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# Two Analogues of a Classical Sequence

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**Abstract**

*We compute exponential generating functions for the numbers of edges in the Hasse diagrams for the* B- *and* D-*analogues of the partition lattices.*

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## Introduction

When one looks up the sequence 1, 6, 31, 160, 856, 4802, 28 337, 175 896, ... in one of Sloane's integer sequence identifiers [HIS, EIS, OIS], one learns that these numbers are the numbers of driving-point impedances of an $n$-terminal network for $n = 2, 3, 4, 5, 6, 7, 8, 9, \dots$ as described in an old article by Riordan [Ri].

In combinatorics there are two common ways of generalizing classical enumerative facts. One such generalization arises by replacing the set $[n] = \{1, \dots, n\}$ by an $n$-dimensional vector space over the finite field $\mathbb{F}_q$ to get a $q$-analogue. The other generalization or extension is by considering "B- and D-analogues" of an "A-case". This terminology stems from Lie theory. (There is no "C-case" here since it coincides with the “B-case”.) Of course one may try to combine the two approaches and supply $q$-B- and $q$-D-analogues.

In this note I shall describe B- and D-analogues of the numbers of driving-point impedances of an $n$-terminal network. To assuage any possible curiosity about how these sequences look, here are their first few terms:

| | |
|---|---|
| B-analogue | 1, 8, 58, 432, 3396, 28 384, 252 456, 2 385 280, ... |
| D-analogue | 0, 4, 31, 240, 1931, 16 396, 147 589, 1 408 224, ... |

I should probably emphasize that I will only give mathematical arguments and will not attempt to provide a physical realization of B- and D-networks.

We start from certain classical hyperplane arrangements. A hyperplane arrangement defines a family of subspaces, namely those subspaces which can be written as intersections of some of the hyperplanes in the arrangement. For each such subspace we will choose a normal form that represents the subspace. Such a normal form consists of an equivalence class of partial $\{\pm 1\}$-partitions in the terminology of Dowling [Do]. Dowling actually constructed $G$-analogues of the partition lattices for any finite group $G$. Using the concept of voltage graphs (or signed graphs for $|G| = 2$) or more generally biased graphs, Zaslavsky gave a far-reaching generalization of Dowling's work. It is amusing to see that not only the network but also the mathematical treatment of hyperplane arrangements carries a graph-theoretical flavour. Here we will stick to the normal form and not translate things into the framework of graph theory, despite the success this approach has had for example in [BjSa]. In some sense the normal form approach pursues a strategy opposite to that of Zaslavsky's graphs.

Whitney numbers and characteristic polynomials for hyperplane arrangements or more generally for subspace arrangements, that is, the numbers of vertices with fixed rank in the Hasse diagrams and the Möbius functions, have been studied by many authors. Apparently little attention has been paid so far to the numbers of edges in the Hasse diagrams.

There is another point worth mentioning. It concerns a dichotomy among the A-, B-, and D-series. We will see that everything is very easy for the first two series whereas for the D-series we must work a little harder. Such a dichotomy between the A- and B-series on the one hand and the D-series on the other also occurs in other contexts, e. g., in the problem of counting reduced decompositions of the longest element in the corresponding Coxeter groups (see [St] for the initial paper). In contrast, in the Lie theory one has a different dichotomy, namely, between the simply laced (like $\mathsf{A}$ and $\mathsf{D}$) and the non-simply-laced (like $\mathsf{B}$ and $\mathsf{C}$) types.

Finally, an obvious generalization, which, however, we do not go into, concerns hyperplane arrangements for the infinite families of unitary reflection groups.

## Hyperplane arrangements and their intersection lattices

Let $\mathcal{A} = \{H_1, \dots, H_N\}$ be a collection of subspaces of codimension 1 in the vector space $\mathbb{R}^n$. We let $L(\mathcal{A})$ denote the poset of all intersections $H_{i_1} \cap \cdots \cap H_{i_r}$, ordered by reverse inclusion. This poset $L(\mathcal{A})$ is actually a geometric lattice. Its bottom element $\widehat{0}$ is the intersection over the empty index set, i. e., $\mathbb{R}^n$. The atoms are the hyperplanes $H_1, \dots, H_N$, and the top element $\widehat{1}$ is $H_1 \cap \cdots \cap H_N$. For many further details the reader is referred to Cartier's Bourbaki talk [Ca], Björner's exposition [Bj] for more general subspace arrangements, and the monograph by Orlik and Terao [OT] for a thorough exposition of the theory.

A theorem due to Orlik and Solomon states that for a finite irreducible Coxeter group $W$ with Coxeter arrangement $\mathcal{A} = \mathcal{A}(W)$ we have the equality

$$|\mathcal{A}^H| = |\mathcal{A}| + 1 - h \tag{1}$$

where $H \in \mathcal{A}$ is any hyperplane of the arrangement, $h$ is the Coxeter number of $W$, and $\mathcal{A}^H$ is the hyperplane arrangement in $H$ with the hyperplanes $H \cap H'$ for $H' \in \mathcal{A} - \{H\}$. In other words, (1) says that each atom in the intersection lattice $L(\mathcal{A})$ is covered by $|\mathcal{A}| + 1 - h$ elements. One may wonder what can be said about the number of elements that cover an arbitrary element in $L(\mathcal{A})$.

The intersection lattices that concern us here come from the following hyperplanes in $\mathbb{R}^n$.

| type of $\mathcal{A}$ | elements of $\mathcal{A}$ |
|---|---|
| $(\mathsf{A}_1)^n$ | $\{x_a = 0\}_{a=1,\dots,n}$ |
| $\mathsf{A}_{n-1}$ | $\{x_b = x_c\}_{1\leqslant b<c\leqslant n}$ |
| $\mathsf{B}_n$ | $\{x_a = 0\}_{a=1,\dots,n},\ \{x_b = x_c\}_{1\leqslant b<c\leqslant n},\ \{x_b = -x_c\}_{1\leqslant b<c\leqslant n}$ |
| $\mathsf{D}_n$ | $\{x_b = x_c\}_{1\leqslant b<c\leqslant n},\ \{x_b = -x_c\}_{1\leqslant b<c\leqslant n}$ |

Note that $\bigcap\limits_{H\in\mathcal{A}} H$ is the line $x_1 = \cdots = x_n$ for type $\mathsf{A}_{n-1}$ (so the rank is $n-1$ in this case if $n > 0$) whereas for the other types the hyperplanes only meet in the zero vector. We agree to let $\mathsf{A}_{-1}$ denote the empty hyperplane arrangement in 0. So the intersection lattices for $\mathsf{A}_{-1}$ and

$\mathsf{A}_0$ are isomorphic. Also there is a slight abuse of notation for type $\mathsf{A}_1$ because it can be considered as $(\mathsf{A}_1)^1$ or as $\mathsf{A}_{2-1}$. But this will not cause trouble.

For each subspace $E \in L(\mathcal{A})$ we define the subset $B_E \subseteq [n] = \{1, \dots, n\}$ by the property that

$$C_E := \bigcap_{a \in [n] - B_E} \{x_a = 0\}$$

is the smallest intersection of coordinate hyperplanes that contains $E$. For instance if $\mathcal{A}$ is of type $\mathsf{A}_{n-1}$, we have $B_E = [n]$ for all $E \in L(\mathcal{A})$. For the hyperplane $E = \{x_1 = x_2\} \cap \{x_2 = x_3\} \cap \{x_1 = -x_3\} \cap \{x_4 = x_7\} \cap \{x_5 = x_8\} \cap \{x_8 = 0\} \subseteq \mathbb{R}^8$ we get $B_E = \{4, 6, 7\}$.

Regarded as a subspace of $C_E$, $E$ is described by a partition of $B_E$ together with a function $\zeta : B_E \to \{\pm 1\}$. If $\{B_1, \dots, B_k\}$ is a partition of $B_E$ into $k$ blocks, then $E$ is the $k$-dimensional subspace

$$E = \big\{(x_1, \dots, x_n) \in C_E \;\big|\; b, c \in B_j \text{ for some } j \Longrightarrow \zeta(b)\, x_b = \zeta(c)\, x_c\big\}.$$

Clearly, the correspondence between $E$ and $\big(\{B_1, \dots, B_k\}, \zeta\big)$ is 1 to $2^k$ because for each block there is a choice of sign.

This correspondence gives us a convenient notation for the subspaces in $L(\mathcal{A})$. We write down a partition of some $B \subseteq [n]$ and decorate the numbers $a \in B$ with $\zeta(a) = -1$ with an overbar. Having the possibility of choosing an overall sign for each block, we agree that the smallest number in each block does not have an overbar. As an example take the Coxeter arrangement of type $\mathsf{B}_3$. There are 24 subspaces to be considered. Their representations as "signed permutations" are shown in the vertices (boxes) of the following Hasse diagram.

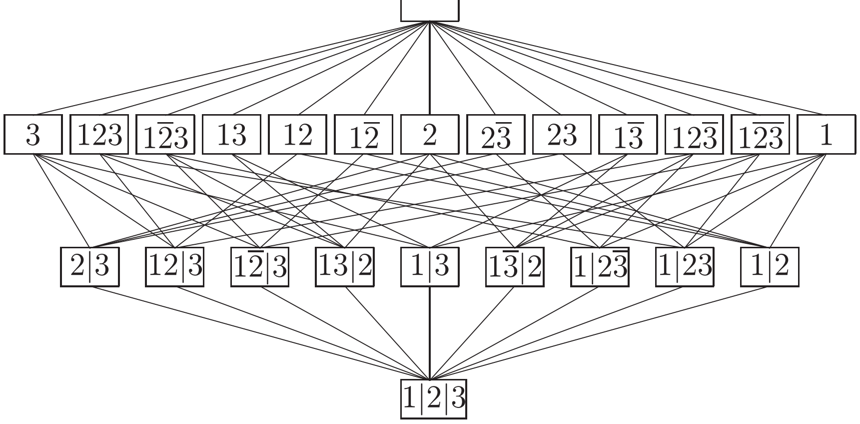


FIGURE 1. Hasse diagram of the $\mathsf{B}_3$ lattice

For instance 3 stands for the line $x_1 = x_2 = 0$, $1\bar{2}3$ is for $x_1 = -x_2 = x_3$, $1|2\bar{3}$ denotes the plane $x_2 = -x_3$, $1|2$ means $x_3 = 0$ etc.

## Vertices in the Hasse diagrams

**Lemma 1.** *For a partition $\{B_1,\dots,B_k\}$ of a subset $B\subseteq[n]$ and a function $\zeta: B\to\{\pm1\}$ the $k$-dimensional subspace*

$$\left\{(x_1,\dots,x_n)\in\mathbb{R}^n \;\middle|\; \begin{array}{l} a\in[n]-B \implies x_a=0 \\ b,c\in B_j \text{ for some } j \implies \zeta(b)\,x_b=\zeta(c)\,x_c \end{array}\right\}$$

*belongs to $L(\mathcal{A})$ according to the following table.*

| type of $\mathcal{A}$ | condition |
|---|---|
| $(\mathsf{A}_1)^n$ | $\lvert B\rvert=k$, $\zeta=1$ |
| $\mathsf{A}_{n-1}$ | $B=[n]$, $\zeta=1$ |
| $\mathsf{B}_n$ | — |
| $\mathsf{D}_n$ | $\big\lvert[n]-B\big\rvert\neq1$ |

*Proof.* The conditions in the table above should be clear. For the types $(\mathsf{A}_1)^n$ and $\mathsf{A}_{n-1}$ we put $\zeta=1$ for simplicity (literally, $\zeta$ must only be constant on each block $B_j$). The condition for $\mathsf{D}_n$ simply takes into account that the hyperplanes $x_a=0$ do not belong to $L(\mathcal{A})$. But for instance $x_1=\dots=x_r=0$ for $r\geqslant2$ can be written as $x_1=-x_2$, $x_1=\dots=x_r$ and hence this subspace is an element of $L(\mathcal{A})$. □

For integers $n,k\geqslant0$ and $b>0$ let $S_b(n,k)$ denote the number of partitions of $[n]$ into $k$ blocks each containing at least $b$ elements. So $S_1(n,k)=S(n,k)$ is a Stirling number of the second kind. Besides $b=1$ we shall only need the case where $b=2$, which one knows from Pólya-Szegő [PS, Part I, Chap. 4, § 3; Part VIII, Chap. 1, No. 22.3]. Nevertheless we state the following more general proposition.

**Proposition 2.** *For every integer $b>0$ the generating function for the numbers $S_b(n,k)$ of partitions of $[n]$ into $k$ blocks of length at least $b$ is*

$$\sum_{n,k\geqslant0} S_b(n,k)\frac{x^n}{n!}\,y^k=\exp\left(y\cdot\left(e^x-1-x-\frac{x^2}{2!}-\dots-\frac{x^{b-1}}{(b-1)!}\right)\right).$$

*Proof.* For $k\geqslant1$ we have the recurrence relation

$$S_b(n,k)=k\,S_b(n-1,k)+\tbinom{n-1}{b-1}\,S_b(n-b,k-1). \tag{2}$$

In fact, to obtain a partition of $[n]$ into $k$ blocks of lengths at least $b$, we can either take a partition of $[n-1]$ into $k$ blocks of lengths at least $b$ and append the element $n$ to any one of the $k$ blocks, or we can take $b-1$ elements from $[n-1]$ which together with $n$ constitute a block with $b$ elements and partition the remaining $n-b$ elements into $k-1$ blocks of lengths at least $b$.

To prove the proposition we must show that for every integer $k \geqslant 0$

(3)
$$f_k(x) := \sum_{n\geqslant 0} S_b(n,k)\frac{x^n}{n!} = \frac{1}{k!}\Big(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{b-1}}{(b-1)!}\Big)^k.$$

This follows by induction on $k$. The case $k=0$ is clear: $S_b(n,0)=\delta_{n,0}$. For $k\geqslant 1$ we get a differential equation for $f_k(x)$, namely

$$\begin{aligned} f_k'(x) &= \sum_n S_b(n,k)\frac{x^{n-1}}{(n-1)!} \\ &\overset{(2)}{=} \sum_n k\, S_b(n-1,k)\frac{x^{n-1}}{(n-1)!} + \sum_n \tbinom{n-1}{b-1}\, S_b(n-b,k-1)\frac{x^{n-1}}{(n-1)!} \\ &= k\, f_k(x) + \frac{x^{b-1}}{(b-1)!}\, f_{k-1}(x) \\ &= k\, f_k(x) + \frac{x^{b-1}}{(b-1)!}\frac{1}{(k-1)!}\Big(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{b-1}}{(b-1)!}\Big)^{k-1} \end{aligned}$$

whose unique solution satisfying $f_k(0)=0$ is in fact given by the right hand side in equation (3). $\square$

The lattices $L(\mathcal{A})$ are graded posets with rank function the codimension. The $r$th Whitney number of the second kind of a graded poset is by definition the number of elements of rank $r$. We begin by making the Whitney numbers quite explicit. We fix one of our hyperplane arrangements $\mathcal{A}$ in $\mathbb{R}^n$ and let $W(n,r)$ be the $r$th Whitney number (of the second kind) of the intersection lattice $L(\mathcal{A})$. The Whitney numbers $W(n,n-k)$ when written in an array can be seen as a generalization of Pascal's triangle. In fact, Pascal's triangle arises for the Boolean lattices of type $(\mathsf{A}_1)^n$.

Let us digress for a moment to consider such generalized Pascal triangles or arrays. The (upper left) corner in the arrays that follow carry the Whitney number $W(0,0)$, and the entries $(p,q)$ for the other Whitney numbers $W(p,q)$ are in accordance with the following diagram.

$$\begin{array}{ccc} W(n,n-k) & \longrightarrow & W(n+1,n-k+1) \\ \downarrow & & \\ W(n+1,n-k) & & \end{array}$$

- **Pascal arrangements** = Coxeter arrangements of type $(\mathsf{A}_1)^n$. $W(n, n-k) = \binom{n}{k}$.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 3 & \longrightarrow & 6 & \longrightarrow & 10 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 4 & \longrightarrow & 10 & \longrightarrow & 20 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

- **Stirling arrangements** = Coxeter arrangements of type $\mathsf{A}_{n-1}$. $W(n, n-k) = S(n, k)$. For the $\mathsf{A}_{n-1}$ lattices the analogue of the equation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ reads $S(n, k) = k\, S(n-1, k) + S(n-1, k-1)$, the case $b = 1$ of (2).

$$\begin{array}{ccccccccc}
1 & \xrightarrow{\cdot 0} & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \xrightarrow{\cdot 2} & 3 & \xrightarrow{\cdot 2} & 7 & \xrightarrow{\cdot 2} & 15 & \xrightarrow{\cdot 2} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \xrightarrow{\cdot 3} & 6 & \xrightarrow{\cdot 3} & 25 & \xrightarrow{\cdot 3} & 90 & \xrightarrow{\cdot 3} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \xrightarrow{\cdot 4} & 10 & \xrightarrow{\cdot 4} & 65 & \xrightarrow{\cdot 4} & 350 & \xrightarrow{\cdot 4} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

- **2-Dowling arrangements** = Coxeter arrangements of type $\mathsf{B}_n$. $W(n, n-k) = T(n,k)$. For the $\mathsf{B}_n$ lattices the Whitney numbers satisfy the relation $T(n,k) = (2k+1)\,T(n-1,k) + T(n-1,k-1)$ (see Corollary 5).

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \xrightarrow{\cdot 3} & 4 & \xrightarrow{\cdot 3} & 13 & \xrightarrow{\cdot 3} & 40 & \xrightarrow{\cdot 3} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \xrightarrow{\cdot 5} & 9 & \xrightarrow{\cdot 5} & 58 & \xrightarrow{\cdot 5} & 330 & \xrightarrow{\cdot 5} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \xrightarrow{\cdot 7} & 16 & \xrightarrow{\cdot 7} & 170 & \xrightarrow{\cdot 7} & 1520 & \xrightarrow{\cdot 7} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Continuing in the obvious way, one gets Whitney numbers of Dowling lattices corresponding to the complete monomial groups $(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$, the wreath product of the symmetric group of degree $n$ acting on $(\mathbb{Z}/m\mathbb{Z})^n$. This is straightforward, and calculations can be found in [Be1, Be2]. For the $\mathsf{D}_n$ lattices the situation is more subtle. The following table suggests why this is so.

| type | exponents |
|---|---|
| $(\mathsf{A}_1)^n$ | $1, 1, \ldots, 1$ |
| $\mathsf{A}_n$ | $1, 2, \ldots, n$ |
| $\mathsf{B}_n$ | $1, 3, \ldots, 2n-1$ |
| $\mathsf{D}_n$ | $1, 3, \ldots, 2n-3, n-1$ |

The maverick exponent $n-1$ for type $\mathsf{D}_n$ reveals the fact that the determinant of a $2n \times 2n$ skew-symmetric matrix is the square of a polynomial in the matrix entries.

This ends our digression. Also from now on we will neglect the nearly trivial case of type $(\mathsf{A}_1)^n$.

**Proposition 3.** *The Whitney numbers $W(n, n-k)$ are given by the following formulae.*

| type | $W(n, n-k)$ |
|---|---|
| $\mathsf{A}_{n-1}$ | $S(n,k)$ |
| $\mathsf{B}_n$ | $\sum_{j=k}^{n} 2^{j-k}\binom{n}{j}\, S(j,k)$ |
| $\mathsf{D}_n$ | $\sum_{j=k}^{n} 2^{j-k}\binom{n}{j}\, S(j,k) - 2^{n-1-k} n\, S(n-1,k)$ |

*Proof.* The proof follows by elementary combinatorial reasoning from the table in Lemma 1. (Recall that $S(n, k)$ is a Stirling number of the second kind.) ☐

The table in Proposition 3 can also be found in the last corollary of [Za].

**Theorem 4.** *The generating functions for the Whitney numbers are as given in the following table.*

| type | $\sum_{n,k\geqslant 0} W(n, n-k)\frac{x^n}{n!}\, y^k$ |
|---|---|
| $\mathsf{A}$ | $\exp\big(y\cdot(e^x - 1)\big)$ |
| $\mathsf{B}$ | $e^x \exp\Big(\frac{y}{2}\cdot\big(e^{2x} - 1\big)\Big)$ |
| $\mathsf{D}$ | $(e^x - x)\exp\Big(\frac{y}{2}\cdot\big(e^{2x} - 1\big)\Big)$ |

*Proof.* For type $\mathsf{A}$ this is Proposition 2 with $b = 1$. For type $\mathsf{B}$ the coefficients

$$a_n(y) = \sum_{k\geqslant 0}\sum_{j=k}^{n} 2^{j-k}\binom{n}{j}\, S(j,k)\, y^k \in \mathbb{Z}[y]$$

are the binomial transforms of

$$b_j(y) = \sum_{k\geqslant 0} 2^{j-k} S(j,k)\, y^k \in \mathbb{Z}[y].$$

Hence

$$\begin{aligned}\sum_{n\geqslant 0} a_n(y)\frac{x^n}{n!} &= e^x \sum_{j\geqslant 0} b_j(y)\frac{x^j}{j!}\\ &= e^x \sum_{j,k} S(j,k)\frac{(2x)^j}{j!}\left(\frac{y}{2}\right)^k = e^x \exp\Big(\frac{y}{2}\cdot\big(e^{2x} - 1\big)\Big).\end{aligned}$$

Finally, for type $\mathsf{D}$ we need to subtract

$$\begin{aligned}\sum_{n,k} 2^{n-1-k} n\, S(n-1,k)\frac{x^n}{n!}\, y^k &= x\sum_{n,k} S(n-1,k)\frac{(2x)^{n-1}}{(n-1)!}\Big(\frac{y}{2}\Big)^k \\ &= x\exp\Big(\frac{y}{2}\cdot\big(e^{2x}-1\big)\Big)\end{aligned}$$

from the generating function for type $\mathsf{B}$. $\square$

Setting $y = 1$ in Theorem 4 we get the exponential generating function for the numbers of vertices in the Hasse diagrams. The coefficients in this exponential generating function are the Bell numbers for type $\mathsf{A}$ and the Dowling numbers for type $\mathsf{B}$. For type $\mathsf{D}$ these numbers are apparently unnamed.

**Corollary 5.** *The Whitney numbers $T(n,k) = W(n,n-k)$ for the 2-Dowling arrangements satisfy the recurrence relation*

$$T(n,k) = (2k+1)\,T(n-1,k) + T(n-1,k-1).$$

*Proof.* $\Big(\frac{\partial}{\partial x} - 2\,y\,\frac{\partial}{\partial y} - 1 - y\Big)\, e^x \exp\Big(\frac{y}{2}\cdot\big(e^{2x}-1\big)\Big) = 0.$ $\square$

## Edges in the Hasse diagrams

There are two obvious ways to count edges in a Hasse diagram. Namely, go through all vertices and add up the numbers of edges that go upwards, or, dually, that go downwards. As the result of Orlik and Solomon for the elements of rank 1 suggests, it is easier here to count edges corresponding to vertices that cover a given vertex than to count those edges corresponding to vertices that are covered by a given vertex.

An edge in the Hasse diagram for $L(\mathcal{A})$ emanating in an upward direction from $E \in L(\mathcal{A})$ corresponds to a subspace $E' \in L(\mathcal{A})$ of codimension 1 in $E$. We shall count how many such subspaces are contained in $E$.

Schematically, we have

$$\big(\{B_1,\dots,B_k\},\zeta\big) \rightsquigarrow \big(\{B'_1,\dots,B'_{k-1}\},\zeta'\big)$$

with

$$E = \left\{(x_1,\dots,x_n)\in\mathbb{R}^n \;\middle|\; \begin{array}{l} a\in[n]-B \implies x_a = 0 \\ b,c\in B_j \text{ for some } j \implies \zeta(b)\,x_b = \zeta(c)\,x_c\end{array}\right\}$$

where $B = B_1\cup\cdots\cup B_k$, and $E'$ is obtained by imposing a further equation,

$$E' = \left\{(x_1,\dots,x_n)\in\mathbb{R}^n \;\middle|\; \begin{array}{l} a\in[n]-B' \implies x_a = 0 \\ b,c\in B'_j \text{ for some } j \implies \zeta'(b)\,x_b = \zeta'(c)\,x_c\end{array}\right\}$$

where $B' = B'_1 \cup \cdots \cup B'_{k-1}$.

Imposing a further equation may have two different types of incarnations in terms of normal forms. (As usual, $\widehat{B_k}$ means that $B_k$ is omitted.)

- **Fusing two blocks**. Choose $1 \leqslant i < j \leqslant k$, $\varepsilon \in \{\pm 1\}$.

$$\left|\begin{array}{l}\{B'_1, \ldots, B'_{k-1}\} := \{B_1, \ldots, \widehat{B_i}, \ldots, \widehat{B_j}, \ldots, B_k, B_i \cup B_j\} \\ \zeta'(a) := \begin{cases} \zeta(a) & \text{if } a \in B - B_j \\ \varepsilon \cdot \zeta(a) & \text{if } a \in B_j \end{cases}\end{array}\right.$$

- **Dropping one block**. Choose $1 \leqslant i \leqslant k$.

$$\left|\begin{array}{l}\{B'_1, \ldots, B'_{k-1}\} := \{B_1, \ldots, \widehat{B_i}, \ldots, B_k\} \\ \zeta'(a) := \zeta(a) \text{ for all } a \in B - B_i\end{array}\right.$$

**Lemma 6.** *For*

$$\big(\{B_1, \ldots, B_k\}, \zeta\big) \rightsquigarrow \big(\{B'_1, \ldots, B'_{k-1}\}, \zeta'\big)$$

*with fixed* $\big(\{B_1, \ldots, B_k\}, \zeta\big)$ *there are the following numbers of possibilities for fusing two blocks or dropping one block.*

| type | conditions | fusing | dropping |
|---|---|---|---|
| $\mathsf{A}_{n-1}$ | $B = [n]$, $\zeta = 1$<br>$B' = [n]$, $\zeta' = 1$ | $\binom{k}{2}$ | $0$ |
| $\mathsf{B}_n$ | — | $\binom{k}{2} \cdot 2$ | $k$ |
| $\mathsf{D}_n$ | $\big\|[n] - B\big\| \neq 1$<br>$\big\|[n] - B'\big\| \neq 1$ | $\binom{k}{2} \cdot 2$ | $\begin{cases} k & \text{if } B \neq [n] \\ \#\{i \mid \lvert B_i \rvert \geqslant 2\} & \text{if } B = [n] \end{cases}$ |

The total number of subspaces of dimension $k - 1$ in $L(\mathcal{A})$ lying in some fixed subspace $E \in L(\mathcal{A})$ of dimension $k$ is thus $\binom{k}{2}$ for type $\mathsf{A}$ and $k^2$ for type $\mathsf{B}$, while for type $\mathsf{D}$ this number is not specified by the dimension alone and can vary between $k^2 - k$ and $k^2$.

The following diagrams give a rough idea of how the Hasse diagrams look for the first few lattices in the $\mathsf{D}$-series. The first diagram abbreviates the relevant piece of information for the Hasse diagram of the $\mathsf{B}_3$ lattice, whose full form was given earlier. For instance the Hasse diagram for $\mathsf{D}_4$ contains

$$1 \cdot 12 + 12 \cdot 7 + 16 \cdot 3 + 18 \cdot 4 + 24 \cdot 1 + 1 \cdot 0 = 240$$

edges.

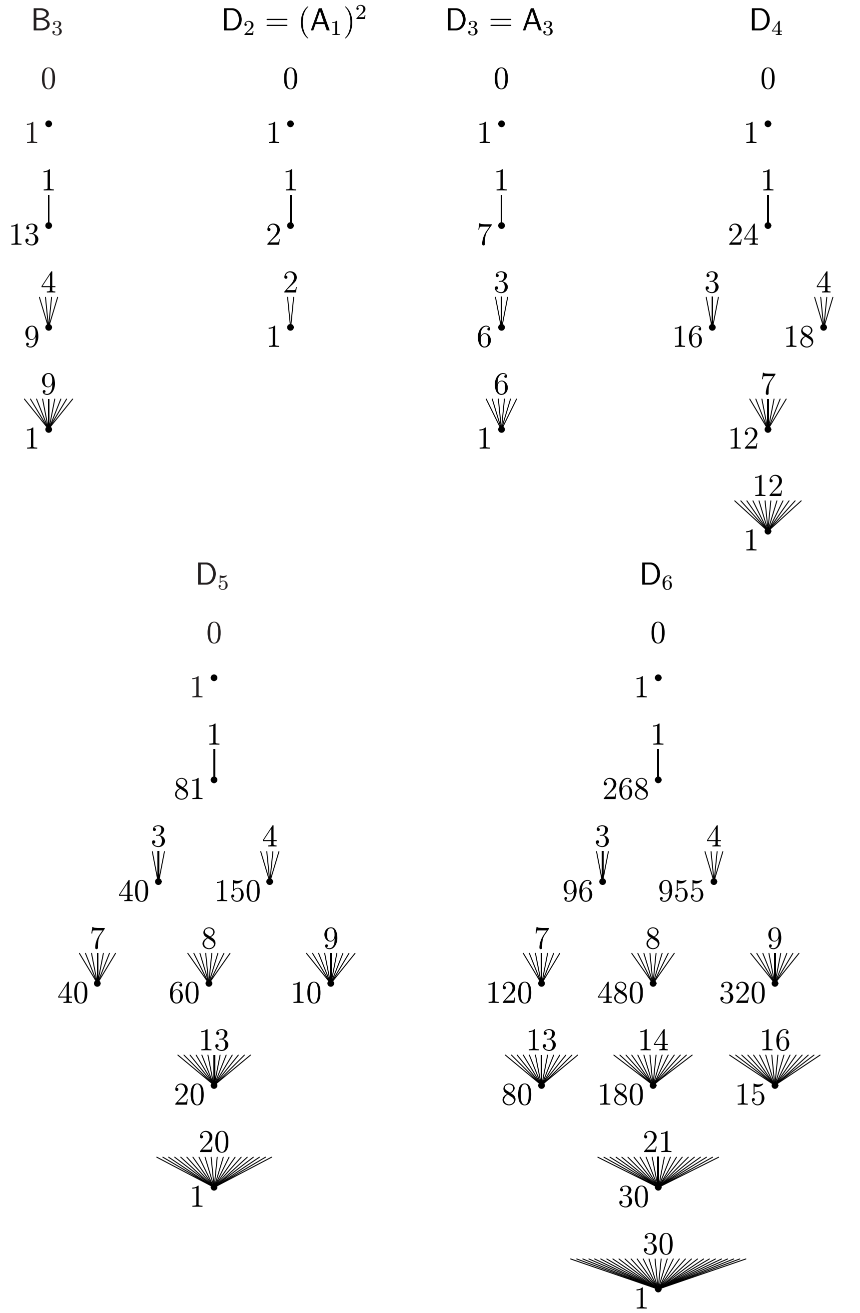


FIGURE 2. Abbreviated Hasse diagrams

**Theorem 7.** *The exponential generating functions for the numbers of edges in the Hasse diagrams for types* $(\mathsf{A}_{n-1})_{n\geqslant 0}$*,* $(\mathsf{B}_n)_{n\geqslant 0}$*, and* $(\mathsf{D}_n)_{n\geqslant 0}$

*are as given in the following table.*

| type | exponential generating function |
|---|---|
| $\mathsf{A}$ | $\frac{1}{2}(e^x-1)^2\exp(e^x-1)$ |
| $\mathsf{B}$ | $e^x\frac{1}{4}\big(e^{4x}-1\big)\exp\Big(\frac{1}{2}\big(e^{2x}-1\big)\Big)$ |
| $\mathsf{D}$ | $(e^x-1-x)\frac{1}{4}\big(e^{4x}-1\big)\exp\Big(\frac{1}{2}\big(e^{2x}-1\big)\Big)$ $+\,e^x\frac{1}{4}\,\big(e^{4x}-1-4x\big)\exp\Big(\frac{1}{2}\big(e^{2x}-1-2x\big)\Big)$ |

*Proof.* For type $\mathsf{A}_{n-1}$ there are $S(n,k)$ $k$-dimensional subspaces each containing $\binom{k}{2}$ subspaces in $L(\mathcal{A})$ of codimension 1. Thus we get the generating function

$$\sum_{n,k} S(n,k)\tbinom{k}{2}\frac{x^n}{n!} = \frac{1}{2}\frac{\partial^2}{\partial y^2}\sum_{n,k} S(n,k)\frac{x^n}{n!}\,y^k\Big|_{y=1}$$
$$= \frac{1}{2}\frac{\partial^2}{\partial y^2}\exp\big(y\cdot(e^x-1)\big)\Big|_{y=1} = \frac{1}{2}(e^x-1)^2\exp(e^x-1).$$

For type $\mathsf{B}_n$ there are $\sum\limits_{j=k}^{n} 2^{j-k}\binom{n}{j}\,S(j,k)$ $k$-dimensional subspaces each containing $k^2$ subspaces in $L(\mathcal{A})$ of codimension 1. Thus we get the generating function

$$\sum_{n,k}\sum_{j=k}^{n} 2^{j-k}\tbinom{n}{j}\,S(j,k)\,k^2\frac{x^n}{n!}$$
$$= \sum_{j,n}\frac{x^{n-j}}{(n-j)!}\frac{\partial}{\partial y}\,y\frac{\partial}{\partial y}\sum_{k} 2^{j-k}S(j,k)\frac{x^j}{j!}\,y^k\Big|_{y=1}$$
$$= \sum_{m\geqslant 0}\frac{x^m}{m!}\frac{\partial}{\partial y}\,y\frac{\partial}{\partial y}\sum_{j,k} 2^{j-k}S(j,k)\frac{x^j}{j!}\,y^k\Big|_{y=1}$$
$$= \sum_{m\geqslant 0}\frac{x^m}{m!}\frac{\partial}{\partial y}\,y\frac{\partial}{\partial y}\exp\Big(\frac{y}{2}\cdot\big(e^{2x}-1\big)\Big)\Big|_{y=1}$$
$$= \sum_{m\geqslant 0}\frac{x^m}{m!}\frac{1}{4}\big(e^{4x}-1\big)\exp\Big(\frac{1}{2}\big(e^{2x}-1\big)\Big)$$
$$= e^x\frac{1}{4}\big(e^{4x}-1\big)\exp\Big(\frac{1}{2}\big(e^{2x}-1\big)\Big).$$

The reader may wonder why we did not insert the formula for

$$\sum_{n,k}\sum_{j} 2^{j-k}\binom{n}{j}\, S(j,k)\frac{x^n}{n!}\, y^k$$

directly. The reason for going through the seemingly arcane substitution $m = n - j$ is that we can then use this calculation for type $\mathsf{D}$. Namely, for type $\mathsf{D}$ we must subtract the terms for $j = n-1$ and $j = n$, that is, for $m = 1$ and $m = 0$ in the generating function for $\mathsf{B}$ and then add the modified term corresponding to $j = n$.

Let us direct our attention to the case $B = [n]$ for $\mathsf{D}_n$. We get a partition of $[n]$ into $k$ blocks with exactly $h$ blocks of length 1 by choosing $h$ elements from $[n]$ and partitioning the remaining set of $n - h$ elements into $k - h$ blocks of lengths at least 2. Taking into account also the choice of $\zeta : [n] \to \{\pm 1\}$, we have

$$2^{n-k}\binom{n}{h}\, S_2(n-h, k-h)$$

elements of rank $n - k$ in the Hasse diagram for $\mathsf{D}_n$ which are covered by $k^2 - h$ elements. The modified term corresponding to $j = n$ is thus

$$\begin{aligned}
&\sum_{n,k}\sum_{h} 2^{n-k}\binom{n}{h}\, S_2(n-h,k-h)\,(k^2-h)\frac{x^n}{n!}\\
&\quad= \sum_{n,k}\sum_{h}\frac{x^h}{h!}\, 2^{n-k} S_2(n-h,k-h)\,(k^2-h)\frac{x^{n-h}}{(n-h)!}\, y^k\Big|_{y=1}\\
&\quad= \sum_{h}\frac{x^h}{h!}\Big(\frac{\partial}{\partial y}\, y\frac{\partial}{\partial y} - h\Big) y^h \sum_{n,k} 2^{n-k} S_2(n-h,k-h)\frac{x^{n-h}}{(n-h)!}\, y^{k-h}\Big|_{y=1}\\
&\quad= \sum_{h}\frac{x^h}{h!}\Big(\frac{\partial}{\partial y}\, y\frac{\partial}{\partial y} - h\Big) y^h \exp\Big(\frac{y}{2}\cdot\big(e^{2x}-1-2x\big)\Big)\Big|_{y=1}\\
&\quad= \sum_{h}\frac{x^h}{h!}\Big(h^2 - h + (2h+1)\frac{1}{2}\big(e^{2x}-1-2x\big) + \frac{1}{4}\big(e^{2x}-1-2x\big)^2\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \exp\Big(\frac{1}{2}\big(e^{2x}-1-2x\big)\Big)\\
&\quad= e^x\Big(x^2 + (2x+1)\frac{1}{2}\big(e^{2x}-1-2x\big) + \frac{1}{4}\big(e^{2x}-1-2x\big)^2\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \exp\Big(\frac{1}{2}\big(e^{2x}-1-2x\big)\Big)\\
&\quad= e^x\frac{1}{4}\big(e^{4x}-1-4x\big)\exp\Big(\frac{1}{2}\big(e^{2x}-1-2x\big)\Big).
\end{aligned}$$

The exponential generating function for the numbers of edges for the D-series therefore takes the form

$$(e^x-1-x)\frac{1}{4}(e^{4x}-1)\exp\Big(\frac{1}{2}(e^{2x}-1)\Big)+e^x\frac{1}{4}(e^{4x}-1-4x)\exp\Big(\frac{1}{2}(e^{2x}-1-2x)\Big).$$

□

## A curious determinant

Apparently it was A. Lenard who discovered that the Hankel determinant with the Bell numbers as entries is a superfactorial (see the reference in [We]). Let us compute its B-analogue. So let the Dowling numbers $D_n$ be given by

$$\sum_{n\geqslant 0} D_n\frac{x^n}{n!}=e^x\exp\Big(\frac{1}{2}(e^{2x}-1)\Big).$$

**Proposition 8.**

$$\begin{vmatrix} D_0 & D_1 & \dots & D_n \\ D_1 & D_2 & \dots & D_{n+1} \\ \vdots & \vdots & & \vdots \\ D_n & D_{n+1} & \dots & D_{2n} \end{vmatrix} = 2^{n(n+1)/2}\prod_{k=1}^{n} k!$$

We shall prove the following generalization which involves the numbers $G_n$ (for $l=0$) that occurred in Kerber's note [Ke, (7)] in connexion with multiply transitive groups and also in M. Bernstein's and Sloane's "eigen-sequence paper" [BeSl, Table 1(a)] in a new setting.

**Proposition 9.** *Define the sequence of generalized Bell numbers $(G_n)_{n\geqslant 0}$ depending on $l$ and $m$ by*

$$\sum_{n\geqslant 0} G_n\frac{x^n}{n!}=e^{lx}\exp\Big(\frac{1}{m}(e^{mx}-1)\Big). \tag{4}$$

*Then*

$$\begin{vmatrix} G_0 & G_1 & \dots & G_n \\ G_1 & G_2 & \dots & G_{n+1} \\ \vdots & \vdots & & \vdots \\ G_n & G_{n+1} & \dots & G_{2n} \end{vmatrix} = m^{n(n+1)/2}\prod_{k=1}^{n} k! \tag{5}$$

*Proof.* The statement in [Ko, p. 113/114] can be rephrased by saying that a Hankel determinant does not change its value when the matrix entries are subject to a binomial transform. Hence the determinant in (5) is independent of $l \in \mathbb{Z}$ and consequently also independent of $l$ when $l$ is considered as an indeterminate. Therefore we will assume that $l = 0$ in the definition (4) of the numbers $G_n$.

As an aside let us mention that the invariance under binomial transform gives the following identity between Hankel determinants with Bell numbers as entries.

$$\begin{vmatrix} B_0 & B_1 & \dots & B_n \\ B_1 & B_2 & \dots & B_{n+1} \\ \vdots & \vdots & & \vdots \\ B_n & B_{n+1} & \dots & B_{2n} \end{vmatrix} = \begin{vmatrix} B_1 & B_2 & \dots & B_{n+1} \\ B_2 & B_3 & \dots & B_{n+2} \\ \vdots & \vdots & & \vdots \\ B_{n+1} & B_{n+2} & \dots & B_{2n+1} \end{vmatrix}$$

To compute the determinant (5) we proceed by induction. Let us first define $H_{n,k} \in \mathbb{Q}[m]$ by

$$\sum_{n \geqslant 0} H_{n,k} \frac{y^n}{n!} = \frac{1}{k!} e^{-y} \frac{1}{m^k} \left(\log(1+my)\right)^k \qquad (k = 0, 1, 2, \dots). \tag{6}$$

Note that $H_{n,n} = 1$. Hence with

$$I_{h,n} = \sum_{k=0}^{n} G_{h+k} H_{n,k} \tag{7}$$

we have

$$\begin{vmatrix} G_0 & G_1 & \dots & G_n \\ G_1 & G_2 & \dots & G_{n+1} \\ \vdots & \vdots & & \vdots \\ G_n & G_{n+1} & \dots & G_{2n} \end{vmatrix} = \begin{vmatrix} G_0 & \dots & G_{n-1} & I_{0,n} \\ \vdots & & \vdots & \vdots \\ G_{n-1} & \dots & G_{2n-2} & I_{n-1,n} \\ G_n & \dots & G_{2n-1} & I_{n,n} \end{vmatrix}. \tag{8}$$

From

$$\sum_{h,n} I_{h,n} \frac{x^h}{h!} \frac{y^n}{n!} = \exp\Big(\frac{1}{m}\left(e^{mx} - 1\right)\Big) \exp\Big(y \cdot \left(e^{mx} - 1\right)\Big) \tag{9}$$

we see that $I_{0,n} = \cdots = I_{n-1,n} = 0$ and $I_{n,n} = m^n \cdot n!$. Hence (5) follows from (8) by induction.

We must finally prove (9). So let us compute:

$$\sum_{h,n} I_{h,n} \frac{x^h}{h!} \frac{y^n}{n!} \overset{(7)}{=} \sum_{h,k,n} G_{h+k} H_{n,k} \frac{x^h}{h!} \frac{y^n}{n!}$$

$$
\begin{aligned}
&\overset{(6)}{=} \sum_{h,k} G_{h+k} \frac{x^h}{h!} \frac{1}{k!} e^{-y} \frac{1}{m^k} \big(\log(1+my)\big)^k \\
&= e^{-y} \sum_{h,k} G_{h+k} \frac{1}{(h+k)!} \binom{h+k}{h} x^h \frac{1}{m^k} \big(\log(1+my)\big)^k \\
&= e^{-y} \sum_{n} G_n \frac{1}{n!} \Big(x + \frac{1}{m}\log(1+my)\Big)^n \\
&\overset{(4)}{=} e^{-y} \exp\bigg(\frac{1}{m}\Big(e^{m\left(x+\frac{1}{m}\log(1+my)\right)} - 1\Big)\bigg) \\
&= \exp\Big(\frac{1}{m}\big(e^{mx}-1\big)\Big) \exp\Big(y \cdot \big(e^{mx}-1\big)\Big).
\end{aligned}
$$

We have thus verified equation (9). □

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(Concerned with sequences A003128, A039755, A039756, A039757, A039758, A039759, A039760, A039761, A039762, A039763, A039764, A039765.)

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