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On Partition Functions and Divisor Sums

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Abstract

Let n, r be natural numbers, with $r \geq 2$. We present convolution-type formulas for the number of partitions of n that are (1) not divisible by r; (2) coprime to r. Another result obtained is a formula for the sum of the odd divisors of n.

1 Introduction

We derive several convolution-type identities linking partition functions to divisor sums, thereby extending some prior results. In addition, we obtain a Lambert series-like identity for sums of odd divisors.

2 Preliminaries

Let $A \subset N$, the set of all natural numbers. Let n, m, $r \in N$ with
$r \geq 2$, $m \geq 2$, m squarefree. Let $x \in C$, $|x| < 1$.

Definition 1 Let $p_A(n)$ denote the number of partitions of n into parts that belong to A.

Definition 2 Let $\sigma_A(n)$ denote the sum of the divisors, d , of n such that $d \in A$.

Definition 3 Let $p(n)$ denote the number of partitions of n.

Definition 4 Let $q(n)$ denote the number of partitions of n into distinct parts (or into odd parts).

Definition 5 Let $q_0(n)$ denote the number of partitions of n into distinct odd parts (the number of self-conjugate partitions of n).

Definition 6 Let $b_r(n)$ denote the number of *r-regular* partitions of n (the number of partitions of n such that no part is a multiple of r or such that no part occurs r or more times).

Remark: Note that $b_2(n) = q(n)$.

Definition 7 Let $f_m(n)$ denote the number of partitions of n such that all parts are coprime to m.

Definition 8 Let $\sigma_r(n)$ denote the sum of the divisors, d , of n such that d does not divide r.

Definition 9 Let $\sigma_m^*(n)$ denote the sum of the divisors, d , of n such that d is coprime to m.

Definition 10 Let $\phi(n)$ denote Euler's totient function.

Remark: If p is prime, then $f_p(n) = b_p(n)$ and $\sigma_p^*(n) = \sigma_p(n)$.

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{n=1}^{\infty}(1+x^n) \tag{1}$$

Proposition 1 Let $f : A \to N$ be a function such that

$$F_A(x) = \prod_{n\in A}(1-x^n)^{-f(n)/n} = 1 + \sum_{n=1}^{\infty} p_{A,f}(n)x^n$$

and

$$G_A(x) = \sum_{n\in A} \frac{f(n)}{n} x^n$$

converge absolutely and represent analytic functions in the unit disc: $|x| < 1$. Let $p_{A,f}(0) = 1$ and

$$f_A(k) = \sum\{f(d) : d|k,\, d \in A\} \quad .$$

Then

$$np_{A,f}(n) = \sum_{k=1}^{n} p_{A,f}(n-k)f_A(k) \quad .$$

Remarks: Proposition 1 is Theorem 14.8 in [1]. If we let $A = N$, $f(n) = n$, then we obtain

$$np(n) = \sum_{k=1}^{n} p(n-k)\sigma(k) \quad .$$

(See [1, p. 323]). If we let $A = N - 2N$ (the set of odd natural numbers) and $f(n) = n$, we obtain

$$nq(n) = \sum_{k=1}^{n} q(n-k)\sigma_2(k) \quad . \tag{2}$$

This is given as Theorem 1 in [2], and is a special case of Theorem 1(a) below.

3 The Main Results

Theorem 1

$$nb_r(n) = \sum_{k=1}^{n} b_r(n-k)\sigma_r(k) \tag{3}$$

$$nf_m(n) = \sum_{k=1}^{n} f_m(n-k)\sigma_m^*(k) \tag{4}$$

Proof: We apply Proposition 1 with $f(n) = n$. If we let $A = N - rN$ (the set of natural numbers not divisible by r) then (3) follows. If we let $A = \{n \in N : (m,n) = 1\}$, then (4) follows. ■

Next, we present a theorem regarding odd divisors of n.

Theorem 2 Let $f : N \to N$ be a multiplicative function. Let $n = 2^k m$, where $k \geq 0$ and m is odd. Then

$$\sum_{d|n} (-1)^{d-1} f(\frac{n}{d}) = \{f(2^k) - \sum_{j=0}^{k-1} f(2^j)\} \sum_{d|n\,,\,2\not| d} f(d) \quad . \tag{5}$$

Proof: If $d|n$, then by hypothesis, $d = 2^i r$ where $0 \leq i \leq k$, $r|m$. Now

$$\sum_{d|n} (-1)^{d-1} f(\frac{n}{d}) = \sum_{d|n\,,\,2\not| d} f(\frac{n}{d}) - \sum_{2|d|n} f(\frac{n}{d})$$

$$= \sum_{r|m} f(2^k m/r) - \sum_{r|m} \sum_{i=1}^{k} f(2^{k-i} m/r) = f(2^k) \sum_{r|m} f(r) - \sum_{i=1}^{k} f(2^{k-i}) \sum_{r|m} f(r)$$

$$= \{f(2^k) - \sum_{i=1}^{k} f(2^{k-i})\} \sum_{r|m} f(r) = \{f(2^k) - \sum_{j=0}^{k-1} f(2^j)\} \sum_{d|n\,,\,2\not| d} f(d) \quad . \quad \blacksquare$$

Corollary 1

$$\sum_{d|n} (-1)^{d-1} \frac{n}{d} = \sum_{d|n\,,\,2\not| d} d \tag{6}$$

$$\sum_{d|n} (-1)^{d-1} \phi(\frac{n}{d}) = 0 \tag{7}$$

Proof: If f is multiplicative and $n = 2^k m$, where $k \geq 0$ and m is odd, let

$$g(f, k) = \{f(2^k) - \sum_{j=0}^{k-1} f(2^j)\}$$

Theorem 2 may be written as:

$$\sum_{d|n} (-1)^{d-1} f(\frac{n}{d}) = g(f, k) \sum_{d|n\,,\,2\not| d} f(d) \tag{8}$$

Now each of the functions: $f(n) = n$, $f(n) = \phi(n)$ is multiplicative, so Theorem 1 applies. Furthermore,

$$g(n, k) = 2^k - \sum_{j=0}^{k-1} 2^j = 1 \tag{9}$$

$$g(\phi(n), k) = \phi(2^k) - \sum_{j=0}^{k-1} \phi(2^j) = 0 \tag{10}$$

We see that (6) follows from (8) and (9), and (7) follows from (8) and (10). $\blacksquare$

Theorem 3

$$\sum_{n=1}^{\infty} \sigma_2(n) x^n = \sum_{n=1}^{\infty} \frac{n x^n}{1 + x^n}$$

First Proof:

$$\sum_{m=1}^{\infty} \frac{m x^m}{1 + x^m} = \sum_{m,k=1}^{\infty} (-1)^{k-1} m x^{km}$$

$$= \sum_{n=1}^{\infty} x^n (\sum_{d|n} (-1)^{d-1} \frac{n}{d} = \sum_{n=1}^{\infty} \sigma_2(n) x^n$$

by (6) . $\blacksquare$

Second Proof: (2) implies

$$\sum_{n=0}^{\infty} (\sum_{k=0}^{n} q(n-k) \sigma_2(k)) x^n = \sum_{n=0}^{\infty} n q(n) x^n$$

so that

$$(\sum_{n=0}^{\infty} q(n) x^n)(\sum_{n=0}^{\infty} \sigma_2(n) x^n) = \sum_{n=0}^{\infty} n q(n) x^n \tag{11}$$

Now (1) implies

$$\frac{d}{dx} (\sum_{n=0}^{\infty} q(n) x^n) = \frac{d}{dx} (\prod_{n=1}^{\infty} (1 + x^n))$$

that is,

$$\sum_{n=1}^{\infty} n q(n) x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1} \prod_{m \neq n} (1 + x^m)$$

hence

$$\sum_{n=0}^{\infty} n q(n) x^n = \sum_{n=0}^{\infty} \frac{n x^n}{1 + x^n} \prod_{n=1}^{\infty} (1 + x^n)$$

$$= \sum_{n=0}^{\infty} \frac{n x^n}{1 + x^n} \sum_{n=0}^{\infty} q(n) x^n$$

by (1). The conclusion now follows from (11) . $\blacksquare$

Remarks: Theorem 3 may be compared to the well-known Lambert series identity:

$$\sum_{n=1}^{\infty} \sigma(n) x^n = \sum_{n=1}^{\infty} \frac{n x^n}{1 - x^n}$$

In [2], Theorem 2, part (b), we obtained an explicit formula for $\sigma_2(n)$ in terms of $q(n)$ and $q_0(n)$, namely:

$$\sigma_2(n) = \sum_{k=1}^{n} (-1)^{k-1} k q_0(k) q(n-k)$$

References

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2. N. Robbins, Some identities connecting partition functions to other number theoretic functions, *Rocky Mountain J. Math* **29** (1999), 335–345.

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