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Dyck Paths With No Peaks At Height k

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Abstract

A Dyck path of length $2n$ is a path in two-space from $(0,0)$ to $(2n,0)$ which uses only steps $(1,1)$ (north-east) and $(1,-1)$ (south-east). Further, a Dyck path does not go below the x-axis. A peak on a Dyck path is a node that is immediately preceded by a north-east step and immediately followed by a south-east step. A peak is at height k if its y-coordinate is k. Let $G_k(x)$ be the generating function for the number of Dyck paths of length $2n$ with no peaks at height k with $k \geq 1$. It is known that $G_1(x)$ is the generating function for the Fine numbers (sequence A000957 in [6]). In this paper, we derive the recurrence

$$G_k(x) = \frac{1}{1 - xG_{k-1}(x)},\ k \geq 2,\ G_1(x) = \frac{2}{1 + 2x + \sqrt{1-4x}}.$$

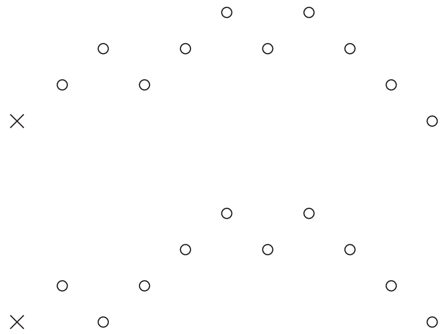
It is interesting to see that in the case $k = 2$ we get $G_2(x) = 1 + xC(x)$, where $C(x)$ is the generating function for the ubiquitous Catalan numbers (A000108). This means that the number of Dyck paths of length $2n+2$, $n \geq 0$, with no peaks at height 2 is the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$. We also provide a combinatorial proof for this last fact by introducing a bijection between the set of all Dyck paths of length $2n+2$ with no peaks at height 2 and the set of all Dyck paths of length $2n$.

Keywords: Dyck paths, Catalan number, Fine number, generating function.

1 Introduction

In [1] it was shown that Fine numbers (A000957) count Dyck paths with no peaks at height 1. One of the results of this paper is that the Catalan numbers (A000108) count Dyck paths with no peaks at height 2. This provides yet another combinatorial setting for the Catalan numbers (cf. [4], [5], [6], [7]).

A Dyck path is a path in two-space which starts at the origin, stays above the x-axis, and allows only steps of $(1,1)$ (i.e. north-east) and $(1,-1)$ (i.e. south-east). A Dyck path ends on the x-axis. A Dyck path therefore has even length with the number of north-east steps equal to the number of south-east steps. A lattice point on the path is called a peak if it is immediately preceded by a north-east step and immediately followed by a south-east step. A peak is at height k if its y-coordinate is k. Here are two Dyck paths each of length 10:



The first path has one peak at height 2 and two peaks at height 3. It has no peaks at height 1. The second path has one peak at height 1 and two at height 3. It has no peaks at height 2. Reference [1] contains much information about Dyck paths. It is known that the number of Dyck paths of length $2n$ is c_n, the n^{th} Catalan number, given by

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

We will prove that the number of these paths with no peaks at height 2 is c_{n-1}. It is known [1] that the number of these paths with no peaks at height 1 is f_n, the n^{th} Fine number with generating function

$$F(x) = \frac{1}{1-x^2C^2(x)} = 1 + x^2 + 2x^3 + 6x^4 + 18x^5 + 57x^6 + 186x^7 + O\left(x^8\right)$$

where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers. See [1], [2], and [3] for further information about the Fine numbers. Theorem 2 below contains a proof that the Fine numbers count Dyck paths with no peaks at height 1. In Theorem 1, we obtain the recurrence

$$G_k(x) = \frac{1}{1-xG_{k-1}(x)}\,,\ k \geq 2,$$

where $G_k(x)$ is the generating function for the number of Dyck paths of length $2n$ with no peaks at height k. In Section 3 we introduce a bijection between the set of all Dyck paths of length $2n$ and the set of all Dyck paths of length $2n+2$ with no peaks at height 2. This bijection provides a combinatorial proof that $G_2(x) = 1 + xC(x)$.

2 Theorems

We will use the fact that

$$F(x) = \sum_{n\geq 0} f_n x^n = \frac{C(x)}{1+xC(x)}.$$

Theorem 1: Let $G_m(x) = \sum_{n\geq 0} g(m,n)x^n$ be the generating function for Dyck paths of length $2n$ with no peaks at height m, $m \geq 1$. Then

$$G_m(x) = \frac{1}{1 - xG_{m-1}(x)} \quad ; \quad m \geq 2 \, .$$

Proof. The set of all Dyck paths of length $2n$, $n \geq 0$, with no peaks at height m consists of the trivial path (the origin) and paths with general form shown in the diagram.

A

$\times$ B

It starts with a north-east step followed by a segment labeled A which represents any Dyck path of length $2k$, $0 \leq k \leq n-1$, with no peaks at height $m-1$. A is followed by a south-east step followed by a segment labeled B which represents any Dyck path of length $2n-2-2k$ with no peaks at height m. Therefore

$$g(m,0) = 1,\, g(m,n) = \sum_{k=0}^{n-1} g(m-1,k)g(m,n-1-k) = \left[x^{n-1}\right]\{G_{m-1}(x)G_m(x)\}.$$

$$i.e. \quad g(m,0) = 1, \quad g(m,n) = [x^n]\{xG_{m-1}(x)G_m(x)\} \; ; \quad n \geq 1,$$

where $[x^k]$ denotes "coefficient of x^k in ". That is,

$$G_m(x) = 1 + x\, G_{m-1}(x)G_m(x) \, ,$$

or equivalently,

$$G_m(x) = \frac{1}{1 - xG_{m-1}(x)}$$

■

Theorem 2: The number of Dyck paths of length $2n$ with no peaks at height 1 is the Fine number f_n for $n \geq 0$.

Proof. With the notation of Theorem 1, we will prove that

$$G_1 = \sum_{n=0}^{\infty} g(1,n)x^n = \frac{1}{1 - x^2C^2}$$

Obviously, $g(1,0) = 1$ and $g(1,1) = 0$. For $n \geq 2$, a Dyck path of length $2n$ with no peaks at height 1 has the form of the diagram in the proof of Theorem 1 with A any Dyck path of length $2k$, $1 \leq k \leq n-1$, and B a Dyck path of length $2n-2k-2$ with no peaks at height 1. Therefore, for $n \geq 2$, we have

$$\begin{aligned} g(1,n) &= \sum_{k=1}^{n-1} c_k\, g(1,n-k-1) = [x^{n-1}]\{C(x)G_1(x)\} - g(1,n-1) \\ &= [x^n]\{xC(x)G_1(x)\} - g(1,n-1) \end{aligned}$$

Therefore

$$\begin{aligned} G_1(x) &= 1 + \sum_{n\geq 2} g(1,n)x^n = 1 + xC(x)G_1(x) - x - xG_1(x) + x \\ &= 1 + xG_1(x)\left(C(x) - 1\right) = 1 + xG_1(x)xC^2(x) \end{aligned}$$

That is,

$$G_1(x) = \frac{1}{1 - x^2C^2(x)}$$

■

Theorem 3: The number of Dyck paths of length $2n$ with no peaks at height 2 is the Catalan number c_{n-1} , for $n \geq 1$.

PROOF. From Theorem 1,

$$G_2(x) = \frac{1}{1 - xG_1(x)} = \frac{1}{1 - x\frac{C(x)}{1+xC(x)}} = 1 + xC(x)$$

■

Remark: In [1] it was shown that

$$\frac{f_{n-1}}{c_n} \to \frac{1}{9} \quad as \quad n \to \infty$$

Therefore

$$\frac{f_n}{c_n} \to \frac{4}{9} \quad as \quad n \to \infty$$

Since

$$\frac{c_{n-1}}{c_n} \to \frac{1}{4} \quad as \quad n \to \infty$$

we see that, for sufficiently large n, approximately $\frac{4}{9}$ of the Dyck paths of length $2n$ have no peaks at height 1, while approximately $\frac{1}{4}$ have no peaks at height 2.

Remark: $G_3(x) = \frac{2}{2-3x+x\sqrt{(1-4x)}} = 1 + x + 2x^2 + 4x^3 + 9x^4 + 22x^5 + 58x^6$
$+163x^7 + 483x^8 + 1494x^9 + O\left(x^{10}\right)$ (sequence A059019 in [6]).

3 A bijection between two Catalan families

We end with a bijection between the two Catalan families mentioned in this paper. Let Φ be the set of all Dyck paths of length $2n$ and let Ψ be the set of all Dyck paths of length $2n + 2$ with no peaks at height 2. We define a bijection between Φ and Ψ as follows. First, starting with a Dyck path P from Φ, we obtain a Dyck path $\widehat{P}$ from Ψ using the following steps.

(1) Attach a Dyck path of length 2 to the left of P to produce P^*.

(2) Let S^* be a maximal sub-Dyck path of P^* with S^* having no peaks at height 1. To each such S^* add a north-east step at the beginning and a south-east step at the end to produce sub-Dyck path $\widetilde{S}$. This step produces a Dyck path $\widetilde{P}$.

(3) From $\widetilde{P}$ eliminate each Dyck path of length 2 that is to the immediate left of each $\widetilde{S}$. We now have a unique element $\widehat{P}$ of Ψ.

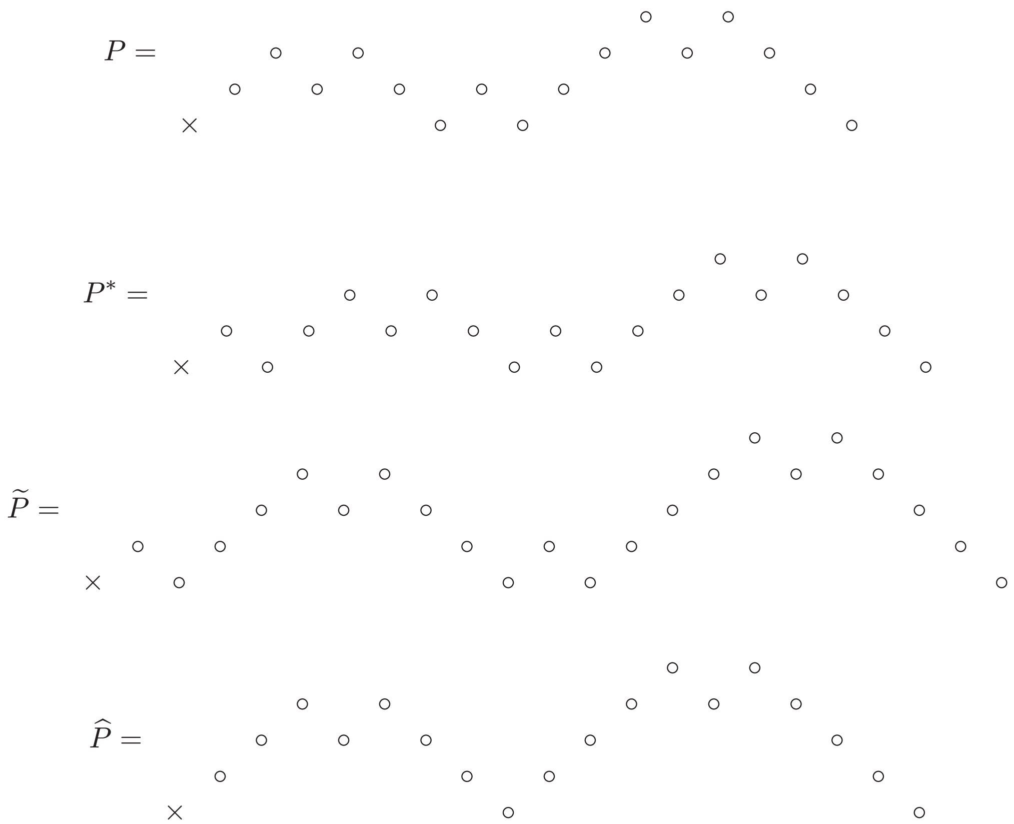
To obtain P from $\widehat{P}$, we reverse the steps as follows:

(1) Let $\widehat{S}$ be a sub-Dyck path of $\widehat{P}$ between two consecutive points on the x-axis with $\widehat{S}$ having no peaks at height 1. To each $\widehat{S}$ add a Dyck path of length 2 immediately to the left. This step produces a Dyck path $\widetilde{P}$.

(2) Let $\widetilde{S}$ be a maximal sub-Dyck path of $\widetilde{P}$. From each such $\widetilde{S}$ remove the left-most north-east step and the right-most south-east step to produce a sub-Dyck path S^*. This step produces a Dyck path P^* of length 2n+2.

(3) From P^* , remove the left-most Dyck path of length 2 to produce P.

For example, we obtain a Dyck path of length 18 with no peaks at height 2 starting with a Dyck path of length 16 as follows:



It is now easy to show that the Catalan numbers count paralellogram polynominoes (or Fat Path Pairs) with no columns at height 2 (see [7], p. 257).

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(Concerned with sequences A000108, A000957, A059019, A059027.)

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