a_{n-1} b_2 - \cdots - a_{m+2} b_{n-m-1}.$$

For $n \geq m+2$ (but not generally for $n = m+1$), the recurrence (1) gives

$$\sum_{k=1}^{m} u_k b_{n-k} = -\sum_{j=1}^{n-m-1} b_j \sum_{k=1}^{m} u_k a_{n-k-j+1} - U,$$

where

$$U=\sum_{k=1}^{m-1}u_k\sum_{j=2}^{m-k+1}a_jb_{n-k-j+1}.$$

Then

$$\begin{aligned}\sum_{k=1}^{m}u_kb_{n-k} &= -\sum_{j=1}^{n-m-1}b_j(a_{n+1-j}-r_{n+1-j-m})-U\\ &= s_n+\sum_{j=1}^{n-m-1}b_jr_{n+1-j-m}-U\\ &= b_n+\sum_{j=2}^{m+1}a_jb_{n+1-j}+\sum_{j=1}^{n-m-1}b_jr_{n+1-j-m}-U,\end{aligned}$$

so that

$$b_n=\sum_{k=1}^{m}u_kb_{n-k}-\sum_{j=2}^{m+1}a_jb_{n+1-j}-\sum_{j=1}^{n-m-1}b_jr_{n+1-j-m}+U. \tag{4}$$

Now put $n=m+1$ into (3) and substitute in (4) for a_{m+1}. The resulting coefficient of b_{n-m} is $-r_1$, and (4) simplifies to

$$\begin{aligned}b_n &= \sum_{k=1}^{m-1}u_kb_{n-k}-\sum_{j=2}^{m}a_jb_{n+1-j}+\sum_{k=1}^{m-2}u_k\sum_{j=2}^{m-k}a_jb_{n-k-j+1}-\sum_{j=1}^{n-m}b_jr_{n+1-j-m}\\ &= \sum_{k=1}^{m-1}b_{n-k}(-a_{k+1}+\sum_{j=1}^{k}u_ja_{k+1-j})-\sum_{j=1}^{n-m}b_jr_{n+1-j-m}. \qquad \blacksquare\end{aligned}$$

Corollary 1. If the recurrence for A in (3) is homogeneous of order $m\geq 2$, then the recurrence for the sequence $(b_4,b_5,b_6,\ldots)$ is of order $m-1$. If $m=2$, then the convolutory inverse of A is the sequence

$$(b_1,b_2,b_3,\ldots)=(1,\ -a_2,\ f,\ (u_1-a_2)f,\ (u_1-a_2)^2f,\ (u_1-a_2)^3f,\ldots),$$

where $f=a_2^2-a_3$.

Proof: Homogeneity of a means that $r_n=0$ for $n\geq 1$, so that $b_n=\sum_{k=1}^{m-1}c_kb_{n-k}$ for $n\geq m+2$. $\blacksquare$

Example 1. The Fibonacci sequence, $A=(1,1,2,3,5,8,\ldots)$, has inverse $(1,-1,-1,0,0,0,0,0,\ldots)$.

Example 2. The Lucas sequence, $A=(1,3,4,7,11,18,\ldots)$, has inverse, $(1,-3,5,-10,20,-40,80,\ldots)$, recurrent with order 1 beginning at the third term.

Example 3. Let A be the 2nd-order nonhomogeneous sequence given by $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2} + n - 2$ for $n \geq 3$. The inverse of A is the sequence $B = (1, -1, -2, -1, 1, 4, 6, 4, -4, -11, \ldots)$ given for $n \geq 4$ by

$$b_n = -B_{n-2} \circledast R_{n-2} = -(b_1, b_2, \cdots, b_{n-2}) \star (1, 2, 3, \ldots, n-2).$$

Example 4. Suppose that A and C are sequences in $\mathcal{G}$. Since $\mathcal{G}$ is a group, there exists B in $\mathcal{G}$ such that $A = B \star C$. For example, if A and C are the Fibonacci and Lucas sequences of Examples 1 and 2, then

$$B = A \star C^{-1} = (1, -2, 4, -8, 16, \ldots),$$

a 1st-order sequence.

Theorem 2. *Let $B = (1, b_2, b_3, \ldots)$ be the convolutory inverse of $A = (1, a_2, a_3, \ldots)$, and let $\widehat{A} = (1, -a_2, a_3, -a_4, a_5, -a_6, \ldots)$. Then the convolutory inverse of $\widehat{A}$ is the sequence $\widehat{B} = (1, -b_2, b_3, -b_4, b_5, -b_6, \ldots)$.*

Proof: Apply (2) to the subgroup $\mathbb{G}^{(1)}$ of sequential matrices. ■

Example 5. Let A be the sequence given by $a_n = \lfloor n\tau \rfloor$, where $\tau = (1 + \sqrt{5})/2$. Then

$$A = (1, 3, 4, 6, 8, 9, 11, 12, \ldots) \quad \text{and} \quad A^{-1} = (1, -3, 5, -9, 17, -30, 52, -90, \ldots).$$

Let A be the sequence given by $a_n = (-1)^{n-1} \lfloor n\tau \rfloor$. Then

$$A = (1, -3, 4, -6, 8, -9, 11, -12, \ldots) \quad \text{and} \quad A^{-1} = (1, 3, 5, 9, 17, 30, 52, 90, \ldots).$$

Example 6. Let A be the Catalan sequence, given by $a_n = \frac{1}{n} \binom{2n-2}{n-1}$. Then

$$\begin{aligned} A &= (1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots) \\ A^{-1} &= (1, -1, -1, -2, -5, -14, -42, -132, \ldots). \end{aligned}$$

Example 7. Let A be the sequence of central binomial coefficients, given by $a_n = \binom{2n-2}{n-1}$, Then

$$A = (1, 2, 6, 20, 70, 252, 924, \ldots) \quad \text{and} \quad A^{-1} = (1, -2, -2, -4, -10, -28, -84, -264, \ldots),$$

with obvious connections to the Catalan sequence.

Certain operations on sequences in $\mathcal{G}$ are easily expressed in terms of convolution. Two of these operations are given as follows. Suppose x is an integer, and $A = (1, a_2, a_3, \ldots)$ is a sequence in $\mathcal{G}$, with inverse $B = (1, b_2, b_3, \ldots)$. Then

$$(1,\ xa_2,\ xa_3,\ xa_4, \ldots) = (1,\ (1-x)b_2,\ (1-x)b_3,\ (1-x)b_4, \ldots) \star A$$

and

$$(1,\ x,\ a_2,\ a_3,\ldots) = (1,\ x+b_2,\ (x-1)b_2+b_3,\ (x-1)b_3+b_4,\ldots)\star A.$$

Stated in terms of power series

$$a(t) = 1 + a_2t + a_3t^2 + \cdots \quad \text{and} \quad 1/a(t) = b(t) = 1 + b_2t + b_3t^2 + \cdots ,$$

the two operations correspond to the identities

$$\begin{aligned} xa(t) + 1 - x &= [(1-x)b(t) + x]a(t); \\ ta(t) + 1 + (x-1)t &= \{b(t) + [(x-1)b(t) + 1]t\}a(t). \end{aligned}$$

3 The group $(\mathbb{G}^{(m)}, \cdot)$

Recall that the set $\mathbb{G}$ consists of the lower triangular infinite integer matrices with all diagonal entries 1. Define $'$ on $\mathbb{G}$ as follows: if $A \in \mathbb{G}$, then A' is the matrix that remains when row 1 and column 1 of A are removed. Clearly $A' \in \mathbb{G}$. Define

$$A^{(0)} = A, \qquad\qquad A^{(n)} = (A^{(n-1)})'$$

for $n \geq 1$. Let

$$\mathbb{G}^{(m)} = \{A \in \mathbb{G} : A^{(m)} = A\}$$

for $m \geq 0$. Note that $(\mathbb{G}^{(1)}, \cdot)$ is the group of sequential matrices introduced in Section 1, and $\mathbb{G}^{(m)} \subset \mathbb{G}^{(d)}$ if and only if $d|m$.

Theorem 3. *$(G^{(m)}, \cdot)$ is a group for $m \geq 0$.*

Proof: $(\mathbb{G}^{(0)}, \cdot)$ is the group $(\mathbb{G}, \cdot)$. For $m \geq 1$, first note that $(AB)' = A'B'$, so that, inductively, $(AB)^{(q)} = A^{(q)}B^{(q)}$ for all $q \geq 1$. In particular, if A and B are in $\mathbb{G}^{(m)}$, then

$$(AB)^{(m)} = A^{(m)}B^{(m)} = AB,$$

so that $AB \in \mathbb{G}^{(m)}$. Moreover,

$$(A^{-1})^{(m)} = (A^{(m)})^{-1} = A^{-1},$$

so that $A^{-1} \in \mathbb{G}^{(m)}$. ■

4 The group $(\mathbb{G}^{(2)}, \cdot)$

Suppose that A, B, C, D are sequences in $\mathcal{G}$. Let $\langle A; B\rangle$ denote the matrix in $\mathbb{G}^{(2)}$ whose first column is $A = (a_1, a_2, \ldots)$ and whose second column is $(0, b_1, b_2, \ldots)$, where $a_1 = b_1 = 1$. We shall see that the product $\langle A; B\rangle \cdot \langle C; D\rangle$ is given by certain "mixed convolutions." Write $\langle A; B\rangle \cdot \langle C; D\rangle$ as $\langle U; V\rangle$. Then

$$u_n = \begin{cases} (a_1, b_2, a_3, \ldots, b_{n-1}, a_n) \star (c_1, c_2, \ldots, c_n), & \text{if } n \text{ is odd;} \\ (b_1, a_2, b_3, \ldots, b_{n-1}, a_n) \star (c_1, c_2, \ldots, c_n), & \text{if } n \text{ is even;} \end{cases}$$

$$v_n = \begin{cases} (b_1, a_2, b_3, \ldots, a_{n-1}, b_n) \star (d_1, d_2, \ldots, d_n), & \text{if } n \text{ is odd;} \\ (a_1, b_2, a_3, \ldots, a_{n-1}, b_n) \star (d_1, d_2, \ldots, d_n), & \text{if } n \text{ is even.} \end{cases}$$

In particular $\langle A; B\rangle \cdot \langle B; A\rangle$ is the sequential matrix of the sequence $A \star B$.

Recursive formulas for columns of $\langle A; B\rangle^{-1}$ can also be given: write $\langle A; B\rangle^{-1}$ as $\langle X; Y\rangle$, so that $\langle A; B\rangle \cdot \langle X; Y\rangle$ is the identity matrix. Each nondiagonal entry of $\langle A; B\rangle \cdot \langle X; Y\rangle$ is zero, so that, solving inductively for $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ gives

$$x_n = \begin{cases} -a_n - b_{n-1}x_2 - a_{n-2}x_3 - \cdots - b_2 x_{n-1}, & \text{if } n \text{ is odd;} \\ -a_n - b_{n-1}x_2 - a_{n-2}x_3 - \cdots - a_2 x_{n-1}, & \text{if } n \text{ is even;} \end{cases} \tag{5}$$

$$y_n = \begin{cases} -b_n - a_{n-1}y_2 - b_{n-2}y_3 - \cdots - a_2 y_{n-1}, & \text{if } n \text{ is odd;} \\ -b_n - a_{n-1}y_2 - b_{n-2}y_3 - \cdots - b_2 y_{n-1}, & \text{if } n \text{ is even.} \end{cases} \tag{6}$$

Example 8. Example 6 shows that the Catalan sequence satisfies the equation

$$(1, a_2, a_3, \ldots)^{-1} = (1, -1, -a_2, -a_3, \ldots),$$

which we abbreviate as $A^{-1} = (1, -A)$. It is natural to ask whether there are sequences A and B for which

$$\langle A; B\rangle^{-1} = \langle 1, -A; B\rangle . \tag{7}$$

This problem is solved as follows. Write the first and second columns of $\langle 1, -A; B\rangle$ as $(1, x_2, x_3, \ldots)$ and $(0, 1, y_2, y_3, \ldots)$, respectively. Equation (7) implies $x_n = -a_{n-1}$ and $y_n = b_n$ for $n \geq 2$. Thus, $b_2 = y_2$, but also, by (6), $y_2 = -b_2$, so that $b_2 = 0$. Inductively, (6) and (7) imply $b_n = 0$ for all $n \geq 3$, so that B is the convolutory identity sequence: $B = (1, 0, 0, 0, \ldots)$. Using this fact together with (5) gives

$$x_n = \begin{cases} -a_n - a_{n-2}x_3 - a_{n-4}x_5 - \cdots - a_2 x_{n-1}, & \text{if } n \text{ is even;} \\ -a_n - a_{n-2}x_3 - a_{n-4}x_5 - \cdots - a_3 x_{n-2}, & \text{if } n \text{ is odd;} \end{cases}$$

so that, substituting $x_k = -a_{k-1}$, we have a recurrence for A:

$$a_n = \begin{cases} a_{n-1} + a_{n-2}a_2 + a_{n-4}a_4 + \cdots + a_2 a_{n-2} & \text{if } n \text{ is even;} \\ a_{n-1} + a_{n-2}a_2 + a_{n-4}a_4 + \cdots + a_3 a_{n-3} & \text{if } n \text{ is odd;} \end{cases}$$

with intial values $a_1 = 1$, $a_2 = 1$. This sequence, listed as A047749 in [4], is given by

$$a_n = \begin{cases} \frac{1}{2m+1}\binom{3m}{m}, & \text{if } n = 2m; \\ \frac{1}{2m+1}\binom{3m+1}{m+1}, & \text{if } n = 2m+1. \end{cases}$$

Example 9. Let $a_n = 1$ and $b_n = F_n$ for $n \geq 1$, where F_n denotes the Fibonacci sequence in Example 1. Let C be the sequence given by $c_1 = 1$, $c_2 = -1$, $c_3 = 0$, $c_4 = 1$, and $c_n = 2^{\lfloor (n-5)/2 \rfloor}$ for $n \geq 5$. Let D be the sequence given by $d_1 = 1$, $d_2 = -1$, $d_3 = -1$, and $d_n = -c_{n+1}$ for $n \geq d_4$. Then $\langle A; B\rangle^{-1} = \langle C; D\rangle$.

Theorem 4. *If any three of four sequences A, B, C, D in G are given, then the fourth sequence is uniquely determined by the condition that $\langle A; B\rangle \cdot \langle C; D\rangle$ be a sequential matrix.*

Proof: The requirement that $\langle A; B\rangle \cdot \langle C; D\rangle$ be a sequential matrix is equivalent to an infinite system of equations, beginning with

$$\begin{aligned} d_1 &= 1 \\ b_2 + d_2 &= a_2 + c_2 \\ b_3 + a_2 d_2 + d_3 &= a_3 + b_2 c_2 + c_3. \end{aligned}$$

For $n \geq 3$, the system can be expressed as follows:

$$\begin{aligned} & b_n + a_{n-1} d_2 + b_{n-2} d_3 + \cdots + h_2 d_{n-1} + d_n \\ = \; & a_n + b_{n-1} c_2 + a_{n-2} c_3 + \cdots + h_2' c_{n-1} + c_n, \end{aligned} \tag{8}$$

where $h_2 = a_2$ if n is odd, $h_2 = b_2$ if n is even; and $h_2' = b_2$ if n is odd, $h_2' = a_2$ if n is even.

Equations (8) show that each of the four sequences is determined by the other three. ■

Example 10. By (8), D is determined by A, B, C in accord with the recurrence

$$\begin{aligned} d_n = \; & a_n + c_2 b_{n-1} + c_3 a_{n-2} + c_4 b_{n-3} + \cdots + c_{n-1} h_2' + c_n \\ & - b_n - d_2 a_{n-1} - d_3 b_{n-2} - d_4 a_{n-3} \cdots - d_{n-1} h_2. \end{aligned} \tag{9}$$

Suppose $a_n = b_n = c_{n-2} = 0$ for $n \geq 3$. Then by (9),

$$d_n = \begin{cases} -b_2 d_{n-1} - a_3 d_{n-2}, & \text{if } n \text{ is even;} \\ -a_2 d_{n-1} - b_3 d_{n-2}, & \text{if } n \text{ is odd;} \end{cases}$$

for $n \geq 4$, with $d_1 = 1$, $d_2 = a_2 - b_2$, $d_3 = a_3 - a_2 d_2 - b_3 d_1$. If $(a_1, a_2, a_3) = (1, -1, -1)$ and $(b_1, b_2, b_3) = (1, -2, -1)$ and $c_1 = 1$, then

$$D = (1, 1, 1, 3, 4, 11, 15, 41, 56, 153, \ldots),$$

which, except for the initial 1, is the sequence of denominators of the convergents to $\sqrt{3}$, indexed in [4] as A002530. In this example, $\langle A; B\rangle \cdot \langle C; D\rangle$ is the sequential matrix with first three terms $1, -1, -1$ and all others zero.

Theorem 5. *If A, B, C in G are given and $|a_2| = 1$, then there exists a unique sequence D in G such that $\langle A; B\rangle \cdot \langle C; D\rangle = \langle C; D\rangle \cdot \langle A; B\rangle$.*

Proof: Write $\langle A; B\rangle \cdot \langle C; D\rangle$ as (s_{ij}) and $\langle C; D\rangle \cdot \langle A; B\rangle$ as (t_{ij}). Equating $s_{n+1,1}$ and $t_{n+1,1}$ and solving for d_n give

$$d_n = \frac{1}{a_2}(u_n - v_n) \tag{10}$$

for $n \geq 3$, where

$$u_n = \begin{cases} c_2 b_n + c_3 a_{n-1} + c_4 b_{n-2} + \cdots + c_n a_2, & \text{if } n \text{ is odd;} \\ c_2 b_n + c_3 a_{n-1} + c_4 b_{n-2} + \cdots + c_n b_2, & \text{if } n \text{ is even;} \end{cases}$$

$$v_n = \begin{cases} a_3 c_{n-1} + a_4 d_{n-2} + \cdots + a_n c_2, & \text{if } n \text{ is odd;} \\ a_3 c_{n-1} + a_4 d_{n-2} + \cdots + a_n d_2, & \text{if } n \text{ is even;} \end{cases}$$

with $d_1 = 1$, $d_2 = b_2 c_2 / a_2$. A sequence D is now determined by (10); we shall refer to the foregoing as part 1.

It is necessary to check that the equations $s_{n+1,2} = t_{n+1,2}$ implied by

$$\langle A; B \rangle \cdot \langle C; D \rangle = \langle C; D \rangle \cdot \langle A; B \rangle$$

do not impose requirements on the sequence D that are not implied by those already shown to determine D. In fact, the equations $s_{n+1,2} = t_{n+1,2}$ with initial value $d_1 = 1$ determine exactly the same sequence D. To see that this is so, consider the mapping $\langle A; B \rangle' = \langle B; A \rangle$. It is easy to prove the following lemma:

$$(\langle A; B \rangle \cdot \langle C; D \rangle)' = \langle B; A \rangle \cdot \langle D; C \rangle .$$

By part 1 applied to $\langle B; A \rangle \cdot \langle D; C \rangle$ and $\langle D; C \rangle \cdot \langle B; A \rangle$, the first column of $\langle B; A \rangle \cdot \langle D; C \rangle$ equals the first column of $\langle D; C \rangle \cdot \langle B; A \rangle$. Therefore, by the lemma, the second column of $\langle A; B \rangle \cdot \langle C; D \rangle$ equals the second column of $\langle C; D \rangle \cdot \langle A; B \rangle$, which is to say that the equations $s_{n+1,2} = t_{n+1,2}$ hold. ■

Example 11. Let $a_1 = 1$, $a_2 = 1$, and $a_n = 0$ for $n \geq 3$. Let B be the Fibonacci sequence. Let $C = (1, 1, 0, 1, 0, 0, \ldots)$, with $c_n = 0$ for $n \geq 5$. Then D is given by $d_1 = 1$, $d_2 = 1$, $d_3 = 2$, and $d_n = L_{n-1}$ for $n \geq 4$, where (L_n) is the Lucas sequence, as in Example 1. Writing $\langle A; B \rangle \cdot \langle C; D \rangle$ as $\langle U, V \rangle$, we have $\langle U, V \rangle = \langle C; D \rangle \cdot \langle A; B \rangle$, where $U = (1, 2, 1, 3, 4, 7, 11, 18, \ldots)$ and $V = (1, 2, 5, 9, 20, 32, 66, 105, 207, \ldots)$.

5 Generalization of Theorem 4

It is natural to ask what sort of generalization Theorem 4 has for $m \geq 3$. The notation $\langle A; B \rangle$ used for matrices in $\mathbb{G}^{(2)}$ is now generalized in the obvious manner to $\langle A_1, A_2, \ldots, A_m \rangle$ in $\mathbb{G}^{(m)}$, where A_i is a sequence $(a_{i1}, a_{i2}, \ldots)$ having $a_{i1} = 1$, for $i = 1, 2, \ldots, m$.

Theorem 4A. *Suppose $A_1, A_2, \ldots, A_m$ and B_i for some i satisfying $1 \leq i \leq m$ are given. Then sequences B_j for $j \neq i$ are uniquely determined by the condition that $\langle A_1, A_2, \ldots, A_m \rangle \cdot \langle B_1, B_2, \ldots, B_m \rangle$ be a sequential matrix. Conversely, suppose $B_1, B_2, \ldots, B_m$ and A_i for some i satisfying $1 \leq i \leq m$ are given. Then sequences A_j for $j \neq i$ are uniquely determined by the condition that $\langle A_1, A_2, \ldots, A_m \rangle \cdot \langle B_1, B_2, \ldots, B_m \rangle$ be a sequential matrix.*

Proof: Let $U = AB$. For given A, each column of B uniquely determines the corresponding column of U, and each column of U determines the corresponding column of B. Thus, under the hypothesis that a particular column B_i of B is given, the equation $U = AB$ determines the corresponding column of U. Consequently, as U is a sequential matrix, every column of U is determined, and this implies that every column of B is determined.

For the converse, suppose B, together with just one column A_i of A, are given, and that the product $U = AB$ is sequential. As a first induction step,

$$a_{21}b_{11} + a_{22}b_{21} = a_{32}b_{22} + a_{33}b_{32} = \cdots . \tag{11}$$

As $a_{i+1,i}$ is given, equations (11) show that $a_{h+1,h}$ is determined for all $h \geq 1$. Assume for arbitrary $k \geq 1$ that $a_{h+j,h}$ is determined for all j satisfying $1 \leq j \leq k$, for all $h \geq 1$. As U is sequential,

$$\begin{aligned} & a_{k+1,1}b_{11} + a_{k+1,2}b_{21} + \cdots + a_{k+1,k+1}b_{k+1,1} \\ = \; & a_{k+2,2}b_{22} + a_{k+2,3}b_{32} + \cdots + a_{k+2,k+2}b_{k+2,2} \\ = \; & \cdots . \end{aligned} \tag{12}$$

As $a_{k+i,i}$ is given, equations (12) and the induction hypothesis show that $a_{k+h,h}$ is determined for all $h \geq 1$. Thus, by induction, A is determined. ■

Theorem 4A shows that Theorem 4 extends to $\mathbb{G}^{(m)}$. The method of proof of Theorem 4A clearly applies to $\mathbb{G}$, so that Theorem 4A extends to $\mathbb{G}$.

6 Transformations involving divisors

We return to the general group $(\mathbb{G}, \cdot)$ for a discussion of several specific matrix transformations involving divisors of integers. The first is given by the left summatory matrix,

$$T(n,k) = \begin{cases} 1, & \text{if } k|n; \\ 0, & \text{otherwise.} \end{cases}$$

The inverse of T is the left Möbius transformation matrix. The matrices T and T^{-1} are indexed as A077049 and A077050 in [4], where transformations by T and T^{-1} of selected sequences in $\mathcal{G}$ are referenced. In general, if A is a sequence written as an infinite column vector, then

$$T \cdot A = \{\sum_{k|n} a_k\} \qquad \text{and} \qquad T^{-1} \cdot A = \{\sum_{k|n} \mu(k) a_k\},$$

that is, the summatory sequence of A and the Möbius transform of A, respectively.

Next, define the *left summing matrix* $S = \{s(n,k)\}$ and the *left differencing matrix* $D = \{d(n,k)\}$ by

$$\begin{aligned} s(n,k) &= \begin{cases} 1, & \text{if } k \leq n; \\ 0, & \text{otherwise.} \end{cases} \\ d(n,k) &= \begin{cases} (-1)^{n+k}, & \text{if } k = n \text{ or } k = n-1; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $D = S^{-1}$.

Example 12. Suppose that a sequence $C = (1, c_2, c_3, \ldots)$ in $\mathcal{G}$ is transformed to a sequence $A = (1, a_2, a_3, \ldots)$ by the sums $a_n = \sum_{k=1}^{n} c_k \lfloor n/k \rfloor$. In order to solve this system of equations, let $U(n,k) = \lfloor n/k \rfloor$ for $k \geq 1$, $n \geq 1$. Then $U = S \cdot T$, so that $U^{-1} = T^{-1} \cdot D$, which means that

$$c_n = \sum_{d|n} \mu(d)(a_{n/d} - a_{n/d-1}),$$

where $a_0 := 0$. If $a_n = 1$ for every $n \geq 1$, then $c_n = \mu(n)$. If $a_n = n$, then C is the convolutory identity, $(1, 0, 0, 0, \ldots)$. If $a_n = \binom{n+1}{2}$, then $c_n = \varphi(n)$. If $a_n = \binom{n+2}{3}$, then C is the sequence indexed as A000741 in [4] and discussed in [1] in connection with compositions of integers with relatively prime summands.

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