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# Algorithms for Bernoulli numbers and Euler numbers

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## Abstract

In this paper we investigate some algorithms which produce Bernoulli numbers, Euler numbers, and tangent numbers. We also give closed formulae for Euler numbers and tangent numbers in terms of Stirling numbers of the second kind.

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**Keywords.** Bernoulli number, Euler number, Euler polynomial, Stirling number of the second kind, tangent number

## 1. Introduction

Recently M. Kaneko (ref. [4]) reformulated Akiyama and Tanigawa's algorithm for computing Bernoulli numbers as follows:

**Proposition 1 (ref. [4]).** *Given an initial sequence $a_{0,m}$ $(m = 0, 1, 2, \cdots)$, define sequences $a_{n,m}$ $(n \geq 1)$ recursively by*

$$a_{n,m} = (m+1) \cdot (a_{n-1,m} - a_{n-1,m+1}) \qquad (n \geq 1, m \geq 0).$$

*Then the leading elements are given by*

$$a_{n,0} = \sum_{m=0}^{n} (-1)^m m! \left\{ {n+1 \atop m+1} \right\} a_{0,m}, \tag{1}$$

*where the Stirling numbers of the second kind $\left\{ {n \atop m} \right\}$ are defined by*

$$\frac{(e^x - 1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{x^n}{n!}.$$

Suppose the initial sequence is $a_{0,m} = 1/(m+1)$. Then the Akiyama and Tanigawa algorithm is the following. Begin with the 0-th row 1, 1/2, 1/3, 1/4, 1/5, 1/6, $\cdots$ The recursive rule gives the first row $1 \cdot (1 - 1/2)$, $2 \cdot (1/2 - 1/3)$, $3 \cdot (1/3 - 1/4)$, $4 \cdot (1/4 - 1/5)$, $\cdots$ which is 1/2, 1/3, 1/4, 1/5, $\cdots$. The 2nd row is given by $1 \cdot (1/2 - 1/3)$, $2 \cdot (1/3 - 1/4)$, $3 \cdot (1/4 - 1/5)$, $\cdots$, etc. The Akiyama-Tanigawa matrix $a_{n,m}$ is then

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1/2 | 1/3 | 1/4 | 1/5 | 1/6 | 1/7 | 1/8 | 1/9 | 1/10 | 1/11 | ... |
| 1/2 | 1/3 | 1/4 | 1/5 | 1/6 | 1/7 | 1/8 | 1/9 | 1/10 | 1/11 | ... | |
| 1/6 | 1/6 | 3/20 | 2/15 | 5/42 | 3/28 | 7/72 | 4/45 | 9/110 | ... | | |
| 0 | 1/30 | 1/20 | 2/35 | 5/84 | 5/84 | 7/120 | 28/495 ... | | | | |
| −1/30 | −1/30 | −3/140 | −1/105 | 0 | 1/140 | 49/3960.. | | | | | |
| 0 | −1/42 | −1/28 | −4/105 | −1/28 | −29/924. | | | | | | |
| 1/42 | 1/42 | 1/140 | −1/105 | −5/231... | | | | | | | |
| 0 | 1/30 | 1/20 | 8/165 | ... | | | | | | | |
| −1/30 | −1/30 | 1/220 | ... | | | | | | | | |
| 0 | −5/66 ... | | | | | | | | | | |
| 5/66 | ... | | | | | | | | | | |
| ... | | | | | | | | | | | |

M. Kaneko [4] gave a direct proof that the leading element $a_{n,0}$ in the above array is $B_n(1)$, where the Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!}.$$

Note that Bernoulli numbers $B_n$ can be defined as $B_n(0)$.

In the sequel we denote the above algorithm as the A-algorithm. Let us change the recursive step in the A-algorithm to

$$a_{n,m} = m \cdot a_{n-1,m} - (m+1) \cdot a_{n-1,m+1} \qquad (n \geq 1, m \geq 0).$$

**Proposition 2.** *Given an initial sequence $a_{0,m}$ $(m = 0, 1, 2, \cdots)$, define the sequences $a_{n,m}$ $(n \geq 1)$ recursively by*

$$a_{n,m} = m \cdot a_{n-1,m} - (m+1) \cdot a_{n-1,m+1}, \qquad (n \geq 1, m \geq 0). \tag{2}$$

*Then*

$$a_{n,0} = \sum_{m=0}^{n} (-1)^m m! \left\{ {n \atop m} \right\} a_{0,m}. \tag{3}$$

We call the algorithm in Proposition 2 the B-algorithm. If we again start with the initial sequence $a_{0,m} = 1/(m+1)$, then (cf. Eq. (6.99) or p. 560 of [2])

$$a_{n,0} = \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ {n \atop m} \right\}}{m+1} = B_n = B_n(0).$$

In fact, we have the following theorem:

**Theorem 1.** *Suppose the initial sequence $a_{0,m}$ $(m = 0, 1, 2, \cdots)$ has the ordinary generating function*

$$A(x) = \sum_{m=0}^{\infty} a_{0,m} x^m.$$

*Then the leading elements $a_{n,0}$ $(n = 0, 1, 2, \dots)$ have exponential generating function*

$$B(x) = \sum_{n=0}^{\infty} a_{n,0} \frac{x^n}{n!}$$

*given by $e^x A(1 - e^x)$ for the A-algorithm and $A(1 - e^x)$ for the B-algorithm.*

Consider now the initial sequence $a_{0,m} = 1/2^m$ in the A-algorithm and B-algorithm, respectively. We obtain the leading elements $a_{n,0}$ as $E_n(1)$ and $E_n(0)$, respectively, where the Euler polynomials $E_n(x)$ are defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n(x) t^n}{n!}.$$

Note that the Euler numbers $E_n$ can be defined as $2^n E_n(1/2)$. Alternatively we may define the Euler numbers by

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}.$$

They are closely related to the tangent numbers $T_n$ (cf. [3]), which are defined by

$$\tan x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} T_{2n+1}}{(2n+1)!} x^{2n+1}, \qquad T_0 = 1.$$

Moreover, if we take the initial sequence to be

$$a_{0,m} = (-1)^{[m/4]} \cdot 2^{-[m/2]} \cdot (1 - \delta_{4,m+1}), \qquad \text{where } \delta_{4,i} = \begin{cases} 1, & \text{if } \; 4|i, \\ 0, & \text{otherwise.} \end{cases}$$

in the A-algorithm and B-algorithm, respectively, the leading elements $a_{n,0}$ become $E_n$ and $T_n$, respectively. We now give the proof of the above statements.

## 2. Proof of Proposition 2 and Theorem 1

To prove Proposition 2, we use a similar trick to that used in the proof of Proposition 2 in [4]. Put

$$g_n(t) = \sum_{m=0}^{\infty} a_{n,m} t^m.$$

By the recursion Eq.(2) we have for $n \geq 1$

$$\begin{aligned}
g_n(t) &= \sum_{m=0}^{\infty} (m \cdot a_{n-1,m} - (m+1) \cdot a_{n-1,m+1}) t^m \\
&= \sum_{m=0}^{\infty} (m+1) a_{n-1,m+1} t^{m+1} - \sum_{m=0}^{\infty} (m+1) a_{n-1,m+1} t^m \\
&= (t-1) \sum_{m=0}^{\infty} (m+1) a_{n-1,m+1} t^m \\
&= (t-1) \frac{d}{dt} g_{n-1}(t) \ = \ \left( (t-1) \frac{d}{dt} \right)^n g_0(t).
\end{aligned}$$

Using the recursion for the Stirling numbers of second kind

$$\left\{ {n+1 \atop m+1} \right\} = (m+1) \left\{ {n \atop m+1} \right\} + \left\{ {n \atop m} \right\},$$

and mathematical induction on $n$, we have (ref. p. 310 in [2])

$$\left( (t-1) \frac{d}{dt} \right)^n = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (t-1)^m \left( \frac{d}{dt} \right)^m .$$

Therefore

$$g_n(t) = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (t-1)^m \left( \frac{d}{dt} \right)^m g_0(t).$$

Setting $t = 0$ we get the assertion of Proposition 2

$$a_{n,0} = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (-1)^m m! a_{0,m}. \quad \square$$

Now we give the proof of Theorem 1. In the A-algorithm we use the identity which appeared in Eq. (3) of [4]:

$$\frac{e^x (e^x - 1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n+1 \atop m+1} \right\} \frac{x^n}{n!},$$

and Eq.(1). Then the exponential generating function for the leading elements $a_{n,0}$ is

$$\begin{aligned}
B(x) = \sum_{n=0}^{\infty} a_{n,0} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} (-1)^m m! \left\{ {n+1 \atop m+1} \right\} a_{0,m} \right) \frac{x^n}{n!} \\
&= \sum_{m=0}^{\infty} (-1)^m m! a_{0,m} \sum_{n=m}^{\infty} \left\{ {n+1 \atop m+1} \right\} \frac{x^n}{n!} \\
&= \sum_{m=0}^{\infty} (-1)^m m! a_{0,m} \frac{e^x (e^x - 1)^m}{m!} \\
&= e^x \sum_{m=0}^{\infty} (1 - e^x)^m a_{0,m} \ = \ e^x A(1 - e^x).
\end{aligned}$$

Next we treat the B-algorithm case. Using Eq.(3) we have

$$\begin{aligned}
B(x) = \sum_{n=0}^{\infty} a_{n,0}\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(-1)^m m!\left\{{n \atop m}\right\}a_{0,m}\right)\frac{x^n}{n!} \\
&= \sum_{m=0}^{\infty}(-1)^m m! a_{0,m}\sum_{n=m}^{\infty}\left\{{n \atop m}\right\}\frac{x^n}{n!} \\
&= \sum_{m=0}^{\infty}(-1)^m m! a_{0,m}\frac{(e^x-1)^m}{m!} \\
&= \sum_{m=0}^{\infty}(1-e^x)^m a_{0,m} \;=\; A(1-e^x).
\end{aligned}$$

This completes the proof of Theorem 1. □

## 3. Euler numbers and Tangent numbers

**Theorem 2.** *Set $a_{0,m} = 1/2^m$ for $m \geq 0$ in the A-algorithm and B-algorithm. Then the leading elements $a_{n,0}$ for $n \geq 0$ are given by $E_n(1)$ and $E_n(0)$, respectively.*

*Proof.* In the B-algorithm,

$$\begin{aligned}
A(1-e^x) &= \sum_{m=0}^{\infty}(1-e^x)^m a_{0,m} \\
&= \sum_{m=0}^{\infty}\left(\frac{1-e^x}{2}\right)^m \;=\; \frac{2}{e^x+1}.
\end{aligned}$$

The exponential generating functions for $E_n(0)$ and $E_n(1)$ are $2/(e^x+1)$ and $2e^x/(e^x+1)$, respectively. Using Theorem 1 completes the proof. □

**Theorem 3.** *Set*

$$a_{0,m} = (-1)^{[m/4]}\cdot 2^{-[m/2]}\cdot(1-\delta_{4,m+1}), \qquad \textit{where } \delta_{4,i} = \begin{cases} 1, & \textit{if } \; 4|i, \\ 0, & \textit{otherwise,} \end{cases}$$

*in the A-algorithm and B-algorithm. Then the leading elements $a_{n,0}$ are $E_n$ and $T_n$, respectively.*

*Proof.* The exponential generating functions for $E_n$ and $T_n$ are $2e^x/(e^{2x}+1)$ and $2/(e^{2x}+1)$, respectively. From the results of Theorem 1, we only need to prove that $A(1-e^x) = 2/(e^{2x}+1)$ in the B-algorithm. We have

$$\begin{aligned}
A(1-e^x) &= \sum_{m=0}^{\infty}(1-e^x)^m a_{0,m} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k(1-e^x)^{4k}}{2^{2k}} + \sum_{k=0}^{\infty}\frac{(-1)^k(1-e^x)^{4k+1}}{2^{2k}} + \sum_{k=0}^{\infty}\frac{(-1)^k(1-e^x)^{4k+2}}{2^{2k+1}}.
\end{aligned}$$

Let

$$D(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (1-e^x)^{4k}}{2^{2k}} = \frac{4}{(e^{2x}-4e^x+5)(e^{2x}+1)}.$$

Then

$$\begin{aligned} A(1-e^x) &= D(x) + (1-e^x)D(x) + \frac{(1-e^x)^2}{2} D(x) \\ &= D(x) \cdot (1 + 1 - e^x + \frac{1-2e^x+e^{2x}}{2}) \\ &= \frac{4}{(e^{2x}-4e^x+5)(e^{2x}+1)} \cdot \frac{e^{2x}-4e^x+5}{2} \quad = \quad \frac{2}{e^{2x}+1}. \end{aligned}$$

This completes the proof. $\square$

The following is the matrix generated by Theorem 3 for the Euler numbers $E_n$:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1/2 | 0 | -1/4 | -1/4 | -1/8 | 0 | 1/16 | 1/16 | 1/32 | ... |
| 0 | 1 | 3/2 | 1 | 0 | -3/4 | -7/8 | -1/2 | 0 | 5/16 | ... | |
| -1 | -1 | 3/2 | 4 | 15/4 | 3/4 | -21/8 | -4 | -45/16 | ... | | |
| 0 | -5 | -15/2 | 1 | 15 | 81/4 | 77/8 | -19/2 | ... | | | |
| 5 | 5 | -51/2 | -56 | -105/4 | 255/4 | 1071/8 | ... | | | | |
| 0 | 61 | 183/2 | -119 | -450 | -1683/4 | ... | | | | | |
| -61 | -61 | 1263/2 | 1324 | -585/4 | ... | | | | | | |
| 0 | -1385 | -4155/2 | 5881 | ... | | | | | | | |
| 1385 | 1385 | -47751/2 | ... | | | | | | | | |
| 0 | 50521 | ... | | | | | | | | | |
| -50521 | ... | | | | | | | | | | |
| ... | | | | | | | | | | | |

The following is the matrix generated by Theorem 3 for the tangent numbers $T_n$:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1/2 | 0 | -1/4 | -1/4 | -1/8 | 0 | 1/16 | 1/16 | 1/32 | 0 | ... |
| -1 | 0 | 1 | 1 | 1/4 | -1/2 | -3/4 | -1/2 | -1/16 | 1/4 | 5/16 | ... | |
| 0 | -2 | -1 | 2 | 7/2 | 2 | -1 | -3 | -11/4 | -7/8 | ... | | |
| 2 | 0 | -8 | -8 | 4 | 16 | 15 | 1 | -113/8 | ... | | | |
| 0 | 16 | 8 | -40 | -64 | -10 | 83 | 120 | ... | | | | |
| -16 | 0 | 136 | 136 | -206 | -548 | -342 | ... | | | | | |
| 0 | -272 | -136 | 1232 | 1916 | -688 | ... | | | | | | |
| 272 | 0 | -3968 | -3968 | 11104 | ... | | | | | | | |
| 0 | 7936 | 3968 | -56320 | ... | | | | | | | | |
| -7936 | 0 | 176896 | ... | | | | | | | | | |
| 0 | -353792 | ... | | | | | | | | | | |
| 353792 | ... | | | | | | | | | | | |
| ... | | | | | | | | | | | | |

Using Eq.(1) and Eq.(3) in Theorem 2 and 3, we can give closed formulae for $E_n(0)$, $E_n(1)$, $E_n$, and $T_n$.

**Corollary.**

$$E_n(0) = \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ {n \atop m} \right\}}{2^m}, \qquad E_n = \sum_{m=0}^{n} (-1)^m m! \left\{ {n+1 \atop m+1} \right\} a_{0,m},$$

$$E_n(1) = \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ {n+1 \atop m+1} \right\}}{2^m}, \qquad T_n = \sum_{m=0}^{n} (-1)^m m! \left\{ {n \atop m} \right\} a_{0,m},$$

*where* $\{a_{0,m}\}_{m=0}^{\infty}$ *is the initial sequence in Theorem 3.*

**Remark.** A referee mentions that the B-algorithm may well yield other notable sequences. For instance, the Bell numbers can be obtained from the initial sequence $(-1)^n/n!$, since their exponential generating function is

$$B(x) = A(1-e^x) = \sum_{m=0}^{\infty} \frac{(e^x-1)^m}{m!} = e^{e^x-1}.$$

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