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# A multidimensional version of a result of Davenport-Erdős

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**Abstract**

Davenport and Erdős showed that the distribution of values of sums of the form

$$S_h(x) = \sum_{m=x+1}^{x+h} \left(\frac{m}{p}\right),$$

where $p$ is a prime and $\left(\frac{m}{p}\right)$ is the Legendre symbol, is normal as $h, p \to \infty$ such that $\frac{\log h}{\log p} \to 0$. We prove a similar result for sums of the form

$$S_h(x_1, \ldots, x_n) = \sum_{z_1=x_1+1}^{x_1+h} \cdots \sum_{z_n=x_n+1}^{x_n+h} \left(\frac{z_1 + \cdots + z_n}{p}\right).$$

## 1. Introduction

Given a prime number $p$, an integer $x$ and a positive integer $h$, we consider the sum

$$S_h(x) = \sum_{m=x+1}^{x+h} \left(\frac{m}{p}\right),$$

where here and in what follows $\left(\frac{m}{p}\right)$ denotes the Legendre symbol. The expected value of such a sum is $\sqrt{h}$. If $p$ is much larger than $h$, it is a very difficult problem to show that there is any cancellation in an individual sum $S_h(x)$ as above. The classical inequality of Pólya-Vinogradov (see [8], [10]) shows that $S_h(x) = O(\sqrt{p}\log p)$, and assuming the Generalized Riemann Hypothesis, Montgomery and Vaughan [7] proved that $S_h(x) = O(\sqrt{p}\log\log p)$. The results of Burgess [2] provide cancellation in $S_h(x)$ for smaller values of $h$, as small as $p^{1/4}$. One does expect to have cancellation in $S_h(x)$ for $h > p^\epsilon$, for fixed $\epsilon > 0$ and $p$ large. This would imply the well-known hypothesis of Vinogradov that the smallest positive quadratic nonresidue mod $p$ is $< p^\epsilon$, for any fixed $\epsilon > 0$ and $p$ large enough in terms of $\epsilon$. We mention that Ankeny [1] showed that assuming the Generalized Riemann Hypothesis, the smallest positive quadratic nonresidue mod $p$ is $O(\log^2 p)$. It is much easier to obtain cancellation, even square root cancellation, if one averages $S_h(x)$ over $x$. In fact, Davenport and Erdős [5] entirely solved the problem of the distribution of values of $S_h(x)$, $0 \le x < p$, as $h, p \to \infty$ such that $\frac{\log h}{\log p} \to 0$. Under these growth conditions they showed that the distribution becomes normal. Precisely, they proved that

$$\frac{1}{p}M_p(\lambda) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{1}{2}t^2}dt, \quad \text{as } p \to \infty,$$

where $M_p(\lambda)$ is the number of integers $x$, $0 \le x < p$, satisfying $S_h(x) \le \lambda h^{\frac{1}{2}}$.

For a fixed $n \ge 2$, we consider multidimensional sums of the form

$$S_h(x_1,\dots,x_n) = \sum_{z_1=x_1+1}^{x_1+h}\cdots\sum_{z_n=x_n+1}^{x_n+h}\left(\frac{z_1+\cdots+z_n}{p}\right), \tag{1.1}$$

where $p$ is a prime number, $x_1,\dots,x_n$ are integer numbers, and $h$ is a positive integer. Upper bounds for individual sums of this type have been provided by Chung [3]. In this paper we investigate the distribution of values of these sums, and obtain a result similar to that of Davenport and Erdős. Let

$$c_n := \int_0^n f(t)^2 dt, \tag{1.2}$$

where $f(t)$ is the volume of the region in $\mathbb{R}^{n-1}$ defined by

$$\{(a_1,\dots,a_{n-1}) \in \mathbb{R}^{n-1} : 0 < a_i \le 1,\ i = 1,\dots,n-1;\ t-1 \le a_1+\cdots+a_{n-1} < t\}.$$

We will see that this constant $c_n$ naturally appears as a normalizing factor in our distribution result below. Let $M_{n,p}(\lambda)$ be the number of lattice points $(x_1,\dots,x_n)$ with $0 \le x_1,\dots,x_n < p$, such that

$$S_h(x_1,\dots,x_n) \le \lambda c_n^{\frac{1}{2}} h^{n-\frac{1}{2}}.$$

Then we show that as $h, p \to \infty$ such that $\frac{\log h}{\log p} \to 0$, one has

$$\frac{1}{p^n}M_{n,p}(\lambda) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{t^2}{2}}dt.$$

## 2. Estimating the moments

We now proceed to estimate higher moments of our sums $S_h(x_1, \dots, x_n)$.

**Lemma 1.** *Let $p$ be a prime number and let $h$ and $r$ be positive integers. Then*

$$\sum_{x_1,\dots,x_n (\mathrm{mod}\, p)} S_h^{2r}(x_1, \dots, x_n) = 1 \cdot 3 \cdots (2r-3)(2r-1)$$

$$\cdot \left(c_n h^{2n-1} + O_{n,r}(h^{2n-2})\right)^r \left(p^n + O_r(p^{n-1})\right) + O_r\left(h^{2nr} p^{n-\frac{1}{2}}\right), \quad (2.1)$$

*and*

$$\sum_{x_1,\dots,x_n (\mathrm{mod}\, p)} S_h^{2r-1}(x_1, \dots, x_n) = O_r\left(h^{n(2r-1)} p^{n-\frac{1}{2}}\right). \quad (2.2)$$

*Proof.* Consider first the case when the exponent is $2r$. We have

$$S_h(x_1, \dots, x_n) = \sum_{a_1=1}^{h} \cdots \sum_{a_n=1}^{h} \left(\frac{x_1 + \cdots + x_n + a_1 + \cdots + a_n}{p}\right).$$

Therefore

$$S_h^{2r}(x_1, \dots, x_n) =$$

$$\sum_{a_{1,1}=1}^{h} \cdots \sum_{a_{n,1}=1}^{h} \cdots \sum_{a_{1,2r}=1}^{h} \cdots \sum_{a_{n,2r}=1}^{h} \left(\frac{(x_1+\cdots+x_n+a_{1,1}+\cdots+a_{n,1})\cdots(x_1+\cdots+x_n+a_{1,2r}+\cdots+a_{n,2r})}{p}\right)$$

and so

$$\sum_{x_1,\dots,x_n (\mathrm{mod}\, p)} S_h^{2r}(x_1, \dots, , x_n)$$

$$= \sum_{\substack{a_{i,j}=1 \\ 1\le i\le n \\ 1\le j\le 2r}}^{h} \sum_{x_1,\dots,x_n (\mathrm{mod}\, p)} \left(\frac{(x_1+\cdots+x_n+a_{1,1}+\cdots+a_{n,1})\cdots(x_1+\cdots+x_n+a_{1,2r}+\cdots+a_{n,2r})}{p}\right).$$

Divide the sets of $n$-tuples $\{(a_{1,i}, \dots, a_{n,i}) : i = 1, \dots, 2r\}$ into two types. If there exists an $i$ such that the number of $j \in \{1, \dots, 2r\}$ for which $a_{1,i} + \cdots + a_{n,i} = a_{1,j} + \cdots + a_{n,j}$ is odd, we say that it is of type 1. The others will be of type 2. First consider the sum of terms of type 1. Since for each fixed $x_2, \dots, x_n$, the product $(x_1 + \cdots + x_n + a_{1,1} + \cdots + a_{n,1}) \cdots (x_1 + \cdots + x_n + a_{1,2r} + \cdots + a_{n,2r})$, as a polynomial in $x_1$, is not congruent mod $p$ to the square of another polynomial, by Weil's bounds [11] we have

$$\sum_{x_2,\dots,x_n (\mathrm{mod}\, p)} \sum_{x_1 (\mathrm{mod}\, p)} \left(\frac{(x_1+\cdots+x_n+a_{1,1}+\cdots+a_{n,1})\cdots(x_1+\cdots+x_n+a_{1,2r}+\cdots+a_{n,2r})}{p}\right)$$

$$= \sum_{x_2,\cdots,x_n (\mathrm{mod}\, p)} O_r(p^{1/2}) = O_r(p^{n-\frac{1}{2}}).$$

So the sum of terms of type 1 is $O_r\left(h^{2nr} p^{n-\frac{1}{2}}\right)$. Now consider the sum of terms of type 2. Since the polynomial $(x_1 + \cdots + x_n + a_{1,1} + \cdots + a_{n,1}) \cdots (x_1 + \cdots + x_n + a_{1,2r} + \cdots + a_{n,2r})$ is a perfect square in this case, the Legendre symbol is 1, except for those values of $x_1, \dots, x_n$

for which this product vanishes mod $p$. Since the product has at most $r$ distinct factors, for any values of $x_2, \dots, x_n$ there are at most $r$ values of $x_1$ for which the product vanishes mod $p$. Thus the sum over $x_1, \dots, x_n$ is at most $p^n$, and at least $(p-r)p^{n-1}$. Hence the contribution of terms of type 2 is

$$F(h,n,r)\left(p^n + O_r(p^{n-1})\right),$$

where $F(h,n,r)$ is the number of sets $\{(a_{1,i},\dots,a_{n,i}) : i = 1,\dots,2r\}$ yielding multinomials of type 2, i.e., sets for which each value of $a_{1,i}+\cdots+a_{n,i}$ occurs an even number of times, as $i$ runs over the set $\{1,2,\dots,2r\}$. For any integer $m$ with $n \le m \le nh$, let $N_m(h,n)$ be the number of $n$-tuples $(a_{1,i},\dots,a_{n,i})$ for which $1 \le a_{1,i},\dots,a_{n,i} \le h$ and $a_{1,i}+\cdots+a_{n,i} = m$. Then the number of pairs of $n$-tuples $(a_{1,i},\dots,a_{n,i})$, $(a_{1,j},\dots,a_{n,j})$, with $a_{1,i}+\cdots+a_{n,i} = a_{1,j}+\cdots+a_{n,j}$, is $\sum_m (N_m(h,n))^2$. In what follows we write simply $N_m$ instead of $N_m(h,n)$. The number of ways to choose $r$ such pairs of $n$-tuples (not necessarily distinct) is $\left(\sum_m N_m^2\right)^r$, and the number of ways to arrange these pairs in $2r$ places is $(2r-1)(2r-3)\cdots 3\cdot 1$. Hence,

$$F(h,n,r) \le 1\cdot 3\cdots(2r-3)(2r-1)\left(\sum_m N_m^2\right)^r.$$

On the other hand, the number of ways of choosing $r$ pairs of distinct sums is at least

$$\begin{aligned}\left(\sum_m N_m^2\right)\left(\sum_m N_m^2 - \max_m\{N_m^2\}\right)\cdots\left(\sum_m N_m^2 - (r-1)\max_m\{N_m^2\}\right)\\ \ge \left(\sum_m N_m^2 - r\max_m\{N_m^2\}\right)^r,\end{aligned}$$

and the number of different ways to arrange them in $2r$ places is $(2r-1)(2r-3)\cdots 3\cdot 1$. Thus

$$\begin{aligned}1\cdot 3\cdots(2r-3)(2r-1)\left(\sum_m N_m^2 - r\max_m N_m^2\right)^r &\le F(h,n,r)\\ &\le 1\cdot 3\cdots(2r-3)(2r-1)\left(\sum_m N_m^2\right)^r.\end{aligned}$$

Next, we estimate the number $N_m(h,n) = N_m$. It is clear that for any $m$ with $0 < m \le nh$, $N_m$ is the number of lattice points in the region $R_m$ in $\mathbb{R}^{n-1}$ given by

$$R_m := \begin{cases} 0 < a_i \le h, & \text{for } i = 1,\dots,n-1;\\ m-h \le a_1+\cdots+a_{n-1} < m. \end{cases}$$

We send the region $R_m$ to the unit cube in $\mathbb{R}^{n-1}$ via the map $\mathbf{x} \mapsto \frac{\mathbf{x}}{h}$. Then we have

$$\overline{R_m} := \begin{cases} 0 < a_i \le 1, & \text{for } i = 1,\dots,n-1;\\ \frac{m}{h}-1 \le a_1+\cdots+a_{n-1} < \frac{m}{h}. \end{cases}$$

By the Lipschitz principle [4] we know that

$$N_m = vol(R_m) + O_n(h^{n-2}) = h^{n-1}vol(\overline{R}_m) + O_n(h^{n-2}).$$

With $f$ defined as in the Introduction, we may write $vol(\overline{R}_m) = f(\frac{m}{h})$. Then

$$\begin{aligned}\sum_{0<m\leq nh} N_m^2 &= \sum_{0<m\leq nh} h^{2n-2}\left(f\left(\frac{m}{h}\right)\right)^2 + \sum_{0<m\leq nh} O_n(h^{2n-3})\\ &= h^{2n-1}\sum_{0<m\leq nh}\left(f\left(\frac{m}{h}\right)\right)^2\frac{1}{h} + O_n(h^{2n-2})\\ &= h^{2n-1}\int_0^n (f(t))^2\,dt + O_n(h^{2n-2})\\ &= h^{2n-1}c_n + O_n(h^{2n-2}), \quad \text{as } h\to\infty.\end{aligned}$$

Hence

$$F(h,n,r) = 1\cdot 3\cdots(2r-3)(2r-1)\left(c_n h^{2n-1} + O_{n,r}(h^{2n-2})\right)^r,$$

and (2.1) follows. It is clear that (2.2) holds, since there are no sets of type 2 in this case. This completes the proof of the lemma.

□

## 3. Main results

By using the estimates for the higher moments of $S_h(x_1,\ldots,x_n)$ given in Lemma 1, we show that under appropriate growth conditions on $h, p$, the distribution of our sums $S_h(x_1,\ldots,x_n)$ is normal.

**Theorem 1.** *Let $h$ be any function of $p$ such that*

$$h\to\infty,\quad \frac{\log h}{\log p}\to 0 \quad \textit{as } p\to\infty. \tag{3.1}$$

*Let $M_{n,p}(\lambda)$ denote the number of lattice points $(x_1,\ldots,x_n)$, $0\leq x_1,\ldots,x_n<p$, such that*

$$S_h(x_1,\ldots,x_n) \leq \lambda c_n^{\frac{1}{2}} h^{n-\frac{1}{2}},$$

*with $S_h(x_1,\ldots,x_n)$ defined by (1.1) and $c_n$ defined by (1.2). Then*

$$\frac{1}{p^n}M_{n,p}(\lambda) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{t^2}{2}}\,dt, \quad \textit{as } p\to\infty.$$

*Proof.* We consider the sum

$$\frac{1}{p^n}\sum_{x_1,\ldots,x_n (\mathrm{mod}\ p)}\left(\frac{1}{c_n^{1/2}h^{n-1/2}}S_h(x_1,\ldots,x_n)\right)^r. \tag{3.2}$$

It follows from the above lemma that for each fixed $r$ and $n$, if $r$ is even, then the quantity from (3.2) is

$$1\cdot 3\cdots(r-3)(r-1)\left(1+O_{n,r}\left(\frac{1}{h}\right)\right)^r\left(1+O_r\left(\frac{1}{p}\right)\right) + O_{n,r}(h^{\frac{r}{2}}p^{-\frac{1}{2}}),$$

while if $r$ is odd, the quantity from (3.2) is $O_{n,r}(h^{\frac{r}{2}}p^{-\frac{1}{2}})$. Using (3.1), we have that for each positive integer $r$,

$$\frac{1}{p^n}\sum_{x_1,\ldots,x_n (\mathrm{mod}\ p)}\left(\frac{1}{c_n^{1/2}h^{n-1/2}}S_h(x_1,\ldots,x_n)\right)^r \to \mu_r, \quad \text{as } p\to\infty, \tag{3.3}$$

where $\mu_r = \begin{cases} 1\cdot 3\cdots(r-1), & \text{if } r \text{ is even;} \\ 0, & \text{if } r \text{ is odd.} \end{cases}$

Let $N_{n,p}(s)$ be the number of $n$-tuples $(x_1,\dots,x_n)$ with $0 \le x_i < p$, $i=1,\dots,n$ such that $S_h(x_1,\dots,x_n) \le s$. Then $N_{n,p}(s)$ is a non-decreasing function of $s$ with discontinuities at certain integral values of $s$. We also note that $N_{n,p}(s)=0$ if $s<-h^n$, $N_{n,p}(s)=p^n$ if $s \ge h^n$, and $M_{n,p}(\lambda) = N_{n,p}(\lambda c_n^{\frac{1}{2}} h^{n-\frac{1}{2}})$. We write (3.3) in the form

$$\frac{1}{p^n}\sum_{s=-h^n}^{h^n}\left(\frac{s}{c_n^{\frac{1}{2}}h^{n-\frac{1}{2}}}\right)^r (N_{n,p}(s)-N_{n,p}(s-1)) \to \mu_r, \quad \text{as } p\to\infty. \tag{3.4}$$

This is similar to relation (26) of Davenport-Erdős [5]. Following their argument, if we set

$$\Phi_{n,p}(t) = \frac{1}{p^n}N_{n,p}(tc_n^{\frac{1}{2}}h^{n-\frac{1}{2}}) = \frac{1}{p^n}M_{n,p}(t),$$

and

$$\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{1}{2}u^2}\,du,$$

we obtain

$$\int_{-\infty}^{\infty} t^r\, d\Phi_{n,p}(t) \to \int_{-\infty}^{\infty} t^r\, d\Phi(t), \quad \text{as } p\to\infty, \tag{3.5}$$

for any fixed positive integer $r$, which is the analogue of relation (28) from [5]. It now remains to show that, for each real number $\lambda$,

$$\Phi_{n,p}(\lambda) \to \Phi(\lambda), \quad \text{as } p\to\infty. \tag{3.6}$$

The assertion of (3.6) follows from the well-known fact (see [6]) in the theory of probability that if $F_k$ and $F$ are probability distributions with finite moments $m_{k,r}$, $m_r$ of all orders, respectively, and if $F$ is the unique distribution with the moments $m_r$ such that $m_{k,r}\to m_r$ for all $r$ as $k\to\infty$, then $F_k \to F$ as $k\to\infty$. We give the outline of the proof following the argument of Davenport-Erdős [5]. Suppose that (3.6) fails for some $\lambda$. Then we can find a subsequence $\{\Phi_{n,p'}\}$ and a $\delta>0$ such that

$$|\Phi_{n,p'}(\lambda)-\Phi(\lambda)| \ge \delta, \quad \text{for all } p'. \tag{3.7}$$

By the two theorems of Helly (see the introduction to [9]) there exists a subsequence $\{\Phi_{n,p''}\}$ of $\{\Phi_{n,p'}\}$ which converges to a distribution $\Psi$ at every point of continuity, and

$$\int_{-\infty}^{\infty} t^r\, d\Psi(t) = \lim_{p''\to\infty}\int_{-\infty}^{\infty} t^r\, d\Phi_{n,p''} = \int_{-\infty}^{\infty} t^r\, d\Phi(t).$$

Since $\Phi$ is the only distribution with these special moments $\mu_1,\mu_2,\dots$, we have $\Psi(t)=\Phi(t)$ for all $t$. This contradicts (3.7). Hence one concludes that, as $p\to\infty$,

$$\frac{1}{p^n}M_{n,p}(\lambda) = \Phi_{n,p}(\lambda) \to \Phi(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{1}{2}t^2}\,dt,$$

which completes the proof of the theorem. $\square$

We remark that $c_n$ can be explicitly computed for any given value of $n$. The following proposition provides an equivalent formulation of $c_n$, which allows for easier computations in higher dimensions. For any $n$, consider the polynomial in two variables

$$g_n(X,Y)=\sum_{l=0}^{n-1}\left(\sum_{k=0}^{l}(-1)^k\binom{n}{k}\binom{X+(l-k)Y+n-1}{n-1}\right)^2.$$

Note that the total degree of $g_n(X,Y)$ is at most $2n-2$.

**Proposition 1.** *For any $n$,*

$$c_n=\sum_{k=0}^{2n-2}\frac{a_{n,k}}{k+1},$$

*where $a_{n,k}$ is the coefficient of $X^kY^{2n-2-k}$ in $g_n(X,Y)$.*

*Proof.* We know that for fixed $n$ and $h\to\infty$,

$$\sum_m N_m^2=h^{2n-1}c_n+O_n(h^{2n-2}),$$

where $N_m=N_m(h,n)$ is the number of $n$-tuples $(a_1,\dots,a_n)$ such that $a_1+\cdots+a_n=m$, with $1\le a_i\le h$. Replacing $m$ by $m'=m-n$ and each $a_i$ by $b_i=a_i-1$, we get $\sum_m N_m^2=\sum_{m'}(N'_{m'})^2$, where $N'_{m'}$ is the number of $n$-tuples $(b_1,\dots,b_n)$ such that $b_1+\cdots+b_n=m'$, with $0\le b_i\le h-1$.

Now, the number of ways to obtain a sum of $m'$ from $n$ non-negative integers, with no restrictions, is $\binom{m'+n-1}{n-1}$. If we restrict any fixed $b_i$ to satisfy the inequality $b_i\ge h$, then the number of ways drops to $\binom{m'-h+n-1}{n-1}$. If we restrict any two $b_i,b_j$ to satisfy $b_i,b_j\ge h$ then we have $\binom{m'-2h+n-1}{n-1}$ ways, and so on.

Since for each $k$, there are $\binom{n}{k}$ ways to choose exactly $k$ of the $b_i$'s to be greater than $h$, we obtain by the inclusion-exclusion principle,

$$N'_{m'}=\sum_{0\le k\le m'/h}(-1)^k\binom{n}{k}\binom{m'-kh+n-1}{n-1}.$$

So we have, for $lh\le m'<(l+1)h$, $0\le l\le n-1$,

$$N'_{m'}=\sum_{k=0}^{l}(-1)^k\binom{n}{k}\binom{m'-kh+n-1}{n-1}.$$

Replacing $m'$ by $s+lh$, with $0\le s\le h-1$, we get

$$N'_{s+lh}=\sum_{k=0}^{l}(-1)^k\binom{n}{k}\binom{s+(l-k)h+n-1}{n-1}.$$

Therefore

$$\begin{aligned}\sum_{m'}(N'_{m'})^2&=\sum_{s=0}^{h-1}\sum_{l=0}^{n-1}\left(\sum_{k=0}^{l}(-1)^k\binom{n}{k}\binom{s+(l-k)h+n-1}{n-1}\right)^2\\&=\sum_{s=0}^{h-1}g_n(s,h).\end{aligned}$$

It follows that

$$\sum_{s=0}^{h-1} g_n(s,h) = h^{2n-1}c_n + O_n(h^{2n-2}). \tag{3.8}$$

Now, the main contribution in $g_n(s,h)$ comes from the terms where the exponents of $s$ and $h$ add up to $2n-2$. Since for any $0 \le k \le 2n-2$,

$$\sum_{s=0}^{h-1} s^k = \frac{1}{k+1}h^{k+1} + O_n(h^k),$$

we obtain

$$\begin{aligned}\sum_{s=0}^{h-1} g_n(s,h) &= \sum_{s=0}^{h-1}\left(\sum_{k=0}^{2n-2} a_{n,k}s^k h^{2n-2-k} + \text{lower order terms}\right)\\ &= \sum_{k=0}^{2n-2}\sum_{s=0}^{h-1} a_{n,k}s^k h^{2n-2-k} + O_n(h^{2n-2})\\ &= \sum_{k=0}^{2n-2}\frac{a_{n,k}}{k+1}h^{2n-1} + O_n(h^{2n-2}).\end{aligned}$$

By combining this with (3.8), we obtain the desired result. □

For $n = 2,3,4,5,6$, one finds that $c_2 = \frac{2}{3}$, $c_3 = \frac{11}{20}$, $c_4 = \frac{151}{315}$, $c_5 = \frac{15619}{36288}$, $c_6 = \frac{655177}{1663200}$. The numerator and the denominator of $c_n$ grow rapidly as $n$ increases. For instance, for $n = 10$ and $n = 25$ we have

$$c_{10} = \frac{37307713155613}{121645100408832},$$

and

$$c_{25} = \frac{675361967823236555923456864701225753248337661154331976453}{346599352726078382263391546052020157770685374005248000000}.$$

One can also work with boxes instead of cubes, and obtain similar distribution results. For example, in dimension two, we may consider the sum

$$S_{h,k}(x,y) = \sum_{u=x+1}^{x+h}\sum_{v=y+1}^{y+k}\left(\frac{u+v}{p}\right),$$

where $x$, $y$ are any integers and $h$, $k$ are positive integers, with $h \ge k$, say. Then, by using the same arguments as in the proof of Theorem 1, one can prove the following result.

**Theorem 2.** *Let $h$, $k$ be functions of $p$ such that*

$$h \ge k, \quad \frac{h}{k} \to \alpha, \quad k \to \infty, \quad \frac{\log k}{\log p} \to 0, \quad \textit{as } p \to \infty.$$

*Denote $\beta = \sqrt{\alpha - \frac{1}{3}}$ and $\beta' = \sqrt{1 - \frac{1}{3\alpha}}$. Let $M_p(\lambda)$ be the number of pairs $(x,y)$ with $0 \le x, y < p$, $x$, $y$ integers, such that $S_{h,k}(x,y) \le \lambda\beta k^{\frac{3}{2}}$. Let $M_p{}'(\lambda)$ be the number of pairs $(x,y)$ with $0 \le x,y < p$, $x$, $y$ integers, such that $S_{h,k}(x,y) \le \lambda\beta' h^{\frac{1}{2}}k$. Then, as $p \to \infty$,*

$$\frac{1}{p^2}M_p(\lambda) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{1}{2}x^2}\,dx,$$

*and*

$$\frac{1}{p^2}M_p{}'(\lambda) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{1}{2}x^2}\,dx.$$

We remark that when $h$ is much larger than $k$, $S_{h,k}(x,y)$ is close to $k$ times the 1-dimensional sum $S_h(x+y)$. Also, in this case $\alpha$ is large, $\beta'$ is close to 1, and the above statement for $M_p\prime(\lambda)$ approaches the 1-dimensional result of Davenport and Erdős. Note also that in case $\alpha = 1$, we have $\beta = \sqrt{2/3} = \sqrt{c_2}$ , and the statement of Theorem 2 for $M_p(\lambda)$ coincides with that of Theorem 1 for $n = 2$.

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