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On Powers of 2 Dividing the Values of Certain Plane Partition Functions

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Abstract

We consider two families of plane partitions: totally symmetric self-complementary plane partitions (TSSCPPs) and cyclically symmetric transpose complement plane partitions (CSTCPPs). If $T(n)$ and $C(n)$ are the numbers of such plane partitions in a $2n \times 2n \times 2n$ box, then

$$ord_2(T(n)) = ord_2(C(n))$$

for all $n \geq 1$. We also discuss various consequences, along with other results on $ord_2(T(n))$.

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1 Introduction

In his book "Proofs and Confirmations," David Bressoud [2] discusses the rich history of the Alternating Sign Matrix conjecture and its proof. One of the themes of the book is the connection between alternating sign matrices and various families of plane partitions. (Reference [3] gives a synopsis of this work.)

Pages 197–199 of [2] list ten families of plane partitions which have been extensively studied. The last family in this list is the set of totally symmetric self–complementary plane partitions (TSSCPPs) which fit in a $2n \times 2n \times 2n$ box. In 1994, Andrews [1] proved that the number of such partitions is given by

$$T(n) = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{1}$$

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(This formula also gives the number of $n \times n$ alternating sign matrices. See [9]. The values $T(n)$ can be found as sequence A005130 in [8].)

Another family mentioned by Bressoud is the set of cyclically symmetric transpose complement plane partitions (CSTCPPs). We will let $C(n)$ denote the number of such partitions that fit in a $2n \times 2n \times 2n$ box. (The values $C(n)$ make up sequence A051255 in [8].) In 1983, Mills, Robbins, and Rumsey [7] proved that

$$\begin{aligned} C(n) &= \prod_{j=0}^{n-1} \frac{(3j+1)(6i)!(2i)!}{(4i+1)!(4i)!} \\ &= \prod_{j=0}^{n-1} \frac{(3j+1)!(6i)!(2i)!}{(3j)!(4i+1)!(4i)!}. \end{aligned} \tag{2}$$

The goal of this note is to consider arithmetic properties of, and relationships between, the two functions $T(n)$ and $C(n)$. In particular, we will prove that, for all $n \geq 1$,

$$ord_2(T(n)) = ord_2(C(n))$$

where $ord_2(m)$ is the highest power of 2 dividing m. This is the gist of Section 2 below. Using this fact and additional tools developed in Section 3, we will prove the following congruences:

1. For all $n \geq 0$,
$$T(n) \equiv C(n) \pmod 4.$$
2. For all $n \geq 0$, n not a Jacobsthal number,
$$T(n) \equiv C(n) \pmod{16}.$$

(The Jacobsthal numbers $\{J_n\}_{n=0}^{\infty}$ are the numbers satisfying $J_0 = J_1 = 1$ and $J_{n+2} = J_{n+1} + 2J_n$ for $n \geq 0$. The values J_n make up sequence A001045 in [8].)

Indeed, if we ignore those values of n which are Jacobsthal numbers, we will prove that, for fixed $k \geq 1$,

$$T(n) \equiv C(n) \pmod{2^k}$$

for all but a finite set of values of n. Moreover, the values of n for which this congruence does not hold must satisfy $n \leq J_{2k+1}$. This extends earlier work of the authors [4].

The above results imply some interesting arithmetic properties of $C'(n)$, the number of CSTCPPs in a $2n \times 2n \times 2n$ box which are not TSSCPPs. In particular, we have

$$C'(n) \equiv 0 \pmod 4$$

for all $n \geq 0$. Moreover, we have, for fixed $k \geq 1$,

$$C'(n) \equiv 0 \pmod{2^k}$$

for all but finitely many non-Jacobsthal numbers. There does not appear to be any obvious reason why this property should hold, nor why $C'(n)$ behaves differently when n is a Jacobsthal number.

2 The 2-adic Relationship Between $C(n)$ and $T(n)$

Throughout this note, we make use of the following lemma:

Lemma 2.1. *For any prime p and positive integer N,*

$$ord_p(N!) = \sum_{k\geq 1} \left\lfloor \frac{N}{p^k} \right\rfloor .$$

Proof. For a proof, see [6, Theorem 2.29]. $\square$

The goal of this section is to prove the following theorem:

Theorem 2.2. *For all $n \geq 1$,*

$$ord_2(C(n)) = ord_2(T(n)).$$

Proof. The proof simply involves a manipulation of various sums using Lemma 2.1. Given (2), we have

$$\begin{aligned}
ord_2(C(n)) &= \sum_{j=0}^{n-1}\sum_{k\geq 1} \left\lfloor \frac{3j+1}{2^k} \right\rfloor + \left\lfloor \frac{6j}{2^k} \right\rfloor - \left\lfloor \frac{3j}{2^k} \right\rfloor + \left\lfloor \frac{2j}{2^k} \right\rfloor - \left\lfloor \frac{4j}{2^k} \right\rfloor - \left\lfloor \frac{4j+1}{2^k} \right\rfloor \\
&= \sum_{j=0}^{n-1}\sum_{k\geq 1} \left\lfloor \frac{3j+1}{2^k} \right\rfloor + \left(\left\lfloor \frac{3j}{2^{k-1}} \right\rfloor - \left\lfloor \frac{3j}{2^k} \right\rfloor \right) + \left\lfloor \frac{2j}{2^k} \right\rfloor - 2\left\lfloor \frac{4j}{2^k} \right\rfloor \\
&= \sum_{j=0}^{n-1}\sum_{k\geq 1} \left\{ \left\lfloor \frac{3j+1}{2^k} \right\rfloor - \left(2\left\lfloor \frac{2j}{2^{k-1}} \right\rfloor - \left\lfloor \frac{j}{2^{k-1}} \right\rfloor \right) \right\} + \sum_{j=0}^{n-1} 3j \\
&= \sum_{j=0}^{n-1}\sum_{k\geq 1} \left\{ \left\lfloor \frac{3j+1}{2^k} \right\rfloor - \left(2\left\lfloor \frac{2j}{2^{k}} \right\rfloor - \left\lfloor \frac{j}{2^{k}} \right\rfloor + 2(2j) - j \right) \right\} + \sum_{j=0}^{n-1} 3j \\
&= \sum_{j=0}^{n-1}\sum_{k\geq 1} \left\{ \left\lfloor \frac{3j+1}{2^k} \right\rfloor - \left(2\left\lfloor \frac{2j}{2^{k}} \right\rfloor - \left\lfloor \frac{j}{2^{k}} \right\rfloor \right) \right\} + \sum_{j=0}^{n-1} (3j-4j+j) \\
&= \sum_{j=0}^{n-1}\sum_{k\geq 1} \left\{ \left\lfloor \frac{3j+1}{2^k} \right\rfloor - \left(\left\lfloor \frac{2j}{2^{k}} \right\rfloor + \left\lfloor \frac{2j+1}{2^{k}} \right\rfloor - \left\lfloor \frac{j}{2^{k}} \right\rfloor \right) \right\} \\
&= \sum_{j=0}^{n-1}\sum_{k\geq 1} \left\lfloor \frac{3j+1}{2^k} \right\rfloor - \sum_{j=0}^{2n-1}\sum_{k\geq 1} \left\lfloor \frac{j}{2^k} \right\rfloor + \sum_{j=0}^{n-1}\sum_{k\geq 1} \left\lfloor \frac{j}{2^k} \right\rfloor \\
&= \sum_{j=0}^{n-1}\sum_{k\geq 1} \left\lfloor \frac{3j+1}{2^k} \right\rfloor - \sum_{j=n}^{2n-1}\sum_{k\geq 1} \left\lfloor \frac{j}{2^k} \right\rfloor \\
&= \sum_{j=0}^{n-1}\sum_{k\geq 1} \left\lfloor \frac{3j+1}{2^k} \right\rfloor - \sum_{j=0}^{n-1}\sum_{k\geq 1} \left\lfloor \frac{n+j}{2^k} \right\rfloor \\
&= ord_2(T(n))
\end{aligned}$$

again using Lemma 2.1 and (1). $\square$

3 A Finiteness Result

In this section, we show that there is an upper bound on the values of n for which $ord_2(T(n)) = k$ for any positive integer k. To do this, we use insight obtained from our work in [4]. In that paper, we studied the functions

$$N_k^{\#}(n) = \sum_{j=0}^{n-1} \left\lfloor \frac{3j+1}{2^k} \right\rfloor \text{ and } D_k^{\#}(n) = \sum_{j=0}^{n-1} \left\lfloor \frac{n+j}{2^k} \right\rfloor$$

which implicitly appear in

the second-to-last line of the string of equalities in the proof of Theorem 2.2.

Definition 3.1. *We define the function* $c_k(n) := N_k^{\#}(n) - D_k^{\#}(n)$ *for any positive integer* n*, so that* $ord_2(T(n)) = \sum_{k\geq 1} c_k(n)$.

Theorem 3.2. *Suppose* $0 \leq \rho_k < 2^k$*. Then*

$$c_k(\rho_k) = \begin{cases} 0 & \textit{if} \quad 0 \leq \rho_k \leq J_{k-1} \\ \rho_k - J_{k-1} & \textit{if} \quad J_{k-1} < \rho_k \leq 2^{k-1} \\ J_k - \rho_k & \textit{if} \quad 2^{k-1} \leq \rho_k < J_k \\ 0 & \textit{if} \quad J_k \leq \rho_k < 2^k. \end{cases}$$

Moreover,

$$c_k(n + 2^k) = c_k(n) \textit{ for all } n, k \textit{ in } \mathbb{N}.$$

Furthermore, if $n = 2^k q_k + \rho_k$*, then*

$$c_k(n) = c_k(2^k(q_k + 1) - \rho_k).$$

Proof. This follows from Lemmas 5.1 through 5.4 of [4] for n which are not Jacobsthal numbers (though in the case of Lemmas 5.2 and 5.3 of [4] one has to look inside the proof to get this stronger result), and Theorem 4.1 of [4] for n which are Jacobsthal numbers. □

Since the submission of this article, the authors have found a simpler proof of Theorem 3.2, which will appear in [5]. It is clear from Theorem 3.2 that the values of the function c_k increase in increments of 1 beginning at J_{k-1}, reach a peak at 2^{k-1}, and then decrease in increments of 1 between 2^{k-1} and J_k. Propositions 3.3 and 3.4 show us that when the parities of i and j are the same, the ascents for c_i and c_j "line up" in such a way that if say $j > i$, c_i is beginning one of its ascents at the same point that c_j is beginning an ascent, so that there is an interval where the two agree. Of course c_i will reach its peak first, so beyond that point, the two will fail to agree for some time. However, given the periodic nature of these functions, they will realign at some later point. See the table in the Appendix for a demonstration of this phenomenon.

We use this insight to achieve a lower bound for $ord_2(T(n))$ when n is between two Jacobsthal numbers.

Proposition 3.3. *For* $0 \leq i \leq \left\lceil \frac{k}{2} - 1 \right\rceil$,

$$J_k = J_{k-2i} + J_{2i-1} \cdot 2^{k-2i+1}.$$

Proof. Recall from [4] that, for all $m \geq 0$, $J_m = \frac{2^{m+1} + (-1)^m}{3}$. Then

$$\begin{aligned} J_{k-2i} + J_{2i-1} \cdot 2^{k-2i+1} &= \frac{2^{k-2i+1} + (-1)^{k-2i} + 2^{2i} \cdot 2^{k-2i+1} + (-1)^{2i-1} 2^{k-2i+1}}{3} \\ &= \frac{2^{k+1} + (-1)^k}{3} \text{ upon simplification} \\ &= J_k. \end{aligned}$$

□

Proposition 3.3 allows us to show that, for a given n, the functions c_k are equal to each other for several values of k.

Proposition 3.4. *Suppose $J_k \leq n \leq 2^k$, say $n = J_k + r$ where $r \geq 0$. If $0 \leq i \leq \lceil \frac{k}{2} - 1 \rceil$ and $0 \leq r \leq J_{k-1-2i}$,*

then

$$c_{k+1-2i}(n) = r.$$

Proof. This follows from Theorem 3.2 and Proposition 3.3. □

A symmetric result is true when $2^k \leq n \leq J_{k+1}$.

Corollary 3.5. *Suppose $2^k \leq n \leq J_{k+1}$, say $n = J_{k+1} - r$ where $r \geq 0$.*

If $0 \leq i \leq \lceil \frac{k}{2} - 1 \rceil$ and $0 \leq r \leq J_{k-1-2i}$, then

$$c_{k+1-2i}(n) = r.$$

Proof. This result follows directly from the fact stated in Theorem 3.2 which says that, if $n = 2^k q_k + \rho_k$, then

$$c_k(n) = c_k(2^k(q_k + 1) - \rho_k).$$

In our case, we replace k by $k+1$ and note that $q_{k+1} = 0$, so we have

$$\begin{aligned}
c_{k+1}(n) &= c_{k+1}(J_{k+1} - r) \\
&= c_{k+1}(2^{k+1} - (J_{k+1} - r)) \\
&= c_{k+1}(J_k + r) \text{ using [4, Lemma 3.2]} \\
&= r \text{ by Proposition 3.4.}
\end{aligned}$$

□

Theorem 3.6. *Let $i, k \in \mathbb{N}$ such that $0 \leq i \leq \lceil \frac{k}{2} - 1 \rceil$ and $k - i$ odd. If $(J_i + 1) \leq r \leq 2^k - (J_i + 1)$, then*

$$ord_2(T(J_k + r)) \geq (J_i + 1)\left(\frac{k - i - 1}{2}\right).$$

Proof. If $r \leq J_{k-2i-1}$, where $0 \leq i \leq \lceil \frac{k}{2} - 1 \rceil$, then by Proposition 3.4 or Corollary 3.5,

$$c_{k+1-2i}(J_k + r) = r.$$

Now, suppose i is such that $k - i$ is odd and let $2j + 1 = k - i$. If $J_i + 1 \leq r \leq J_{k-1}$, then $r \not\leq J_i = J_{k-(k-i)} = J_{k-1-2j}$, but $r \leq J_{i+2} = J_{k-(k-i-2)} = J_{k-1-2(j-1)}$. Hence

$$c_{k+1}(J_k + r) = c_{k-1}(J_k + r) = \cdots = c_{k+1-2(j-1)}(J_k + r) = r.$$

Thus, $ord_2(T(J_k + r))$ has $j = \frac{k-i-1}{2}$ summands of value r, so that

$$ord_2(T(J_k + r)) \geq r\left(\frac{k - i - 1}{2}\right) \geq (J_i + 1)\left(\frac{k - i - 1}{2}\right).$$

□

Corollary 3.7. *If $J_{m-1} < n < J_m$, then*

$$ord_2(T(n)) \geq \left\lfloor \frac{m}{2} \right\rfloor.$$

Proof. We break the proof into two cases. First, assume that $m = 2k$, so $n = J_{2k-1}+r$ where $0 < r < 2J_{2k-2}$. Using Theorem 3.6 with $i = 0$ yields

$$\begin{aligned} ord_2(T(n)) &\geq (J_0 + 1)(k-1) \quad \text{if } 2 \leq r \leq 2J_{2k-2} - 2 \\ &> k. \end{aligned}$$

If $r = 1$ or $r = 2J_{2k-2} - 1$ then

$$c_{2k}(n) = c_{2k-2}(n) = \cdots = c_2(n) = 1$$

noting that $2 = 2k - 2(k-1)$, so

$$ord_2(T(n)) \geq k.$$

Since $k = \left\lfloor \frac{m}{2} \right\rfloor$, we have our result.

Next, assume that $m = 2k+1$, so $n = J_{2k} + r$ where $0 < r < 2J_{2k-1}$. Using Theorem 3.6 with $i = 1$ yields

$$\begin{aligned} ord_2(T(n)) &\geq (J_1 + 1)(k-1) \quad \text{if } 2 \leq r \leq 2J_{2k-1} - 2 \\ &> k. \end{aligned}$$

If $r = 1$ or $r = 2J_{2k-1} - 1$ then

$$c_{2k+1}(n) = c_{2k-1}(n) = \cdots = c_3(n) = 1$$

noting that $3 = 2k + 1 - 2(k-1)$, so

$$ord_2(T(n)) \geq k.$$

Since $k = \left\lfloor \frac{m}{2} \right\rfloor$, we have our result. □

Corollary 3.8. *If $ord_2(T(n)) = k$ with $k \geq 1$, then $n < J_{2k+1}$.*

Proof. Suppose $ord_2(T(n)) = k$. From [4, Theorem 4.1], n is not a Jacobsthal number since $ord_2(T(J_i)) = 0$ for all i. Moreover, by Corollary 3.7, if $J_{2j-1} < n < J_{2j+1}$, then $ord_2(T(n)) \geq j$. So if $j > k$, n is not between J_{2j-1} and J_{2j+1}. The largest number remaining is $J_{2k+1} - 1$ and, in fact, $ord_2(T(J_{2k+1} - 1)) = k$. □

4 Implications

It is clear from Theorem 2.2 that

$$T(n) \equiv C(n) \pmod 2.$$

However, much more can be said.

Theorem 4.1. *For all $n \geq 1$,*

$$T(n) \equiv C(n) \pmod 4.$$

Proof. Given Theorem 2.2, it is clear that Theorem 4.1 is automatically true for those values of n for which $ord_2(T(n)) \geq 1$. Hence, we only need to focus on those n for which $ord_2(T(n)) = 0$.

As noted in [4], $ord_2(T(n)) = 0$ if and only if n is a Jacobsthal number. Via straightforward calculations based on (1) and (2), it can be proved that, for all $m \geq 1$,

$$T(J_m) \equiv C(J_m) \equiv (-1)^{m-1} \pmod{4}.$$

□

Theorem 4.2. *For all $n \geq 1$, n not a Jacobsthal number,*

$$T(n) \equiv C(n) \pmod{16}.$$

Proof. We need only check those values of n for which $1 \leq ord_2(T(n)) \leq 2$, so, from Corollary 3.8, only $1 \leq n \leq J_5 - 1$ or $1 \leq n \leq 20$. This is straightforward using Maple and (1) and (2). □

One last congruential implication is noted here.

Theorem 4.3. *For all positive integers k and all but finitely many $n \geq 1$, n not a Jacobsthal number,*

$$T(n) \equiv C(n) \pmod{2^k}.$$

Proof. This is quickly proved since all that must be checked are those values of n for which $1 \leq ord_2(T(n)) \leq k-2$. Corollary 3.8 implies that the only nonJacobsthal positive integers n for which $ord_2(T(n)) \leq k-2$ satisfy $1 \leq n \leq J_{2k+1} - 1$. □

Finally, we note that results analogous to Corollary 3.8 and Theorem 4.3 do not appear to hold for primes $p > 2$. We have confirmed this computationally in regards to Theorem 4.3, and have proved that the finiteness result in Corollary 3.8 can only hold for $p = 2$. In fact, we [5] have proved the following:

Theorem 4.4. *If p is a prime greater than 3, then for each nonnegative integer k there exist infinitely many positive integers n for which $ord_p(A(n)) = k$.*

A result similar to Theorem 4.4 can be proved for $p = 3$, although it is a bit weaker. See [5].

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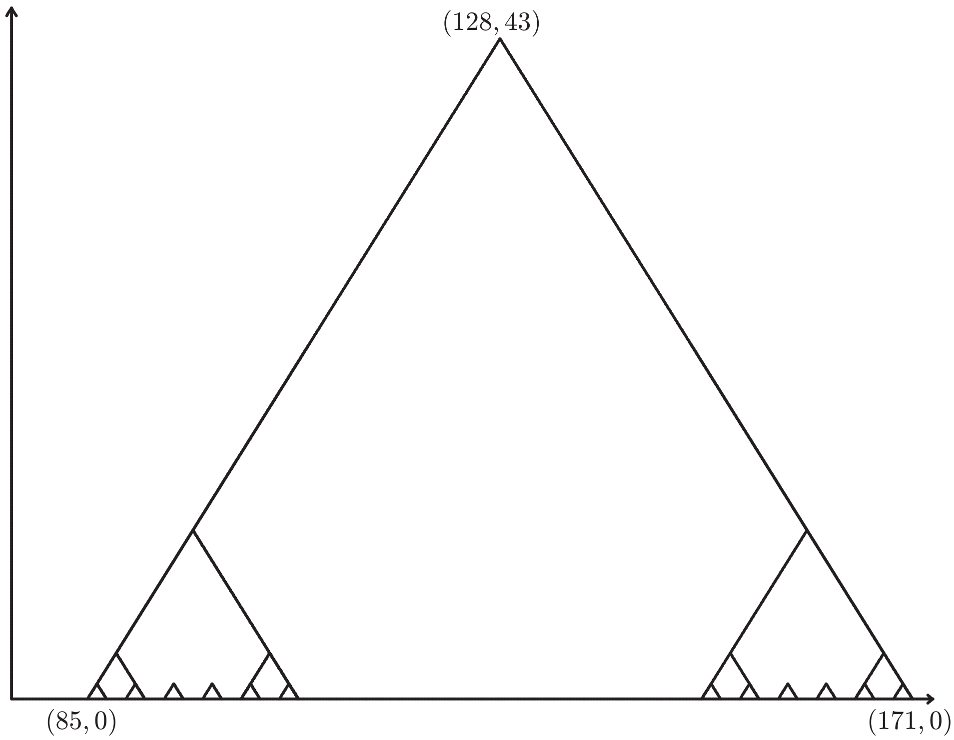
Appendix

The table below includes the values of the functions $c_2(n)$, $c_4(n)$, $c_6(n)$ and $c_8(n)$ for n between 85 and 171, which are the Jacobsthal numbers J_8 and J_9. We provide this table to show the periodic nature of the functions $c_k(n)$, and to motivate Proposition 3.3. It should be noted that, if we were to build a similar table for values of n between J_{2m-1} and J_{2m}, then we would focus attention on functions $c_k(n)$ where k is odd rather than even.

n	$c_2(n)$	$c_4(n)$	$c_6(n)$	$c_8(n)$
85	0	0	0	0
86	1	1	1	1
87	0	2	2	2
88	0	3	3	3
89	0	2	4	4
90	1	1	5	5
91	0	0	6	6
92	0	0	7	7
93	0	0	8	8
94	1	0	9	9
95	0	0	10	10
96	0	0	11	11
97	0	0	10	12
98	1	0	9	13
99	0	0	8	14
100	0	0	7	15
101	0	0	6	16
102	1	1	5	17
103	0	2	4	18
104	0	3	3	19
105	0	2	2	20
106	1	1	1	21
107	0	0	0	22
108	0	0	0	23
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
128	0	0	0	43
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
148	0	0	0	23
149	0	0	0	22
150	1	1	1	21
151	0	2	2	20
152	0	3	3	19
153	0	2	4	18
154	1	1	5	17
155	0	0	6	16
156	0	0	7	15
157	0	0	8	14
158	1	0	9	13
159	0	0	10	12
160	0	0	11	11
161	0	0	10	10
162	1	0	9	9
163	0	0	8	8
164	0	0	7	7
165	0	0	6	6
166	1	1	5	5
167	0	2	4	4
168	0	3	3	3
169	0	2	2	2
170	1	1	1	1
171	0	0	0	0

The following figure gives plots of the functions $c_2, c_4, c_6,\ c_8$ on the same set of axes.



Values of $c_2, c_4, c_6,\ c_8$
for $n = 85$ to $n = 171$

(Concerned with sequences A001045, A005130 and A051255.)

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