Journal of Integer Sequences, Vol. 6 (2003),
Article 03.1.1

Derived Sequences

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Abstract

We define a multiplicative arithmetic function D by assigning $D(p^a) = ap^{a-1}$, when p is a prime and a is a positive integer, and, for $n \geq 1$, we set $D^0(n) = n$ and $D^k(n) = D(D^{k-1}(n))$ when $k \geq 1$. We term $\{D^k(n)\}_{k=0}^{\infty}$ the derived sequence of n. We show that all derived sequences of $n < 1.5 \cdot 10^{10}$ are bounded, and that the density of those $n \in \mathbb{N}$ with bounded derived sequences exceeds 0.996, but we conjecture nonetheless the existence of unbounded sequences. Known bounded derived sequences end (effectively) in cycles of lengths only 1 to 6, and 8, yet the existence of cycles of arbitrary length is conjectured. We prove the existence of derived sequences of arbitrarily many terms without a cycle.

1 Introduction

Define a multiplicative arithmetic function D by assigning

$$D(p^a) = ap^{a-1}, \tag{1}$$

when p is a prime and a is a positive integer. The multiplicativity implies $D(1) = 1$ and, for example, $D(p^a q^b) = abp^{a-1}q^{b-1}$, where q^b is another prime power, $q \neq p$. It is only the shape of the definition (1) that encourages us to use freely terms from calculus. (Our derivatives have no relationship to an earlier use of the term, in Apostol [1], for example.) Writing $D^0(n) = n$ and $D^k(n) = D(D^{k-1}(n))$ for $k \geq 1$ and any positive integer n, we call $\{D^k(n)\}_{k=0}^{\infty}$, or $\{n, D(n), D^2(n), D^3(n), \ldots\}$, the derived sequence of n, and denote this by $\mathcal{D}(n)$. We refer to $D^k(n)$ for $k \geq 1$ as the kth derivative of n and we refer to $D^j(n)$ for $0 \leq j < k$ as integrals of $D^k(n)$.

If $n = \prod_{i=1}^{w} p_i^{a_i}$ is the prime decomposition of n then the definition of $D(n)$ may be given differently as

$$D(n) = \frac{n}{C(n)}\,\tau\left(\frac{n}{C(n)}\right),$$

where $\tau(n) = \prod_{i=1}^{w}(a_i + 1)$ is the number of divisors of n and $C(n) = \prod_{i=1}^{w} p_i$ is the core of n.

Our intention is to initiate a study of the ultimate behaviour of derived sequences. Different forms of ultimate behaviour are indicated in the following examples:

$$\mathcal{D}(5^2 17 \cdot 37) = \{5^2 17 \cdot 37, 2 \cdot 5, \underline{1}, \ldots\}, \tag{2}$$

$$\mathcal{D}(2^{25}) = \{2^{25}, 2^{24}5^2, 2^{27}3 \cdot 5, 2^{26}3^3, \underline{2^{26}3^3 13}, \ldots\}, \tag{3}$$

$$\mathcal{D}(13^{16}) = \{13^{16}, 2^4 13^{15}, 2^5 3 \cdot 5 \cdot 13^{14}, 2^5 5 \cdot 7 \cdot 13^{13}, \underline{2^4 5 \cdot 13^{13}}, \underline{2^5 13^{13}}, \ldots\}, \tag{4}$$

$$\mathcal{D}(2^{32}) = \{2^{32}, 2^{36}, \underline{2^{37}3^2}, \underline{2^{37}3 \cdot 37}, \underline{2^{36}37}, \ldots\}, \tag{5}$$

$$\mathcal{D}(3^8) = \{3^8, \underline{2^3 3^7}, \underline{2^2 3^7 \cdot 7}, \underline{2^2 3^6 \cdot 7}, \underline{2^3 3^6}, \ldots\}. \tag{6}$$

The underlined terms in each case form cycles. Precisely: if

$$\mathcal{D}(n) = \{\ldots, D^j(n), D^{j+1}(n), \ldots, D^{j+k-1}(n), D^{j+k}(n), \ldots\}$$

and $D^{j+k}(n) = D^j(n)$, where $j \geq 0$ and $k \geq 1$ is the smallest integer with this property, then $D^j(n), D^{j+1}(n), \ldots, D^{j+k-1}(n)$ is a derived k-cycle, which, if we need to, we describe as being arrived at in $j + k$ iterations of D. For example, in (3), $D(2^{26}3^3 13) = 2^{26}3^3 13$, and, in (6), $D(2^3 3^6) = 2^3 3^7$. We have 1-cycles in (2) and (3), and 2-, 3- and 4-cycles in (4), (5) and (6), respectively. The 3-cycle in (5) is arrived at in four iterations of D. We will refer to the element of a 1-cycle as a fixed point of D.

It is not known whether the ultimate behaviour of $\mathcal{D}(n)$ is a cycle for all n, or whether, for some n, $D^k(n)$ increases without bound as k increases. We will show, however, that cycles result for more than 99.5% of values of n. Many iterations of D may be required before a cycle is reached, if that is to be the case: for example, $\mathcal{D}(5^{63})$ arrives in 531 iterations at the fixed point $2^{1403}3^{329}5^{106}7^{15}23 \cdot 47 \cdot 53 \cdot 61$. Our most impressive example is $\mathcal{D}(17^{35}19^{39})$, which, in 443507 iterations of D, arrives at the fixed point

$$2^{4318267}3^{1370053}5^{525835}7^{159649}11^{33429}13^{20597}17^{1037}19^{1349}23^{299}31^{31}$$
$$\cdot 43 \cdot 61 \cdot 71^2 479 \cdot 1013 \cdot 22807 \cdot 105167 \cdot 1370053 \cdot 4318267.$$

Other instances of sequences of iterated arithmetic functions are given by Guy [3]. Iteration of the function $\sigma(n) - n$, for example, where σ is the sum-of-divisors function, has been studied extensively. The situation is similar: there is an eventual iterate equal to 1, or there is eventually a cycle, or the ultimate behaviour is unknown. Iteration of the function σ itself was studied in Cohen and te Riele [2].

In the following, p, q, r and t, with and without subscripts, denote prime numbers, and s, with and without subscripts, denotes a squarefree number. We include 1 as a squarefree number. Other letters denote positive integers, unless specified otherwise.

2 General results

We are able to give a number of results of a general nature.

First, it is easy to see that n is a fixed point of D if and only if either $n = 1$ or $n = \prod_{i=1}^{w} p_i^{a_i}$, where $\prod_{i=1}^{w} p_i = \prod_{i=1}^{w} a_i$. More generally, we have Proposition 1, below. For simplicity of notation, we will write $n = \prod p_0^{a_0}$ and $D(n) = \prod p_1^{a_1}$ as shorthand for $n = \prod_{i=1}^{w_0} p_{i0}^{a_{i0}}$ and $D(n) = \prod_{i=1}^{w_1} p_{i1}^{a_{i1}}$, and so on.

Proposition 1 *Suppose $n > 1$ and write $n = \prod p_0^{a_0}$, $D(n) = \prod p_1^{a_1}$, $\dots$, $D^{k-1}(n) = \prod p_{k-1}^{a_{k-1}}$. We have $D^k(n) = n$ if and only if $\prod p_0 \prod p_1 \cdots \prod p_{k-1} = \prod a_0 \prod a_1 \cdots \prod a_{k-1}$.*

Proof. Note that

$$\begin{aligned}
n &= \prod p_0^{a_0}, \\
D(n) &= \prod p_1^{a_1}, \qquad \text{so} \quad \prod p_1^{a_1} = \prod a_0 p_0^{a_0-1}, \\
D^2(n) &= \prod p_2^{a_2}, \qquad \text{so} \quad \prod p_2^{a_2} = \prod a_1 p_1^{a_1-1}, \\
&\vdots \\
D^{k-1}(n) &= \prod p_{k-1}^{a_{k-1}}, \quad \text{so} \quad \prod p_{k-1}^{a_{k-1}} = \prod a_{k-2} p_{k-2}^{a_{k-2}-1}.
\end{aligned}$$

If, further, $D^k(n) = n$ then $\prod p_0^{a_0} = \prod a_{k-1} p_{k-1}^{a_{k-1}-1}$, and we have

$$\begin{aligned}
\prod p_0 \prod p_1 \cdots \prod p_{k-1} &= \frac{\prod p_1^{a_1}}{\prod p_0^{a_0-1}} \frac{\prod p_2^{a_2}}{\prod p_1^{a_1-1}} \cdots \frac{\prod p_{k-1}^{a_{k-1}}}{\prod p_{k-2}^{a_{k-2}-1}} \frac{\prod p_0^{a_0}}{\prod p_{k-1}^{a_{k-1}-1}} \\
&= \prod a_0 \prod a_1 \cdots \prod a_{k-1}.
\end{aligned}$$

The converse is also clear. $\square$

It does not seem to be easy to use this result to determine cycles, but we can at least identify those numbers which have a derivative equal to the fixed point 1 of D:

Proposition 2 *The integer $n > 1$ has*

- *first derivative 1 if $n = s$,*
- *second derivative 1 if $n = p^2 s$ where $p > 2$ and $p \nmid s$,*
- *third derivative 1 if $n = p^3 s$ where $p > 3$ and $p \nmid s$, or $n = p^3 q^2 s$ where $p > 3$, $q > 3$, $p \neq q$ and $(s, pq) = 1$.*

There are no other situations in which n has a derivative equal to 1.

The proof is a matter of recognising those situations in which 2^2 or 3^3 might arise as exact factors of terms in the derived sequence, and avoiding them since these factors will persist in subsequent differentiations.

It is also a matter of checking that 2- and 3-cycles are obtained in the following situations.

Proposition 3 *For any p,*

- *if p^2+1 is squarefree, then* $\mathcal{D}(p^{p^2+1}) = \{\underline{p^{p^2+1}}, \underline{(p^2+1)p^{p^2}}, \ldots\}$,
- *if p^3+2 and p^3+1 are squarefree, then*

$$\mathcal{D}(p^{p^3+2}) = \{\underline{p^{p^3+2}}, \underline{(p^3+2)p^{p^3+1}}, \underline{(p^3+1)p^{p^3}}, \ldots\}.$$

The underlined terms are cycles. Notice that 13, 37, 61, ..., are primes p such that p^3+2 and p^3+1 are squarefree. Certainly, 2- and 3-cycles may arise in other ways, as in the examples (4) and (5).

We can also give a general instance that leads to a 4-cycle:

Proposition 4 *Let s be such that $s \equiv 2 \pmod 3$, $4s-1$ is squarefree and $2s-1$ is squarefree. Then $\mathcal{D}(3^{4s})$ results in a 4-cycle.*

Proof. Write $4s-1 = s_1$ and $2s-1 = 3s_2$ (where $3 \nmid s_2$). If s is odd, then

$$\mathcal{D}(3^{4s}) = \{3^{4s}, 2^2 3^{4s-1} s, \underline{2^2 3^{4s-2} s_1}, \underline{2^3 3^{4s-2} s_2}, \underline{2^3 3^{4s-1} s_2}, \underline{2^2 3^{4s-1} s_1}, \ldots\},$$

while if s is even, then

$$\mathcal{D}(3^{4s}) = \{3^{4s}, 2^3 3^{4s-1}(s/2), \underline{2^2 3^{4s-1} s_1}, \underline{2^2 3^{4s-2} s_1}, \underline{2^3 3^{4s-2} s_2}, \underline{2^3 3^{4s-1} s_2}, \ldots\}.$$

The underlined terms in each case are 4-cycles (in fact, algebraically, the same 4-cycle). $\square$

The smallest permissible value of s in this proposition is $s = 2$, as in the example (6); thereafter, s may take the values 11, 17, 26, 29, 35, ...

Other examples of 4-cycles are not difficult to find, and we need not always start at a prime power. For example:

$$\mathcal{D}(2^{10}3^{10}) = \{2^{10}3^{10}, \underline{2^{11}3^9 5^2}, \underline{2^{11}3^{10}5 \cdot 11}, \underline{2^{11}3^9 5 \cdot 11}, \underline{2^{10}3^{10} \cdot 11}, \ldots\}.$$

It was initially more difficult to find examples of derived k-cycles for $k > 4$, but, having found a few, patterns were detected suggesting infinite families of these. Some are described in the following propositions. (We have other, more general, examples.)

Proposition 5 *Let s be such that $(s, 2 \cdot 3 \cdot 5 \cdot 47) = 1$, and suppose $(s_1, 5 \cdot 23 \cdot 47) = 1$, where $s_1 = (3s-1)/2$. Put $n = 2^{3s}3^{45}5^5 23 s_1$. Then n, $D(n)$, ..., $D^4(n)$ is a 5-cycle.*

In this, s may take the values 1, 13, 23, 29, 41, 53, 61, ...

Proposition 6 (a) *Let s be such that $(s, 2\cdot 3\cdot 5\cdot 7\cdot 53) = 1$, and suppose $(s_1, 5\cdot 53\cdot 107) = 1$, where $s_1 = 6s+1$. Put $n = 2^{6s}3^{105}5^6 7s_1$. Then n, $D(n)$, …, $D^5(n)$ is a 6-cycle.*

(b) *Let s be such that $(s, 2\cdot 3\cdot 5\cdot 13\cdot 43) = 1$, and suppose $(s_1, 7\cdot 13\cdot 43\cdot 131) = 1$, where $s_1 = 30s+1$. Put $n = 2^{30s}3^{129}5^6 7\cdot 43 s_1$. Then n, $D(n)$, …, $D^5(n)$ is a 6-cycle.*

Proof. As usual, the proofs are a matter of straightforward verification. We will demonstrate this in the case of Proposition 6(a). We have

$$\begin{aligned}
n &= 2^{6s}3^{105}5^6 7s_1, && \\
D(n) &= 2^{6s+1}3^{107}5^6 7s, && \text{since } (s_1, 2\cdot 3\cdot 5\cdot 7) = 1,\\
D^2(n) &= 2^{6s+1}3^{107}5^5 107s_1, && \text{since } (s, 2\cdot 3\cdot 5\cdot 7) = 1,\\
D^3(n) &= 2^{6s}3^{106}5^5 107s_1, && \text{since } (s_1, 2\cdot 3\cdot 5\cdot 107) = 1,\\
D^4(n) &= 2^{6s+1}3^{106}5^5 53s, && \text{since } (s_1, 2\cdot 3\cdot 5\cdot 107) = 1,\\
D^5(n) &= 2^{6s+1}3^{105}5^5 53s_1, && \text{since } (s, 2\cdot 3\cdot 5\cdot 53) = 1,\\
D^6(n) &= 2^{6s}3^{105}5^6 7s_1, && \text{since } (s_1, 2\cdot 3\cdot 5\cdot 53) = 1.
\end{aligned}$$

But $D^6(n) = n$. We have also used the fact that s and s_1 are squarefree. □

In Proposition 6(a), we may have $s = 11, 13, 17, 23, 31, 37, 41, \ldots$. In Proposition 6(b), s may take the values $1, 7, 11, 19, 23, 37, 41, \ldots$.

Proposition 7 *Let s be such that $s \equiv 7 \pmod{10}$ and $(s, 3\cdot 23\cdot 31\cdot 47\cdot 103\cdot 311) = 1$, and suppose $(s_1, 3\cdot 5\cdot 31\cdot 47\cdot 103\cdot 311) = 1$, where $s_1 = (2s+1)/5$. Put $n = 2^{2s}3^{311}5^{46}103s_1$. Then n, $D(n)$, …, $D^7(n)$ is an 8-cycle.*

In this, s may take the values $17, 107, 167, 197, 227, \ldots$. Proposition 7 was found by observing that $\mathcal{D}(5^{13}29^{54})$ arrives in 428 iterations of D at the 8-cycle beginning with $29^{29}n$, with $s = 557$. Two other examples of 8-cycles turned up in our searches:

$$\begin{array}{ll}
2^{159}3^{16725}5^5 79\cdot 8363, & 2^{158}3^{16726}5^7 53\cdot 223,\\
2^{159}3^{16725}5^6 7\cdot 79\cdot 8363, & 2^{159}3^{16727}5^7 53\cdot 223,\\
2^{158}3^{16727}5^6 7\cdot 43\cdot 53\cdot 389, & 2^{159}3^{16727}5^5 43\cdot 79\cdot 389,\\
2^{158}3^{16727}5^5 43\cdot 53\cdot 389, & 2^{158}3^{16726}5^5 43\cdot 79\cdot 389,
\end{array}$$

and

$$\begin{array}{ll}
2^{87}3^{149325}5^5 43\cdot 197\cdot 379, & 2^{86}3^{149326}5^7 11\cdot 29\cdot 181,\\
2^{87}3^{149325}5^6 7\cdot 43\cdot 197\cdot 379, & 2^{87}3^{149327}5^7 11\cdot 29\cdot 181,\\
2^{86}3^{149327}5^6 7\cdot 29\cdot 31\cdot 4817, & 2^{87}3^{149327}5^5 31\cdot 43\cdot 4817,\\
2^{86}3^{149327}5^5 29\cdot 31\cdot 4817, & 2^{86}3^{149326}5^5 31\cdot 43\cdot 4817.
\end{array}$$

These occur in $\mathcal{D}(3^{16695})$ and $\mathcal{D}(3^{149319})$, respectively. It is not difficult to determine a two-parameter family, containing general exponents on 2 and 3, that includes both of these 8-cycles.

We have no examples of derived k-cycles with $k = 7$ or $k > 8$.

3 Bounded derived sequences

We have two results on the number of bounded derived sequences, the first resulting largely from a direct search, the second of a much more theoretical nature.

Proposition 8 *For all $n < 1.5 \cdot 10^{10}$, the derived sequence $\mathcal{D}(n)$ is bounded.*

Proof. The proof involved a direct incremental investigation of all numbers $n = \prod_{i=1}^{w} p_i^{a_i} < 1.5 \cdot 10^{10}$ for which $\sum_{i=1}^{w}(a_i - 1) \geq 8$. (An initial factorisation of each n determined whether this condition was satisfied.) In all cases, $\mathcal{D}(n)$ resulted in a cycle. We showed also that the same is true of all n with $\sum_{i=1}^{w}(a_i - 1) \leq 7$. For example, suppose $n = p^3q^2s$, where $p \neq q$ and $(s, pq) = 1$. If p and q are both greater than 3, then $D(n) = 2 \cdot 3p^2q$, $D^2(n) = 2p$ and $D^3(n) = 1$ (or use Proposition 2); then the numbers $p^3 2^2 s$ $(p > 3)$, $p^3 3^2 s$ $(p > 3)$, $2^3 q^2 s$ $(q > 3)$, $3^3 q^2 s$ $(q > 3)$, $2^3 3^2 s$ and $3^3 2^2 s$ must be separately and similarly considered. $\square$

It would seem probable that $\mathcal{D}(n)$ is unbounded for some n, and in that case for all numbers ns, where $(n, s) = 1$, as well. Then the set of such numbers would have positive density in $\mathbb{N}$. We show now that this density is less than 0.004. We have computed lower bounds for the densities of 45 classes of integers, including the known result for the set of squarefree numbers, and have shown that integers in these 45 classes have bounded derived sequences. In each case, the density was computed to within 10^{-6} and then truncated to five decimal digits. Those densities (33 of the 45) which gave a positive lower bound (to that number of digits) are given in Table 1. We refer, for example, to the type p^3q^2S as the set of integers of the form p^3q^2s, where $p \neq q$ and $(s, pq) = 1$. How the given densities were obtained will be illustrated shortly by the determination of such for the type p^3q^2S. The 45 classes of integers were all possible types of the form $p^a q^b \cdots S$ such that $(a-1)+(b-1)+\cdots \leq 7$ (precisely as considered in the proof of Proposition 8), and Table 1 shows that their cumulative density is at least 0.996.

We show now how we obtain that the density of the class p^3q^2S is 0.01447, truncated to five decimal digits.

Let $x > 0$ be given. In general, the number of positive integers $n \leq x$, not divisible by the prime squares $p_1^2, p_2^2, \ldots, p_l^2$ and not divisible by the primes $q_1, q_2, \ldots, q_m$ (all these being different primes) is

$$x \cdot \prod_{i=1}^{l}\left(1 - \frac{1}{p_i^2}\right) \cdot \prod_{i=1}^{m}\left(1 - \frac{1}{q_i}\right) + O(1).$$

Note that the positive integer $n \leq x$ is squarefree if $p^2 \nmid n$ for all primes $p \leq \sqrt{x}$. Fix distinct primes p and q. As above, the number of squarefree positive integers $n \leq x/p^3q^2$ which are divisible by neither p nor q is

$$\frac{x}{p^3q^2}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\prod_{\substack{r\le\sqrt{x/p^3q^2}\\ r\ne p,\,q}}\left(1-\frac{1}{r^2}\right)+O(1)$$

$$=\frac{x}{p^3q^2}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{p^2}\right)^{-1}\left(1-\frac{1}{q^2}\right)^{-1}\prod_{r\le\sqrt{x/p^3q^2}}\left(1-\frac{1}{r^2}\right)+O(1)$$

$$=x\cdot\frac{1}{p^2(p+1)}\cdot\frac{1}{q(q+1)}\prod_{r\le\sqrt{x/p^3q^2}}\left(1-\frac{1}{r^2}\right)+O(1),$$

the products being taken over primes r.

In general, for $y>0$ we have $\prod_{r\le y}(1-1/r^2)=6/\pi^2+O(1/y)$. Applying this above, the number of squarefree positive integers $n\le x/p^3q^2$ which are divisible by neither p nor q is then

$$x\cdot\frac{1}{p^2(p+1)}\cdot\frac{1}{q(q+1)}\left(\frac{6}{\pi^2}+O\left(\sqrt{\frac{p^3q^2}{x}}\right)\right)+O(1)$$
$$=\frac{6x}{\pi^2}\cdot\frac{1}{p^2(p+1)}\cdot\frac{1}{q(q+1)}+O\left(\sqrt{\frac{x}{p^3q^2}}\right).$$

To find the number of positive integers $n\le x$ of the form $n=p^3q^2s$, where p, q are any two distinct primes and $(s,pq)=1$, we sum our result above over primes $p\le\sqrt[3]{x}$ and $q\le\sqrt{x/p^3}$. Therefore the proportion of these integers p^3q^2s which are at most x is given by

$$\frac{1}{x}\sum_{p\le\sqrt[3]{x}}\sum_{\substack{q\le\sqrt{x/p^3}\\ q\ne p}}\left(\frac{6x}{\pi^2}\cdot\frac{1}{p^2(p+1)}\cdot\frac{1}{q(q+1)}+O\left(\sqrt{\frac{x}{p^3q^2}}\right)\right)$$
$$=\sum_{p\le\sqrt[3]{x}}\sum_{\substack{q\le\sqrt{x/p^3}\\ q\ne p}}\frac{6}{\pi^2}\cdot\frac{1}{p^2(p+1)}\cdot\frac{1}{q(q+1)}+O\Bigg(\sum_{p\le\sqrt[3]{x}}\sum_{q\le\sqrt{x/p^3}}\frac{1}{\sqrt{xp^3q^2}}\Bigg).$$

In general, $\sum_{p\le y}1/p^{3/2}=O(1)$ and $\sum_{q\le y}1/q=O(\log\log y)$, and so

$$O\Bigg(\sum_{p\le\sqrt[3]{x}}\sum_{q\le\sqrt{x/p^3}}\frac{1}{\sqrt{xp^3q^2}}\Bigg)=O\Bigg(\frac{1}{\sqrt{x}}\sum_{p\le\sqrt[3]{x}}\frac{1}{\sqrt{p^3}}\sum_{q\le\sqrt{x}}\frac{1}{q}\Bigg)=O\Bigg(\frac{\log\log x}{\sqrt{x}}\Bigg).$$

Thus the required density is

$$\lim_{x\to\infty}\sum_{p\le\sqrt[3]{x}}\sum_{\substack{q\le\sqrt{x/p^3}\\ q\ne p}}\frac{6}{\pi^2}\cdot\frac{1}{p^2(p+1)}\cdot\frac{1}{q(q+1)}=\frac{6}{\pi^2}\sum_{p}\frac{1}{p^2(p+1)}\sum_{q\ne p}\frac{1}{q(q+1)}\ge 0.0144.$$

The double sum was estimated on a computer.

We have therefore shown the following:

Type	*Density*	*Truncated density*
S	$\frac{6}{\pi^2}$	0.60792
p^2S	$\frac{6}{\pi^2}\sum_p \frac{1}{p(p+1)}$	0.20075
p^3S	$\frac{6}{\pi^2}\sum_p \frac{1}{p(p+1)}$	0.07417
p^2q^2S	$\frac{6}{\pi^2}\sum_p \frac{1}{p(p+1)}\sum_{q>p} \frac{1}{q(q+1)}$	0.02212
p^4S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^3(p+1)}$	0.03206
p^3q^2S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^2(p+1)}\sum_{q\neq p} \frac{1}{q(q+1)}$	0.01447
$p^2q^2r^2S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p(p+1)}\sum_{q>p} \frac{1}{q(q+1)}\sum_{r>q} \frac{1}{r(r+1)}$	0.00107
p^5S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^4(p+1)}$	0.01474
p^4q^2S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^3(p+1)}\sum_{q\neq p} \frac{1}{q(q+1)}$	0.00586
p^3q^3S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^2(p+1)}\sum_{q>p} \frac{1}{q^2(q+1)}$	0.00216
$p^3q^2r^2S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p^2(p+1)}\sum_{q\neq p} \frac{1}{q(q+1)}\sum_{r>q,\, r\neq p} \frac{1}{r(r+1)}$	0.00091
$p^2q^2r^2t^2S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p(p+1)}\sum_{q>p} \frac{1}{q(q+1)}\sum_{r>q} \frac{1}{r(r+1)}\sum_{t>r} \frac{1}{t(t+1)}$	0.00002
p^6S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^5(p+1)}$	0.00699
p^5q^2S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^4(p+1)}\sum_{q\neq p} \frac{1}{q(q+1)}$	0.00259
p^4q^3S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^3(p+1)}\sum_{q\neq p} \frac{1}{q^2(q+1)}$	0.00163
$p^4q^2r^2S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p^3(p+1)}\sum_{q\neq p} \frac{1}{q(q+1)}\sum_{r>q,\, r\neq p} \frac{1}{r(r+1)}$	0.00035
$p^3q^3r^2S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p^2(p+1)}\sum_{q>p} \frac{1}{q^2(q+1)}\sum_{r\neq q,p} \frac{1}{r(r+1)}$	0.00023
$p^3q^2r^2t^2S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p^2(p+1)}\sum_{q\neq p} \frac{1}{q(q+1)}\sum_{r>q,\, r\neq p} \frac{1}{r(r+1)}\sum_{t>r,\, t\neq p} \frac{1}{t(t+1)}$	0.00002
p^7S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^6(p+1)}$	0.00338
p^6q^2S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^5(p+1)}\sum_{q\neq p} \frac{1}{q(q+1)}$	0.00120
p^5q^3S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^4(p+1)}\sum_{q\neq p} \frac{1}{q^2(q+1)}$	0.00068
$p^5q^2r^2S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p^4(p+1)}\sum_{q\neq p} \frac{1}{q(q+1)}\sum_{r>q,\, r\neq p} \frac{1}{r(r+1)}$	0.00015
p^4q^4S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^3(p+1)}\sum_{q>p} \frac{1}{q^3(q+1)}$	0.00029
$p^4q^3r^2S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p^3(p+1)}\sum_{q\neq p} \frac{1}{q^2(q+1)}\sum_{r\neq q,\, r\neq p} \frac{1}{r(r+1)}$	0.00016
$p^3q^3r^3S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p^2(p+1)}\sum_{q>p} \frac{1}{q^2(q+1)}\sum_{r>q} \frac{1}{r^2(r+1)}$	0.00001
p^8S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^7(p+1)}$	0.00165
p^7q^2S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^6(p+1)}\sum_{q\neq p} \frac{1}{q(q+1)}$	0.00057
p^6q^3S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^5(p+1)}\sum_{q\neq p} \frac{1}{q^2(q+1)}$	0.00030
$p^6q^2r^2S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p^5(p+1)}\sum_{q\neq p} \frac{1}{q(q+1)}\sum_{r>q,\, r\neq p} \frac{1}{r(r+1)}$	0.00006
p^5q^4S	$\frac{6}{\pi^2}\sum_p \frac{1}{p^4(p+1)}\sum_{q\neq p} \frac{1}{q^3(q+1)}$	0.00023
$p^5q^3r^2S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p^4(p+1)}\sum_{q\neq p} \frac{1}{q^2(q+1)}\sum_{r\neq q,\, r\neq p} \frac{1}{r(r+1)}$	0.00006
$p^4q^4r^2S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p^3(p+1)}\sum_{q>p} \frac{1}{q^3(q+1)}\sum_{r\neq q,\, r\neq p} \frac{1}{r(r+1)}$	0.00002
$p^4q^3r^3S$	$\frac{6}{\pi^2}\sum_p \frac{1}{p^3(p+1)}\sum_{q\neq p} \frac{1}{q^2(q+1)}\sum_{r>q,\, r\neq p} \frac{1}{r^2(r+1)}$	0.00001
	Total:	0.99683

Table 1: Densities giving a positive lower bound

Proposition 9 *The density in* $\mathbb{N}$ *of integers* n *for which* $\mathcal{D}(n)$ *is unbounded is less than* 0.004.

Although we cannot exhibit an unbounded derived sequence, we do have the following result.

Proposition 10 *For any positive integer* M*, there exists an integer* n *such that* $\mathcal{D}(n)$ *requires more than* M *iterations of* D *before a cycle is possible.*

Proof. Let p be a large prime, with "large" to be qualified shortly. Take $n = p^{4p}$. Then

$$\mathcal{D}(n) = \{p^{4p}, 2^2p^{4p}, 2^4p^{4p}, 2^7p^{4p}, 2^87p^{4p}, 2^{12}p^{4p}, 2^{15}3p^{4p}, \dots\}.$$

The exact factor p^{4p} will persist in all terms until the exponent on 2 or some other prime (not p) equals p. Until this happens, if it will ever happen, there can be no derived k-cycle in $\mathcal{D}(n)$, for any k. For suppose there were such a cycle and let the ith term of the cycle be $\prod p_i^{a_i}$, $1 \le i \le k$, where the notation is as in Proposition 1. It follows that at most 2^k divides $\prod p_1 \prod p_2 \cdots \prod p_k$, while at least 2^{2k} divides $\prod a_1 \prod a_2 \cdots \prod a_k$. By Proposition 1, this is a contradiction. The proof is non-constructive, in that we can say only that, whatever the value of M, a prime p exists such that p exceeds all exponents on primes other than p resulting from M iterations of D. $\square$

4 Further work

(1) Find examples of integers n for which $\mathcal{D}(n)$ results in a k-cycle for $k = 7$ or $k > 8$. Do derived k-cycles exist for all positive integers k?

(2) Can it be shown that $\mathcal{D}(n)$ is unbounded for some n, as we have conjectured above? We suspect this to be the case for "most" sequences $\mathcal{D}(p^{q^qp})$ $(p \ne q)$, for reasons suggested in the proof of Proposition 10 (where we considered $q = 2$). It is not known whether $\mathcal{D}(7^{4046})$, $\mathcal{D}(11^{1674})$ and $\mathcal{D}(13^{504})$ are bounded, but this is the case for all smaller exponents on the respective primes. (It took a few weeks for referee's comments to be returned. We are grateful for those. In that time we left a program running, checking $\mathcal{D}(31^{124})$ for boundedness. After $k = 48218701$ iterations, we found no cycle and $D^k(31^{124}) = 2^{15162685573}3^{780548532}5^{348780008}\cdots 127^{10414}131^{131}139^{139}149^{141851}\cdots 62763353 \cdot 348779999$, with all primes up to 131 present. The program was then permanently halted.)

(3) As in the calculus, differentiation is a craft, but integration is an art. Can a technique be developed for finding integrals of a given positive integer or of showing that certain integers, the primes being examples, have no integrals? In Proposition 2, we have given all integrals of 1, but even to identify all integrals of 4 seems to be very difficult.

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2000 *Mathematics Subject Classification*: Primary 11Y55; Secondary 11A25, 11B83.
Keywords: Arithmetic functions, density, unbounded sequences, cycles.

Received October 25, 2002; revised version received December 1, 2002. Published in *Journal of Integer Sequences* December 23, 2002. Revised, February 10 2004.

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