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Generating Functions via Hankel and Stieltjes Matrices

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Abstract

When the Hankel matrix formed from the sequence $1, a_1, a_2, \ldots$ *has an* LDL^T *decomposition, we provide a constructive proof that the Stieltjes matrix* S_L *associated with* L *is tridiagonal. In the important case when* L *is a Riordan matrix using ordinary or exponential generating functions, we determine the specific form that* S_L *must have, and we demonstrate, constructively, a one-to-one correspondence between the generating function for the sequence and* S_L*. If* L *is Riordan when using ordinary generating functions, we show how to derive a recurrence relation for the sequence.*

Keywords. Hankel matrix, Stieltjes matrix, ordinary generating function, exponential generating function, Riordan matrix, LDU decomposition, tridiagonal matrix.

1. Introduction

For each sequence in a large class of important combinatorial sequences, we can derive a closed form expression for an ordinary or exponential generating function starting with the associated Hankel matrix or Stieltjes matrix. In this paper we give explicit relationships between the generating function, the Hankel matrix and the Stieltjes matrix. We also provide several illustrative examples. In [3], some work was done using the Hankel matrix approach, but the conditions under which the method would work were not determined, or were only implicitly conjectured. In the present paper we use the Stieltjes matrix to obtain significant improvements in the analysis and application of the method.

Our basic assumption is that the Hankel matrix generated by the sequence has an LDU factorization, where L is a lower triangular matrix with all diagonal elements equal to one, $U = L^T$, and D is a diagonal matrix with all diagonal elements nonzero. The Hankel matrix generated by the sequence $a_0, a_1, a_2, ...$, is given by the infinite matrix

$$H = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & . \\ a_1 & a_2 & a_3 & a_4 & a_5 & . \\ a_2 & a_3 & a_4 & a_5 & a_6 & . \\ a_3 & a_4 & a_5 & a_6 & a_7 & . \\ a_4 & a_5 & a_6 & a_7 & a_8 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Without loss of generality we take $a_0 = 1$. A necessary and sufficient condition for H to have an LDU factorization is that H be positive definite. When L is a Riordan matrix (see Section 2) using ordinary or exponential generating functions, our method will find a closed form expression for the generating function of the sequence $1, a_1, a_2, a_3, ...$ In the the ordinary generating function case we can then use [4] to find a recurrence relation for the sequence.

Example 1. Delannoy numbers: $1, 3, 13, 63, 321, 1683, ...$
This is sequence A1850 in [5]. See also [1, p. 81]. We apply Gaussian elimination to the Hankel matrix to obtain

$$H = \begin{bmatrix} 1 & 3 & 13 & 63 & 321 & . \\ 3 & 13 & 63 & 321 & 1683 & . \\ 13 & 63 & 321 & 1683 & 8989 & . \\ 63 & 321 & 1683 & 8989 & 48639 & . \\ 321 & 1683 & 8989 & 48639 & 265729 & . \\ . & . & . & . & . & . \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & & & . \\ 3 & 1 & & & & . \\ 13 & 6 & 1 & & & . \\ 63 & 33 & 9 & 1 & & . \\ 321 & 180 & 62 & 12 & 1 & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 & & & & & & . \\ & 4 & & & & & . \\ & & 8 & & & & . \\ & & & 16 & & & . \\ & & & & 32 & & . \\ . & . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 1 & 3 & 13 & 63 & 321 & . \\ & 1 & 6 & 33 & 180 & . \\ & & 1 & 9 & 62 & . \\ & & & 1 & 12 & . \\ & & & & 1 & . \\ . & . & . & . & . & . \end{bmatrix} .$$

The Stieltjes matrix S_L associated with L is the matrix $S_L = L^{-1}\overline{L}$, where $\overline{L}$ is obtained from L by deleting the first row. (See Section 2 for more details about the Stieltjes matrix.) In Example 1,

$$S_L = \begin{bmatrix} 3 & 1 & & & & . \\ 4 & 3 & 1 & & & . \\ & 2 & 3 & 1 & & . \\ & & 2 & 3 & 1 & . \\ & & & 2 & 3 & . \\ . & . & . & . & . & . \end{bmatrix} .$$

From its definition S_L gives the rule of formation of L. Specifically, it gives a rule for obtaining the n^{th} row of L from the previous row. In the example, we have for $n \geq 1$

$$l_{n0} = 3l_{n-1,0} + 4l_{n-1,1}$$

$$l_{nk} = l_{n-1,k-1} + 3l_{n-1,k} + 2l_{n-1,k+1} \quad , \quad k \geq 1 \, .$$

It is convenient to define the leftmost column of L to be the zeroth column, and the first row to be the zeroth row. We say that the zeroth column of L has a $\{3, 4\}$ rule of formation and that the k^{th} column, $k \geq 1$, has a $\{1, 3, 2\}$ rule of formation. Notice that the zeroth column of L contains the Delannoy numbers and that S_L is tridiagonal. In Section 2 we prove that whenever $H = LDU$, then S_L is tridiagonal. From Theorem 2 in Section 2 we see that the Delannoy numbers have a closed-form ordinary generating function given by

$$g(x) = \frac{1}{1 - 3x - 4xf} = \frac{1}{\sqrt{1 - 6x + x^2}} \, ,$$

where

$$f(x) = x(1 + 3f + 2f^2) = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x} \ .$$

Since S_L is tridiagonal and L is a Riordan matrix, we can use [4] to obtain for the Delannoy numbers the recurrence

$$\begin{aligned} na_n &= 3(4n-3)a_{n-1} - 19(2n-3)a_{n-2} + 3(4n-9)a_{n-3} - (n-3)a_{n-4}; \\ \text{for } n &\geq 4, \text{ with } a_0 = 1, a_1 = 3, a_2 = 13, a_3 = 63\,. \end{aligned}$$

Example 2. Bell numbers: $1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, ...$
This sequence illustrates the exponential generating function case. It is sequence A110 in [5]. Here

$$L = \begin{bmatrix} 1 & & & & & . \\ 1 & 1 & & & & . \\ 2 & 3 & 1 & & & . \\ 5 & 10 & 6 & 1 & & . \\ 15 & 37 & 31 & 10 & 1 & . \\ . & . & . & . & . & . \end{bmatrix} \text{ and } S_L = \begin{bmatrix} 1 & 1 & & & & . \\ 1 & 2 & 1 & & & . \\ & 2 & 3 & 1 & & . \\ & & 3 & 4 & 1 & . \\ & & & 4 & 5 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

From Theorem 3 in Section 2, the form of S_L indicates that the exponential generating function $g(x)$ of the Bell numbers is given by

$$\ln(g) = \int (1 + f)dx, \quad g(0) = 1,$$

where

$$f'(x) = 1 + f(x), \quad f(0) = 0.$$

So we obtain

$$f(x) = e^x - 1 \quad \text{and} \quad g(x) = e^{e^x - 1}.$$

We have found that the method works for many other important combinatorial sequences. These include

- the Catalan numbers: $1, 1, 2, 5, 14, 42, 132, 429, \ldots$ (sequence A108)
- the shortened Catalan sequence: 1, 2, 5, 14, 42, 132, 429, ...
- the Catalan numbers interspersed with zeros: $1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, \ldots$
- central binomial coefficients: $1, 2, 6, 20, 70, 252, , 924, 3432, \ldots$ (A984)

- central trinomial coefficients: $1, 1, 3, 7, 19, 51, 141, \ldots$ (A2426),
- Schröder's numbers: $1, 2, 6, 22, 90, 394, 1806, \ldots$ (A6318)
- Schröder's second problem: $1, 1, 3, 11, 45, 197, 903, 4279, \ldots$ (A1003)
- gamma numbers or Motzkin sums: $1, 0, 1, 1, 3, 6, 15, 36, 91, 232, \ldots$ (A5043)
- Fine numbers: $1, 0, 1, 2, 6, 18, 57, 186, 622, \ldots$ (A957)
- directed animals: $1, 2, 5, 13, 35, 96, 267, 750, 2123, \ldots$ (A5773)
- telephone numbers, or self-inverse permutations: $1, 1, 2, 4, 10, 26, 76, 232, 764, \ldots$ (A85)
- derangement numbers: $1, 0, 1, 2, 9, 44, 265, 1854, 14833, \ldots$ (A166).

In Section 2 we show that whenever $H = LDU$ then S_L is always tridiagonal, and we give the specific form of S_L. Theorem 2 in that section indicates the specific form that S_L must have for L to be a Riordan matrix with ordinary generating functions. Theorem 3 indicates the specific form that S_L must have for L to be Riordan with exponential generating functions. In Section 3 we give some further examples.

2. Definitions and Theorems

Definition. The **Hankel matrix** $H = (h_{nk})_{n,k\geq 0}$ generated by the sequence $1, a_1, a_2, a_3, ...$ is given by

$$h_{00} = 1, \quad h_{nk} = a_{n+k} \quad \text{for} \quad n \geq 0, \quad k \geq 0.$$

Definition. Let $L = (l_{nk})_{n,k\geq 0}$ be a lower triangular matrix with $l_{ii} = 1$ for all $i \geq 0$. The **Stieltjes matrix** S_L associated with L is given by $S_L = L^{-1}\overline{L}$, where $\overline{L}$ is obtained from L by deleting the first row of L. That is, the element in the n^{th} row and k^{th} column of $\overline{L}$ is given by

$$\overline{l}_{nk} = l_{n+1,k}\,.$$

Remark. We note that S_L is unique, and so

$$S_L = S_{\widetilde{L}} \Leftrightarrow L = \widetilde{L}\,.$$

Remark. If $S_L = (s_{ik})_{i,k\geq 0}$ then

$$l_{nk} = \sum_{i\geq o} s_{ik} l_{n-1,i} \quad \text{for} \quad n \geq 1\,.$$

That is, from S_L , we obtain a rule for computing the n^{th} row of L from the $(n-1)^{th}$ row.

Remark. S_L is tridiagonal if and only if there exist sequences $\{\lambda_k\}_{k\geq 0}$, and $\{\mu_k\}_{k\geq 0}$ such that

$$l_{n0} = \lambda_0 l_{n-1,0} + \mu_0 l_{n-1,1} \quad \text{for} \quad n \geq 1,$$

$$l_{nk} = l_{n-1,k-1} + \lambda_k l_{n-1,k} + \mu_k l_{n-1,k+1} \quad \text{for} \quad k \geq 1 \quad \text{and} \quad n \geq 1,$$

and

$$s_{00} = \lambda_0, \quad s_{10} = \mu_0\ , \quad \text{and for} \quad k \geq 1\ ,\ s_{kk} = \lambda_k\ ,\ s_{k+1,k} = \mu_k\,.$$

Definition. A **Riordan matrix with ordinary generating functions** is a lower triangular matrix for which the generating function for the k^{th} column, $k \geq 0$, is given by $g(x)[f(x)]^k$, where

$$g(x) = 1 + g_1 x + g_2 x^2 + \cdots \quad \text{and} \quad f(x) = x + f_2 x^2 + f_3 x^3 + \cdots$$

Definition. A **Riordan matrix with exponential generating functions** is a lower triangular matrix for which the generating function for the k^{th} column, $k \geq 0$, is given by $\frac{1}{k!} g(x)[f(x)]^k$, where

$$g(x) = 1 + g_1 x + g_2 \frac{x^2}{2!} + g_3 \frac{x^3}{3!} + \cdots \quad \text{and} \quad f(x) = x + f_2 \frac{x^2}{2!} + f_3 \frac{x^3}{3!} + \cdots\ .$$

See [2] for a detailed description of Riordan matrices. In [6] Woodson explores other kinds of Riordan matrices.

Theorem 1. *Let $H = (h_{nk})_{n,k\geq 0}$ be the Hankel matrix generated by the sequence $1, a_1, a_2, a_3, ...$ Assume that $H = LDU$ where*

$$L = (l_{nk})_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & . \\ l_{10} & 1 & & & & . \\ l_{20} & l_{21} & 1 & & & . \\ l_{30} & l_{31} & l_{32} & 1 & & . \\ l_{40} & l_{41} & l_{42} & l_{43} & 1 & . \\ . & . & . & . & . & . \end{bmatrix},$$

$$D = \begin{bmatrix} d_0 & & & & & . \\ & d_1 & & & & . \\ & & d_2 & & & . \\ & & & d_3 & & . \\ & & & & d_4 & . \\ . & . & . & . & . & . \end{bmatrix} \qquad d_i \neq 0 \quad for\ all\ i, \qquad U = L^T .$$

That is,

$$h_{nk} = \sum_{i=0}^{k} d_i l_{ki} l_{ni} .$$

Then the Stieltjes matrix S_L is tridiagonal with the form

$$S_L = \begin{bmatrix} \lambda_0 & 1 & & & & . \\ \mu_0 & \lambda_1 & 1 & & & . \\ & \mu_1 & \lambda_2 & 1 & & . \\ & & \mu_2 & \lambda_3 & 1 & . \\ & & & \mu_3 & \lambda_4 & . \\ . & . & . & . & . & . \end{bmatrix},$$

where

$$\lambda_0 = a_1 , \quad \mu_0 = d_1 , \quad \lambda_k = l_{k+1,k} - l_{k,k-1} , \quad \mu_k = \frac{d_{k+1}}{d_k} , \quad k \geq 1 .$$

Proof. We will prove that

$$l_{n0} = a_1 l_{n-1,0} + d_1 l_{n-1,1}$$

and

$$l_{nk} = l_{n-1,k-1} + \lambda_k l_{n-1,k} + \mu_k l_{n-1,k+1} \quad for\ all\ k \geq 1.$$

We use induction on k. From the definition of the Hankel matrix,

$$h_{nk} = h_{n-1,k+1} \quad \text{for all} \quad k \geq 0 \quad \text{and} \quad n \geq 1$$

$$h_{n0} = h_{n-1,1} \Leftrightarrow d_0 l_{n0} = d_0 l_{n-1,0} l_{10} + d_1 l_{n-1,1} l_{11} \Leftrightarrow l_{n0} = a_1 l_{n-1,0} + d_1 l_{n-1,1} \, .$$

$$h_{n1} = h_{n-1,2} \Leftrightarrow d_0 l_{10} l_{n0} + d_1 l_{11} l_{n1} = d_0 l_{20} l_{n-1,0} + d_1 l_{21} l_{n-1,1} + d_2 l_{22} l_{n-1,2}$$

$$\begin{aligned}
&\Leftrightarrow \quad d_1 l_{n1} = l_{20} l_{n-1,0} - l_{10} l_{n0} + d_1 l_{21} l_{n-1,1} + d_2 l_{n-1,2} \\
&\Leftrightarrow \quad d_1 l_{n1} = d_1 l_{n-1,0} + d_1 (l_{21} - l_{10}) l_{n-1,1} + d_2 l_{n-1,2} \\
&\Leftrightarrow \quad l_{n1} = l_{n-1,0} + \lambda_1 l_{n-1,1} + \mu_1 l_{n-1,2}
\end{aligned}$$

Now assume that

$$l_{ni} = l_{n-1,i-1} + \lambda_i l_{n-1,i} + \mu_i l_{n-1,i+1} \quad for \quad 1 \leq i \leq k-1 \, .$$

Then

$$\begin{aligned}
h_{nk} \quad = \quad & h_{n-1,k+1} \Leftrightarrow \sum_{i=0}^{k} d_i l_{ki} l_{ni} = \sum_{i=0}^{k+1} d_i l_{k+1,i} l_{n-1,i} \\
\Leftrightarrow \quad & \sum_{i=0}^{k-1} d_i l_{ki} l_{ni} - \sum_{i=0}^{k-1} d_i l_{k+1,i} l_{n-1,i} + d_k l_{nk} = d_k l_{k+1,k} l_{n-1,k} + d_{k+1} l_{n-1,k+1} \\
\Leftrightarrow \quad & d_0 l_{k0} l_{n0} + \sum_{i=1}^{k-1} d_i l_{ki} l_{ni} - \left[d_0 l_{k+1,0} l_{n-1,0} + \sum_{i=1}^{k-1} d_i l_{k+1,i} l_{n-1,i} \right] + d_k l_{nk} \\
= \quad & d_k l_{k+1,k} l_{n-1,k} + d_{k+1} l_{n-1,k+1} \\
\Leftrightarrow \quad & d_0 l_{k0} \left[a_1 l_{n-1,0} + d_1 l_{n-1,1} \right] + \sum_{i=1}^{k-1} d_i l_{ki} l_{ni} \\
& - \left[d_0 (a_1 l_{k0} + d_1 l_{k1}) l_{n-1,0} + \sum_{i=1}^{k-1} d_i l_{k+1,i} l_{n-1,i} \right] + d_k l_{nk} \\
= \quad & d_k l_{k+1,k} l_{n-1,k} + d_{k+1} l_{n-1,k+1} \\
\Leftrightarrow \quad & d_1 l_{k0} l_{n-1,1} + \sum_{i=1}^{k-1} d_i l_{ki} l_{ni} - \left[d_1 l_{k1} l_{n-1,0} + \sum_{i=1}^{k-1} d_i l_{k+1,i} l_{n-1,i} \right] + d_k l_{nk} \\
= \quad & d_k l_{k+1,k} l_{n-1,k} + d_{k+1} l_{n-1,k+1} \\
\Leftrightarrow \quad & d_1 l_{k0} l_{n-1,1} + \sum_{i=1}^{k-1} d_i l_{ki} \left[l_{n-1,i-1} + \lambda_i l_{n-1,i} + \frac{d_{i+1}}{d_i} l_{n-1,i+1} \right]
\end{aligned}$$

$$
\begin{aligned}
& - \left[d_1 l_{k1} l_{n-1,0} + \sum_{i=1}^{k-1} d_i l_{n-1,i} \left[l_{k,i-1} + \lambda_i l_{ki} + \frac{d_{i+1}}{d_i} l_{k,i+1} \right] \right] + d_k l_{nk} \\
= \quad & d_k l_{k+1,k} l_{n-1,k} + d_{k+1} l_{n-1,k+1} \\
\Leftrightarrow \quad & d_1 l_{k0} l_{n-1,1} + \sum_{i=1}^{k-1} d_i l_{ki} l_{n-1,i-1} + \sum_{i=1}^{k-1} d_{i+1} l_{ki} l_{n-1,i+1} \\
& - \left[d_1 l_{k1} l_{n-1,0} + \sum_{i=1}^{k-1} d_i l_{k,i-1} l_{n-1,i} + \sum_{i=1}^{k-1} d_{i+1} l_{k,i+1} l_{n-1,i} \right] + d_k l_{nk} \\
= \quad & d_k l_{k+1,k} l_{n-1,k} + d_{k+1} l_{n-1,k+1} \\
\Leftrightarrow \quad & d_1 \left[l_{k0} l_{n-1,1} + l_{k1} l_{n-1,0} - l_{k1} l_{n-1,0} - l_{k0} l_{n-1,1} \right] \\
& + d_2 \left[l_{k2} l_{n-1,1} + l_{k1} l_{n-1,2} - l_{k1} l_{n-1,2} - l_{k2} l_{n-1,1} \right] \\
& + d_3 \left[l_{k3} l_{n-1,2} + l_{k2} l_{n-1,3} - l_{k2} l_{n-1,3} - l_{k3} l_{n-1,2} \right] \\
& ..... \\
& ..... \\
& + d_{k-1} \left[l_{k,k-1} l_{n-1,k-2} + l_{k,k-2} l_{n-1,k-1} - l_{k,k-2} l_{n-1,k-1} - l_{k,k-1} l_{n-1,k-2} \right] \\
& + d_k \left[l_{k,k-1} l_{n-1,k} - l_{kk} l_{n-1,k-1} \right] + d_k l_{nk} \\
= \quad & d_k l_{k+1,k} l_{n-1,k} + d_{k+1} l_{n-1,k+1}
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow \quad & d_k l_{nk} = d_k l_{n-1,k-1} + d_k \left[l_{k+1,k} - l_{k,k-1} \right] l_{n-1,k} + d_{k+1} l_{n-1,k+1} \\
\Leftrightarrow \quad & l_{nk} = l_{n-1,k-1} + \lambda_k l_{n-1,k} + \mu_k l_{n-1,k+1}
\end{aligned}
$$

When S_L has $\lambda_i = \lambda$ and $\mu_i = \mu$ for all $i \geq 1$ we can obtain an ordinary generating function for the sequence $1, a_1, a_2, ...$

Theorem 2. *Let H be the Hankel matrix generated by the sequence $1, a_1, a_2, ...$, and let $H = LDL^T$. Then S_L has the form*

$$
S_L = \begin{bmatrix}
a_1 & 1 & & & & . \\
d_1 & \lambda & 1 & & & . \\
 & \mu & \lambda & 1 & & . \\
 & & \mu & \lambda & 1 & . \\
 & & & \mu & \lambda & . \\
. & . & . & . & . & .
\end{bmatrix},
$$

if and only if the ordinary generating function $g(x)$ *of the sequence* $1, a_1, a_2, ...$ *is given by*

$$g(x) = \frac{1}{1 - a_1 x - d_1 x f}\,,$$

where

$$f = x(1 + \lambda f + \mu f^2)\,, \quad f(0) = 0.$$

Proof. We note that $\mu \neq 0$ and

$$f = \frac{1 - \lambda x - \sqrt{(1-\lambda x)^2 - 4\mu x}}{2\mu x}\,.$$

Consider the lower triangular matrix $\widetilde{L}$ such that the generating function for the k^{th} column is $g(x)[f(x)]^k, \quad k \geq 0$.

$$\begin{aligned} g(x) &= \frac{1}{1 - a_1 x - d_1 x f} \Leftrightarrow g(x) = 1 + a_1 x g(x) + d_1 x g f \\ &\Leftrightarrow \widetilde{l}_{00} = 1 \quad \text{and} \quad [x^n]\, g = a_1\, [x^n]\, xg + d_1\, [x^n]\, xgf \\ &\Leftrightarrow \widetilde{l}_{00} = 1 \quad \text{and} \quad \widetilde{l}_{n0} = a_1 \widetilde{l}_{n-1,0} + d_1 \widetilde{l}_{n-1,1} \quad for \quad n \geq 1. \end{aligned}$$

Also, for $k \geq 1$,

$$\begin{aligned} f &= x(1 + \lambda f + \mu f^2) \Leftrightarrow g f^k = x g f^{k-1} + \lambda x g f^k + \mu x g f^{k+1} \\ &\Leftrightarrow [x^n]\, g f^k = [x^n]\, x g f^{k-1} + \lambda\, [x^n]\, x g f^k + \mu\, [x^n]\, x g f^{k+1} \\ &\Leftrightarrow \widetilde{l}_{nk} = \widetilde{l}_{n-1,k-1} + \lambda \widetilde{l}_{n-1,k} + \mu \widetilde{l}_{n-1,k+1}\,. \end{aligned}$$

Therefore S_L has the given form if and only if $S_L = S_{\widetilde{L}} \Leftrightarrow L = \widetilde{L}$.

We now turn to the exponential generating function case. We get an exponential generating function for the sequence $1, a_1, a_2, ...$ when the sequences $\{\lambda_i\}_{i\geq 0}$ and $\{\frac{\mu_i}{i+1}\}_{i\geq 0}$ are arithmetic sequences.

Theorem 3. *Let* H *be the Hankel matrix generated by the sequence* $1, a_1, a_2, ...$, *and let* $H = LDL^T$. *Then* S_L *has the form given in Theorem 1. If* $\{\lambda_i\}_{i\geq 0}$, *is an arithmetic sequence with common difference* λ *and* $\{\frac{\mu_i}{i+1}\}_{i\geq 0}$ *an arithmetic sequence with common difference* μ , *then the exponential generating function* $g(x)$ *for the sequence* $1, a_1, a_2, ...$ *is given by*

$$\ln(g) = \int (a_1 + d_1 f) dx\,, \qquad g(0) = 1,$$

where f *is given by*

$$f' = 1 + \lambda f + \mu f^2\,, \qquad f(0) = 0\,.$$

Proof. Consider the lower triangular matrix $\widehat{L}$ with $\frac{1}{k!}g(x)[f(x)]^k$ for the exponential generating function of the k^{th} column, $k \geq 0$. We note that $\widehat{L}$ is a Riordan matrix with exponential generating functions.

$$\begin{aligned}\ln(g) &= \int (a_1 + d_1 f)dx \Rightarrow g' = a_1 g + d_1 fg \Rightarrow \left[\frac{x^n}{n!}\right] g' = a_1 \left[\frac{x^n}{n!}\right] g + d_1 \left[\frac{x^n}{n!}\right] fg \\ &\Rightarrow \widehat{l}_{n+1,0} = a_1 \widehat{l}_{n0} + d_1 \widehat{l}_{n1} \Rightarrow \widehat{l}_{n0} = \lambda_0 \widehat{l}_{n-1,0} + \mu_0 \widehat{l}_{n-1,1}\end{aligned}$$

For $k \geq 1$,

$$\begin{aligned}\left(\frac{gf^k}{k!}\right)' &= \frac{g'f^k}{k!} + \frac{gf^{k-1}f'}{(k-1)!} = \frac{a_1 gf^k}{k!} + \frac{d_1 gf^{k+1}}{k!} + \frac{gf^{k-1} + \lambda gf^k + \mu gf^{k+1}}{(k-1)!} \\ &= (a_1 + \lambda k)\frac{gf^k}{k!} + (d_1 + \mu k)\frac{gf^{k+1}}{k!} + \frac{gf^{k-1}}{(k-1)!} \\ &= \frac{gf^{k-1}}{(k-1)!} + \lambda_k \frac{gf^k}{k!} + \frac{\mu_k}{k+1}\frac{gf^{k+1}}{k!}\,.\end{aligned}$$

Therefore

$$\left[\frac{x^n}{n!}\right]\left(\frac{gf^k}{k!}\right)' = \left[\frac{x^n}{n!}\right]\left(\frac{gf^{k-1}}{(k-1)!} + \lambda_k \frac{gf^k}{k!} + \mu_k \frac{gf^{k+1}}{(k+1)!}\right).$$

That is,

$$\begin{aligned}\widehat{l}_{n+1,k} &= \widehat{l}_{n,k-1} + \lambda_k \widehat{l}_{nk} + \mu_k \widehat{l}_{n,k+1}\,, \\ \widehat{l}_{nk} &= \widehat{l}_{n-1,k-1} + \lambda_k \widehat{l}_{n-1,k} + \mu_k \widehat{l}_{n-1,k+1}\,.\end{aligned}$$

Therefore S_L has the given form if and only if $L = \widehat{L}$.

3. Further Examples

Example 3. Derangements: $1, 0, 1, 2, 9, 44, 265, 1854, 14833, ...$
This is sequence A166 in [5]. $H = LDL^T$ and

$$S_L = \begin{bmatrix} 0 & 1 & & & & . \\ 1 & 2 & 1 & & & . \\ & 4 & 4 & 1 & & . \\ & & 9 & 6 & 1 & . \\ & & & 16 & 8 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

This is the exponential case with $\lambda_k = 2k$ and $\mu_k = (k+1)^2$. Therefore $\lambda = 2$ and $\mu = 1$. So $f' = 1 + 2f + f^2$ with $f(0) = 0$. That gives $f = \frac{x}{1-x}$ and $\ln(g) = \int f dx$, $g(0) = 1$. So

$$g(x) = \frac{e^{-x}}{1-x}\,.$$

Example 4. Here we start with a Stieltjes matrix having the form in Theorem 3. The associated sequence is 1,3,10,39,187,1128,8455, ... (sequence A54912 in [5]).

$$S_L = \begin{bmatrix} 3 & 1 & & & & . \\ 1 & 6 & 1 & & & . \\ & 6 & 9 & 1 & & . \\ & & 15 & 12 & 1 & . \\ & & & 28 & 15 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Here S_L has the form in Theorem 3 with $\lambda = 3$ and $\mu = 2$. Therefore the exponential generating function for the sequence in the leftmost column of L is given by $\ln(g) = \int (3+f)dx, \quad g(0) = 1$ where $f' = 1+3f+2f^2, \quad f(0) = 0$. We get $f = \frac{e^x-1}{2-e^x}$ and

$$g(x) = \sqrt{\frac{e^{5x}}{2-e^x}} = 1+3x+10\frac{x^2}{2!}+39\frac{x^3}{3!}+187\frac{x^4}{4!}+1128\frac{x^5}{5!}+8455\frac{x^6}{6!}+O(x^7)$$

We can also use Theorem 1 to construct L and D. Recall that $d_{i+1} = \mu_i d_i$, and that $d_0 = 1$.

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(Concerned with sequences A108, A166, A957, A984, A1003, A1850, A2426, A5773, A6318, A54912.)

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