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# The function ${}_vM_m(s;a,z)$ and some well-known sequences

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**Abstract**

*In this paper we define the function ${}_vM_m(s;a,z)$, and we study the special cases ${}_1M_m(s;a,z)$ and ${}_nM_{-1}(1;1,n+1)$. We prove some new equivalents of Kurepa's hypothesis for the left factorial. Also, we present a generalization of the alternating factorial numbers.*

## 1 Introduction

Studying the Kurepa function

$$K(z) = !z = \int_0^{+\infty} \frac{t^z - 1}{t-1}\, e^{-t}\, dt \quad (\mathrm{Re}\, z > 0),$$

G. V. Milovanović gave a generalization of the function

$$M_m(z) = \int_0^{+\infty} \frac{t^{z+m} - Q_m(t;z)}{(t-1)^{m+1}} e^{-t} dt \quad (\mathrm{Re}\, z > -(m+1)),$$

where the polynomials $Q_m(t;z)$, $m = -1, 0, 1, 2, \ldots$ are given by

$$Q_{-1}(t;z) = 0 \quad Q_m(t;z) = \sum_{k=0}^{m} \binom{m+z}{k} (t-1)^k.$$

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The function $\{M_m(z)\}_{m=-1}^{+\infty}$ has the integral representation

$$M_m(z) = \frac{z(z+1)\cdots(z+m)}{m!}\int_0^1 \xi^{z-1}(1-\xi)^m e^{(1-\xi)/\xi}\Gamma\Big(z, \frac{1-\xi}{\xi}\Big)\, d\xi,$$

where $\Gamma(z,x)$, the incomplete gamma function, is defined by

$$\Gamma(z,x) = \int_x^{+\infty} t^{z-1}e^{-t}\, dt. \tag{1}$$

Special cases include

$$M_{-1}(z) = \Gamma(z) \qquad \text{and} \qquad M_0(z) = K(z), \tag{2}$$

where $\Gamma(z)$ is the gamma function

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}\, dt.$$

The numbers $M_m(n)$ were introduced by Milovanović [10] and Milovanović and Petojević [11]. For non-negative integers $n, m \in \mathbb{N}$ the following identities hold:

$$M_m(0) = 0, \quad M_m(n) = \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} \sum_{k=i}^{n-1} k! \binom{m+n}{k+m+1}.$$

For the numbers $M_m(n)$ the following relations hold:

$$\begin{aligned} M_m(n+1) &= n! + \sum_{\nu=0}^{m} M_\nu(n), \\ \lim_{n\to+\infty} \frac{M_\nu(n)}{M_{\nu-1}(n)} &= 1, \\ \lim_{n\to+\infty} \frac{M_m(n)}{(n-1)!} &= 1. \end{aligned}$$

The generating function of the numbers $\{M_m(n)\}_{n=0}^{+\infty}$ is given by

$$\frac{1}{m!}\left(A_m(x)e^{x-1}\left(\mathrm{Ei}\,(1) - \mathrm{Ei}\,(1-x) + B_m(x)e^x - C_m(x)\right)\right) = \sum_{n=0}^{+\infty} M_m(n)\frac{x^n}{n!}$$

where $A_m(x)$, $B_m(x)$, and $C_m(x)$ are polynomials defined as follows:

$$\begin{aligned} \frac{A_m(x)}{m!} &= \sum_{k=0}^{m} \binom{m}{k} \frac{(x-1)^k}{k!}, \\ \frac{B_m(x)}{m!} &= \sum_{\nu=0}^{m-1} \left( \sum_{k=1}^{m-\nu} \binom{m}{k+\nu} \frac{(-1)^{k-1}}{k} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \right) \frac{(x-1)^\nu}{\nu!}, \\ \frac{C_m(x)}{m!} &= \sum_{j=0}^{m-1} \left( \sum_{\nu=0}^{j} (-1)^\nu \binom{j}{\nu} \sum_{k=j+1}^{m} \frac{(-1)^{k-1}}{k-\nu} \binom{m}{k} \right) \frac{(x-1)^j}{j!}, \end{aligned}$$

Here $\mathrm{Ei}\,(x)$ is the exponential integral defined by

$$\mathrm{Ei}\,(x) = \text{p.v.} \int_{-\infty}^{x} \frac{e^t}{t}\,dt \qquad (x > 0). \tag{3}$$

In this paper, we give a generalization of the function $M_m(z)$ which we denote as ${}_vM_m(s; a, z)$. These generalization are of interest because its special cases include:

$$\begin{aligned} {}_1M_1(1; 1, n+1) &= n!, \\ {}_1M_0(1; 1, n) &= !n, \\ {}_nM_{-1}(1; 1, n+1) &= A_n. \end{aligned}$$

where $n!$, $!n$, and $A_n$ are the *right factorial numbers* (sequence A000142 in [17]), the *left factorial numbers* (sequence A003422 in [17]) and the *alternating factorial numbers* (sequence A005165 in [17]), respectively. They are defined as follows:

$$0! = 1,\ n! = n \cdot (n-1)!; \qquad !0 = 0,\ \ !n = \sum_{k=0}^{n-1} k! \quad \text{and} \quad A_n = \sum_{k=1}^{n} (-1)^{n-k} k!. \tag{4}$$

## 2 Definitions

We now introduce a generalization of the function $M_m(z)$.

**Definition 1** *For $m = -1, 0, 1, 2, \ldots$, and $\mathrm{Re}\, z > v - m - 2$ the function ${}_vM_m(s; a, z)$ is defined by*

$$\begin{aligned} &{}_vM_m(s; a, z) = \\ &= \sum_{k=1}^{v} \frac{(-1)^{2v+1-k}\Gamma(m+z+2-k)}{\Gamma(z+1-k)\Gamma(m+2)} \mathcal{L}[s;\ {}_2F_1(a, k-z, m+2, 1-t)], \end{aligned}$$

*where $v$ is a positive integer, and $s$, $a$, $z$ are complex variables.*

The hypergeometric function ${}_2F_1(a, b; c, x)$ is defined by the series

$${}_2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \qquad (|x| < 1),$$

and has the integral representation

$${}_2F_1(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt,$$

in the $x$ plane cut along the real axis from 1 to $\infty$, if $\operatorname{Re} c > \operatorname{Re} b > 0$.
The symbols $(z)_n$ and $\mathcal{L}[s; F(t)]$ represent the Pochhammer symbol

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)},$$

and Laplace transform

$$\mathcal{L}[s; F(t)] = \int_0^\infty e^{-st} F(t) dt.$$

Table 1: The numbers ${}_1M_m(1; a, n)$ for $m = 1, 2, 3, 4$ and $a = 0, 1, 2$

| $a = 0$ | in [17] | $a = 1$ | in [17] | $a = 2$ | in [17] |
|---|---|---|---|---|---|
| ${}_1M_1(1, 0, n)$ | *A000217* | ${}_1M_1(1, 1, n)$ | *A014144* | ${}_1M_1(1, 2, n)$ | *A007489* |
| ${}_1M_2(1, 0, n)$ | *A000292* | ${}_1M_2(1, 1, n)$ | unlisted | ${}_1M_2(1, 2, n)$ | *A014145* |
| ${}_1M_3(1, 0, n)$ | *A000332* | ${}_1M_3(1, 1, n)$ | unlisted | ${}_1M_3(1, 2, n)$ | unlisted |
| ${}_1M_4(1, 0, n)$ | *A000389* | ${}_1M_4(1, 1, n)$ | unlisted | ${}_1M_4(1, 2, n)$ | unlisted |

The term "unlisted" in Table 1 means that the sequence cannot currently be found in Sloane's on-line encyclopedia of integer sequences[17].

**Lemma 1** *Let $m = -1, 0, 1, 2, \dots$ . Then*

$${}_1M_m(1; 1, z) = M_m(z).$$

*Proof.* The proof presented here is due to Professor G. V. Milovanović.
Since

$$(k+m+1)! = (m+1)!(m+2)_k \quad \text{and} \quad (1-z)_k = \frac{(-1)^k \Gamma(z)}{\Gamma(z-k)},$$

we have

$$\binom{m+z}{k+m+1} = \frac{\Gamma(m+z+1)}{\Gamma(z-k)(k+m+1)!} = \frac{\Gamma(m+z+1)}{\Gamma(z)(m+1)!} \cdot \frac{(1-z)_k(-1)^k}{(m+2)_k},$$

so that

$$\begin{aligned} \frac{t^{z+m} - Q_m(t; z)}{(t-1)^{m+1}} &= \sum_{k=0}^{+\infty} \binom{m+z}{k+m+1} (t-1)^k \qquad (|t-1| < 1) \\ &= \frac{\Gamma(m+z+1)}{\Gamma(z)(m+1)!} \sum_{k=0}^{+\infty} \frac{(1-z)_k (1)_k}{(m+2)_k} \cdot \frac{(1-t)^k}{k!} \\ &= \frac{\Gamma(m+z+1)}{\Gamma(z)(m+1)!} \, {}_2F_1(1, 1-z, m+2; 1-t). \end{aligned}$$

# 3 The function ${}_1M_m(s;a,z)$

## 3.1 The numbers $\{{}_1M_m(1;-n,r)\}_{r=0}^{+\infty}{}_{n=0}^{+\infty}{}_{m=-1}^{+\infty}$

We have

$$ {}_1M_m(1;-n,z) = \frac{\Gamma(m+z+1)}{\Gamma(z)\Gamma(m+2)}\,\mathcal{L}[1;\ {}_2F_1(-n,1-z,m+2;1-t)], $$

starting with the polynomials

$$ \sum_{k=0}^{\infty} \binom{m+z}{k+m+1}\binom{n}{k}(1-t)^k, \quad n \in \mathbb{N}. $$

Since

$$ (-n)_k = (-1)^k \frac{n!}{(n-k)!} $$

we have

$$ \sum_{k=0}^{\infty} \binom{m+z}{k+m+1}\binom{n}{k}(1-t)^k = \frac{\Gamma(m+z+1)}{\Gamma(z)\Gamma(m+2)}\,{}_2F_1(-n,1-z,m+2;1-t) \tag{5} $$

or, by continuation,

$$ = \pi \operatorname{cosec} \pi z \int_0^1 \xi^{1-z}(1-\xi)^{m+z+1}\big(1-(1-t)\xi\big)^n d\xi. $$

Hence, the following definition is reasonable.

**Definition 2** *For $n \in \mathbb{N}$ and $m = -1, 0, 1, 2, \ldots$, the polynomials $z \mapsto {}_mP_n(z)$ are defined by*

$$ {}_mP_n(z) = \mathcal{L}[1;\ {}_2F_1(-n,1-z,m+2;1-t)]. $$

Table 2: The polynomials ${}_mP_2(z)$, ${}_mP_3(z)$ and ${}_mP_4(z)$

| $m$ | ${}_mP_2(z)$ | ${}_mP_3(z)$ | ${}_mP_4(z)$ |
|---|---|---|---|
| $-1$ | $\frac{1}{2}z^2 - \frac{3}{2}z + 2$ | $-\frac{1}{3}z^3 + \frac{7}{2}z^2 - \frac{49}{6}z + 6$ | $\frac{3}{8}z^4 - \frac{61}{12}z^3 + \frac{193}{8}z^2 -$ |
| $0$ | $\frac{1}{6}z^2 - \frac{1}{2}z + \frac{4}{3}$ | $-\frac{1}{12}z^3 + z^2 - \frac{29}{12}z + \frac{5}{2}$ | $-\frac{509}{12}z + 24$ |
| $1$ | $\frac{1}{12}z^2 - \frac{1}{4}z + \frac{7}{6}$ | $-\frac{1}{30}z^3 + \frac{9}{20}z^2 - \frac{67}{60}z + \frac{17}{10}$ | |

The polynomials ${}_mP_n(z)$ can be expressed in terms of the *derangement numbers* (sequence A000166 in [17])

$$ S_k = k! \sum_{\nu=0}^{k} \frac{(-1)^\nu}{\nu!} \qquad (k \geq 0) $$

in the form

**Theorem 1** *For $m = -1, 0, 1, 2, \ldots$ and $n \in \mathbb{N}$ we have*

$$ {}_mP_n(z) = \binom{n+m+1}{n}^{-1} \sum_{k=0}^{\infty} \binom{n+m+1}{k+m+1} \binom{z-1}{k} (-1)^k S_k. $$

*Proof.* Using the relation (5) we have

$$ \begin{aligned} {}_mP_n(z) &= \frac{\Gamma(z)\Gamma(m+2)}{\Gamma(m+z+1)} \int_0^{\infty} e^{-t} \sum_{k=0}^{\infty} \binom{m+z}{k+m+1} \binom{n}{k} (1-t)^k dt \\ &= \frac{\Gamma(z)(m+1)!}{\Gamma(m+z+1)} \sum_{k=0}^{\infty} \frac{\Gamma(m+z+1)}{\Gamma(z-k)(k+m+1)!} \binom{n}{k} \int_0^{\infty} e^{-t}(1-t)^k dt \\ &= \binom{n+m+1}{n}^{-1} \sum_{k=0}^{\infty} \binom{n+m+1}{k+m+1} \binom{z-1}{k} \int_0^{\infty} e^{-t}(1-t)^k dt. \end{aligned} $$

Now use

$$ \mathcal{L}[s; (t+\alpha)^{z-1}] = \frac{e^{\alpha s}\Gamma(z, \alpha s)}{s^z} \qquad (\mathrm{Re}\, s > 0), $$

to obtain

$$ {}_mP_n(z) = \binom{n+m+1}{n}^{-1} \sum_{k=0}^{\infty} \binom{n+m+1}{k+m+1} \binom{z-1}{k} (-1)^k \frac{\Gamma(k+1, -1)}{e}. $$

Here $\Gamma(z, x)$ is the incomplete gamma function. The result follows from

$$ \Gamma(k+1, -1) = eS_k. $$

**Lemma 2** *For $n \in \mathbb{N}$ we have*

$$ {}_{-1}P_n(z) = 1 + n! \sum_{k=1}^{n} \frac{(-1)^k S_k}{(n-k)!\,(k!)^2} \prod_{i=1}^{k} (z-i). $$

*Proof.* Applying Theorem 1 for $m = -1$, we have

$$ \begin{aligned} {}_{-1}P_n(z) &= \sum_{k=0}^{n} \binom{n}{k} \binom{z-1}{k} (-1)^k S_k, \\ &= 1 + n! \sum_{k=1}^{n} \frac{(-1)^k S_k}{(n-k)!(k!)^2} \prod_{i=1}^{k} (z-i). \end{aligned} $$

**Remark 1** *Let the sequence $X_{n,k}$ be defined by*

$$X_{n,k} = \begin{cases} Y_n, & \text{if } n = k, \\ (n-k)X_{n-1,k}, & \text{if } n > k, \end{cases}$$

*where $Y_n = (n!)^2$. Since $X_{n,k} = (n-k)!\,(k!)^2$ we have*

$${}_{-1}P_n(z) = 1 + n! \sum_{k=1}^{n} \frac{(-1)^k S_k}{X_{n,k}} \prod_{i=1}^{k} (z-i).$$

**Theorem 2** *For $m = -1, 0, 1, 2, \ldots$ and $n \in \mathbb{N}$ we have*

$$\begin{aligned} {}_mP_0(z) &= {}_mP_1(z) = 1 \\ {}_mP_n(z) &= \frac{1}{n+m+1} \left[(m+1)\,{}_{m-1}P_n(z) + n\,{}_mP_{n-1}(z)\right], \quad m > -1. \end{aligned}$$

*Proof.* According to Theorem 1 we have

$$\frac{1}{n+m+1} \left[(m+1)\,{}_{m-1}P_n(z) + n\,{}_mP_{n-1}(z)\right] =$$

$$\begin{aligned} & \frac{(m+1)!n!}{(n+m+1)!} \sum_{k=0}^{\infty} \binom{n+m}{k+m} \binom{z-1}{k} (-1)^k S_k + \\ + \; & \frac{(m+1)!n!}{(n+m+1)!} \sum_{k=0}^{\infty} \binom{n+m}{k+m+1} \binom{z-1}{k} (-1)^k S_k. \end{aligned}$$

The recurrence for ${}_mP_n(z)$ now follows from

$$\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}.$$

**Corollary 1** *For $m = -1, 0, 1, 2, \ldots$ and $n \in \mathbb{N}$ we have*

$${}_1M_m(1; -n, z) = \frac{n!\, z(z+1)\ldots(z+m)}{(n+m+1)!} \sum_{k=0}^{\infty} \binom{n+m+1}{k+m+1} \binom{z-1}{k} (-1)^k S_k.$$

**Corollary 2** *For $n \in \mathbb{N}$ the result is as follows*

$$
\begin{aligned}
{}_1M_{-1}(1;-n,z) &= 1 + n!\sum_{k=1}^{n} \frac{(-1)^k S_k}{(n-k)!\,(k!)^2} \prod_{i=1}^{k}(z-i), \\
{}_1M_m(1;0,z) &= {}_1M_m(1;-1,z) = \frac{\Gamma(m+z+1)}{\Gamma(z)\Gamma(m+2)}, \\
{}_1M_m(1;-n,z) &= \frac{1}{n+m+1}\cdot[\,(m+z)\cdot {}_1M_{m-1}(1;-n,z) + \\
&+ \; n\cdot {}_1M_m(1;-n+1,z)], \quad m > -1.
\end{aligned}
$$

The numbers ${}_1M_m(1;-n,r)_{r=0}^{\infty}$ can now be evaluated recursively.

Table 3: The numbers ${}_1M_m(1;-n,r)$ for $m = 0,1,2,3,4$ and $n = 1,2$

| ${}_1M_m(1,-n,r)$ | sequence in [17] | ${}_1M_m(1,-n,r)$ | sequence in [17] |
|---|---|---|---|
| ${}_1M_0(1,-1,r)$ | $0,1,2,3,4...$ | ${}_1M_0(1,-2,r)$ | *A000125* |
| ${}_1M_1(1,-1,r)$ | *A000217* | ${}_1M_1(1,-2,r)$ | *A055795* |
| ${}_1M_2(1,-1,r)$ | *A000292* | ${}_1M_2(1,-2,r)$ | *A027660* |
| ${}_1M_3(1,-1,r)$ | *A000332* | ${}_1M_3(1,-2,r)$ | *A055796* |
| ${}_1M_4(1,-1,r)$ | *A000389* | ${}_1M_4(1,-2,r)$ | *A055797* |

## 3.2 The numbers $\{{}_1M_m(1/n;m+2,r)\}_{r=0}^{+\infty}{}_{n=1}^{+\infty}{}_{m=-1}^{+\infty}$

In Table 4 twelve well-known sequences from [17] are given. These sequences are special cases of the function ${}_vM_m(s;a,r)$ for $v = 1$, $s = 1/n$, and $a = m+2$. The sequences have the following common characteristic.

**Lemma 3** *For $m = -1,0,1,2,...,$ we have*

$$
{}_1M_m(1/n;m+2,z) = \frac{n^z\Gamma(z+m+1)}{(m+1)!}.
$$

*Proof.* Since

$$
{}_2F_1(m+2,1-z,m+2,1-t) = t^{z-1}
$$

we have

$$
\mathcal{L}[1/n;\ {}_2F_1(m+2,1-z,m+2,1-t)] = n^z\Gamma(z).
$$

Table 4: The numbers ${}_1M_m(1/n; m+2, r)$ for $m = -1, 0, 1, 2, 3, 4$ and $n = 1, 2, 3, 4$

| ${}_1M_m(1/n, m+2, r)$ | in [17] | ${}_1M_m(1/n, m+2, r)$ | in [17] |
|---|---|---|---|
| ${}_1M_{-1}(1, 1, r)$ | A000142 | ${}_1M_{-1}(1/2, 1, r)$ | A066318 |
| ${}_1M_{-1}(1/3, 1, r)$ | A032179 | ${}_1M_{-1}(1/4, 1, r)$ | unlisted |
| ${}_1M_0(1, 2, r)$ | A000142 | ${}_1M_0(1/2, 2, r)$ | A000165 |
| ${}_1M_0(1/3, 2, r)$ | A032031 | ${}_1M_0(1/4, 2, r)$ | A047053 |
| ${}_1M_1(1, 3, r)$ | A001710 | ${}_1M_1(1/2, 3, r)$ | A014297 |
| ${}_1M_1(1/3, 3, r)$ | unlisted | ${}_1M_1(1/4, 3, r)$ | unlisted |
| ${}_1M_2(1, 4, r)$ | A001715 | ${}_1M_2(1/2, 4, r)$ | unlisted |
| ${}_1M_2(1/3, 4, r)$ | unlisted | ${}_1M_2(1/4, 4, r)$ | unlisted |
| ${}_1M_3(1, 5, r)$ | A001720 | ${}_1M_3(1/2, 5, r)$ | unlisted |
| ${}_1M_3(1/3, 5, r)$ | unlisted | ${}_1M_3(1/4, 5, r)$ | unlisted |
| ${}_1M_4(1, 6, r)$ | A001725 | ${}_1M_4(1/2, 6, r)$ | unlisted |
| ${}_1M_4(1/3, 6, r)$ | unlisted | ${}_1M_4(1/4, 6, r)$ | unlisted |

### 3.3 Some equivalents of Kurepa's hypothesis

The special values $M_{-1}(z) = \Gamma(z)$ and $M_0(z) = K(z)$ given in (2) yield

$$ {}_1M_{-1}(1, 1, n+1) = n! \quad \text{and} \quad {}_1M_0(1; 1, n) = !n \tag{6} $$

where $n!$ and $!n$ are the right factorial numbers and the left factorial numbers given in (4). The function $n!$ and $!n$ are linked by Kurepa's hypothesis:

**KH hypothesis.** *For $n \in \mathbb{N}\backslash\{1\}$ we have*

$$ \gcd(\,!n,\ n!\,) = 2 $$

*where $\gcd(a, b)$ denotes the greatest common divisor of integers $a$ and $b$.*

This is listed as Problem B44 of Guy's classic book [6]. In [8], it was proved that the KH is equivalent to the following assertion

$$ !p \not\equiv 0 \pmod{p}, \quad \text{for all primes } p > 2. $$

The sequences $a_n$, $b_n$, $c_n$, $d_n$ and $e_n$ (sequences A052169, A051398, A051403, A002467 and A002720 in [17]) are defined as follows:

$$\begin{array}{rrrcl} a_2=1 & a_3=2 & a_n & = & (n-2)a_{n-1}+(n-3)a_{n-2}, \\ & b_3=2 & b_n & = & -(n-3)b_{n-1}+2(n-2)^2, \\ c_1=3 & c_2=8 & c_n & = & (n+2)(c_{n-1}-c_{n-2}), \\ d_0=0 & d_1=1 & d_n & = & (n-1)(d_{n-1}+d_{n-2}), \\ e_0=1 & e_1=2 & e_n & = & 2ne_{n-1}-(n-1)^2e_{n-2}. \end{array}$$

They are related to the left factorial function. For instance, let $p>3$ be a prime number. Then

$$!p \equiv -3a_{p-2} \equiv -b_p \equiv -2c_{p-3} \equiv d_{p-2} \equiv e_{p-1} \pmod{p}.$$

We give the details for the last congruence.

*Proof.* Let

$$L_n^\nu(x) = \sum_{k=0}^{n} \frac{\Gamma(\nu+n+1)}{\Gamma(\nu+k+1)} \frac{(-x)^k}{k!(n-k)!}$$

be the Laguerre polynomials, and set $L_n^0(x) = L_n(x)$. Using the relation $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$, we have

$$L_{p-1}(x) \equiv -(p-1)! \sum_{k=0}^{p-1} \frac{x^k}{k!} \pmod{p}.$$

Wilson's theorem yields

$$!p \equiv -L_{p-1}(-1) \pmod{p}.$$

The recurrence for Laguerre polynomials

$$(n+1)L_{n+1}^\nu(x) = (\nu+2n+1-x)L_n^\nu(x) - (\nu+n)L_{n-1}^\nu(x),$$

for $\nu=0,\ x=-1$ produces

$$h_{p-1}(p-1)! \equiv p \pmod{p},$$

where

$$h_1 = 2 \quad h_2 = \frac{7}{2} \quad h_n = 2h_{n-1} - \frac{n-1}{n}h_{n-2}.$$

The identity $e_n = h_n n!$ finally yields

$$!p \equiv e_{p-1} \pmod{p}.$$

## 4 The numbers ${}_nM_{-1}(1;1,n+1)$

The special values

$$ {}_nM_{-1}(1;1,n+1)=A_n, $$

are the alternating factorial numbers given in (4). This sequence satisfies the recurrence relation

$$ A_0=0,\ A_1=(-1)^{n-1},\ A_n=-(n-1)A_{n-1}+nA_{n-2}. $$

These numbers can be expressed in terms of the gamma function as follows

$$ \begin{aligned} A_n &= \sum_{k=1}^{n}(-1)^{n-k}\Gamma(k+1)=\int_0^\infty e^{-x}\left(\sum_{k=1}^{n}(-1)^{n-k}x^k\right)dx \\ &= \int_0^\infty e^{-x}\frac{x^{n+1}-(-1)^n x}{x+1}dx. \end{aligned} $$

The same relation is now used in order to define the function $A_z$:

**Definition 3** *For every complex number $z$, Re $z>0$, the function $A_z$ is defined by*

$$ A_z \stackrel{\text{def}}{=} \int_0^\infty e^{-x}\frac{x^{z+1}-(-1)^z x}{x+1}dx. $$

The identity $\frac{x^{z+1}-(-1)^z x}{x+1}=x^z-\frac{x^z-(-1)^{z-1}x}{x+1}$ gives

$$ \int_0^\infty e^{-x}\frac{x^{z+1}-(-1)^z x}{x+1}dx=\int_0^\infty e^{-x}x^z\,dx-\int_0^\infty e^{-x}\frac{x^z-(-1)^{z-1}x}{x+1}dx, $$

i.e.,

$$ A_z=\Gamma(z+1)-A_{z-1}. \tag{7} $$

This gives $A_0=\Gamma(2)-A_1=0$ and $A_{-1}=\Gamma(1)-A_0=1$. An inductive argument shows that $A_{-n}$, the residue of $A_z$ at the pole $z=-n$, is given by

$$ res\ A_{-n}=(-1)^n\sum_{k=0}^{n-2}\frac{1}{k!}, \qquad n=2,3,4,\ldots $$

The derivation employs the fact that $\Gamma(z)$ is meromorphic with simple poles at $z=-n$ and residue $(-1)^n/n!$ there.

The function $A_z$ can be expressed in terms of the exponential integral $\mathrm{Ei}\,(x)$ and the incomplete gamma function $\Gamma(z,x)$ by

$$ A_z=\mathcal{L}[1;\frac{t^{z+1}-(-1)^z}{t+1}]=e\Gamma(z+2)\Gamma(-z-1,1)-(-1)^z e\mathrm{Ei}\,(-1)-(-1)^z. $$

## 4.1 The generating function for $A_{n-1}$

The *total number of arrangements of a set with $n$ elements* (sequence A000522 in [17]) is defined (see [3], [5], [15] and [16]) by:

$$a_0 = 1, \quad a_n = na_{n-1} + 1, \quad \text{or} \quad a_n = n! \sum_{k=0}^{n} \frac{1}{k!}. \tag{8}$$

The sequence $\{a_n\}$ satisfies:

$$a_0 = 1, \quad a_1 = 2, \quad a_n = (n+1)a_{n-1} - (n-1)a_{n-2}, \tag{9}$$

and

$$a_0 = 1, \quad a_n = \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n+1-k}(n+1-k)a_k. \tag{10}$$

Relation (9) comes from the theory of continued fractions and (10) follows directly from (8).

We now establish a connection between the sequence $\{a_n\}$ and the alternating factorial numbers $A_n$.

**Lemma 4** *Let $a_{-1} = 1$ and $n \in \mathbb{N}\backslash\{1\}$. Then*

$$A_{n-1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_{k-1}.$$

*Proof.* Using (10) and induction on $n$ we have

$$n! = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k.$$

Inversion yields

$$a_n = n! - \sum_{k=1}^{n} \binom{n}{k-1} (-1)^{n+1-k} a_{k-1}.$$

The relation $\binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1}$, $k \geq 1$ produces

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} a_{k-1} + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_{k-1} = n!.$$

The result now follows from (7).

**Theorem 3** *The exponential generating function for $\{A_{n-1}\}$ is given by*

$$g(x) = e^{1-x}[E_i(-1) - E_i(x-1) + e^{-1}] - 1 = \sum_{n=2}^{\infty} A_{n-1} \frac{x^n}{n!},$$

*where $E_i(x)$ is the exponential integral (3).*

*Proof.* The expansion of the exponential integral

$$E_i(x) = \gamma + \ln(x) + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!},$$

where $\gamma$ is Euler's constant, appears in [1, p. 57, 5.1.10.]. The statement of the theorem can be written as

$$e[E_i(-1) - E_i(x-1) + e^{-1}] = e\left[-\ln(x-1) + \sum_{k=1}^{\infty} \frac{(-1)^k - (x-1)^k)}{k \cdot k!}\right] =$$

$$= \sum_{k=0}^{\infty} a_{k-1} \frac{x^k}{k!}. \tag{11}$$

Expand $e^{-x}$ as a Taylor series to obtain

$$e^{-x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}.$$

Then

$$\begin{aligned} e^{1-x}[E_i(-1) - E_i(x-1) + e^{-1}] - 1 &= \sum_{k=0}^{\infty} a_{k-1} \frac{x^k}{k!} \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!} - 1 \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{k-1} \frac{x^k}{k!} (-1)^{n-k} \frac{x^{n-k}}{(n-k)!} - 1 \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} a_{k-1} (-1)^{n-k} \frac{x^n}{n!} - 1 \\ &= \sum_{n=0}^{\infty} A_{n-1} \frac{x^n}{n!} - 1. \end{aligned}$$

By induction on $n$ we get

**Lemma 5** *Let $n \in N$. The function $g(x)$ in Theorem 4.1 satisfies*

$$g^{(n)}(x) = \frac{d}{dx} g^{(n-1)}(x) = (-1)^n \left[g(x) + 1 + \sum_{k=0}^{n-1} \frac{k!}{(x-1)^{k+1}}\right].$$

This identity gives the algorithm for computing the $n$-th derivation of the function $g(x)$.

## 4.2 The AL hypothesis

In [6, p. 100] the following problem is given:

**Problem B43.** Are there infinitely many numbers $n$ such that $A_{n-1}$ is a prime?

Here

$$A_n = \sum_{k=1}^{n} (-1)^{n-k} k!.$$

If there is a value of $n-1$ such that $n$ divides $A_{n-1}$, then $n$ will divide $A_{m-1}$ for all $m > n$, and there would be only a finite number of prime values. The required condition for the existence of infinitely many numbers $n$ such that $A_{n-1}$ is a prime may be expressed as follows:

**AL hypothesis.** *For every prime number $p$*

$$A_{p-1} \not\equiv 0 \pmod{p},$$

*holds.*

Let $p$ be a prime number and $n, m \in \mathbb{N}\backslash\{1\}$. It is not difficult to prove the following results:

$$A_{n-1} \equiv -1 - \sum_{k=2}^{n} [k - 1 - (-1)^{n-k}]\Gamma(k) \pmod{n},$$

$$\begin{aligned} A_{n-1} &= \frac{\Gamma(n+1) - 1 + \sum_{k=2}^{n-1}[(-1)^{n-k}n - k + 1 - (-1)^{n-k}]\Gamma(k)}{n-1}, \\ &= 1 - !n + 2\sum_{k=1}^{n-1} A_k \\ &= 3 - !n - !(n-1) \cdot 2n + 4\sum_{i=2}^{n}\sum_{k=1}^{i-1} A_k. \end{aligned}$$

$$\sum_{k=1}^{n}\sum_{i=0}^{m-1} (-1)^i (\Gamma(k+1))^{m-i} A_{k-1}^i = \begin{cases} A_n^m, & m \text{ even} \\ A_n^m + 2\sum_{j=1}^{n-1} A_j^m, & m \text{ odd} \end{cases},$$

$$A_{p-1} = -\sum_{i=1}^{p} \frac{1}{\Gamma(i)} + p\sum_{i=1}^{p} \frac{n_i(-1)^{i-1}}{\Gamma(i)}, \quad n_i \in \mathbb{N} \quad (i = 1, 2, ...p)$$

The first step in solving problem B43 is proving the AL hypothesis. Using the previous identities, equivalents for the AL hypothesis can be given.

## 5 Conclusions

The main contribution is to define the function ${}_vM_m(s;a,z)$ by which problems B43 and B44 are connected. The Kurepa hypothesis is an unsolved problem since 1971 and there seems to be no significant progress in solving it, apart from numerous equivalents, such as these in Section 3.3. Further details can be found in [7].

However, apart from $n!$, $!n$, and $A_n$, twenty-five more well-known sequences in [17] are special cases of the function ${}_vM_m(s;a,z)$. The first study of the function gave the author the idea to find an algorithm for computing some special cases (Corollary 2 and Lemma 3) before solving the above mentioned problems.

The definition of the function ${}_vM_m(s;a,z)$ suggest another method of studying the function by using the characteristics of the inverse Laplace transform.

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