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Giuga Numbers and the Arithmetic Derivative

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Abstract

We characterize Giuga numbers as solutions to the equation $n' = an + 1$, with $a \in \mathbb{N}$ and n' being the arithmetic derivative. Although this fact does not refute Lava's conjecture, it does suggest doubts about its veracity.

1 Introduction

1.1 The Arithmetic Derivative

The arithmetic derivative was introduced by Barbeau [3] (see A003415 in Sloane's *On-Line Encyclopedia of Integer Sequences*). The derivative of an integer is defined to be the unique map sending every prime to 1 and satisfying the Leibniz rule, i.e., $p' = 1$, for any prime p, and $(nm)' = nm' + mn'$ for any $n, m \in \mathbb{N}$. This map makes sense and is well-defined [7].

Proposition 1. *If $n = \prod_{i=1}^{k} p_i^{r_i}$ is the factorization of n in prime powers, then the only way to define n' satisfying the desired properties is*

$$n' = n \sum_{i=1}^{k} \frac{r_i}{p_i}.$$

1.2 Giuga numbers

In [4], Giuga numbers were introduced in the following way motivated by previous work by Giuga [6].

Definition 2. A Giuga number is a composite number n such that p divides $\frac{n}{p} - 1$ for every prime divisor p of n.

It follows easily from the definition that every Giuga number is square-free.. There are several characterizations of Giuga numbers. We present the most important in the following proposition (note that we will consider $\mathbb{N} = \{1, 2, \dots\}$).

Proposition 3. *Let n be a composite integer. Then, the following are equivalent:*

i) n is a Giuga number.

ii) Giuga [6]: $\displaystyle\sum_{p|n} \frac{1}{p} - \prod_{p|n} \frac{1}{p} \in \mathbb{N}$.

iii) Borwein et al. [4]: $\displaystyle\sum_{j=1}^{n-1} j^{\phi(n)} \equiv -1 \pmod{n}$, where ϕ is Euler's totient function.

iv) Agoh [1]: $nB_{\phi(n)} \equiv -1 \pmod{n}$, where B is a Bernoulli number.

To date, only thirteen Giuga numbers are known (see A007850 in Sloane's *On-Line Encyclopedia of Integer Sequences*):

- With 3 factors:
$$\mathbf{30} = 2 \cdot 3 \cdot 5.$$

- With 4 factors:
$$\mathbf{858} = 2 \cdot 3 \cdot 11 \cdot 13,$$
$$\mathbf{1722} = 2 \cdot 3 \cdot 7 \cdot 41.$$

- With 5 factors:
$$\mathbf{66198} = 2 \cdot 3 \cdot 11 \cdot 17 \cdot 59.$$

- With 6 factors:
$$\mathbf{2214408306} = 2 \cdot 3 \cdot 11 \cdot 23 \cdot 31 \cdot 47057,$$
$$\mathbf{24423128562} = 2 \cdot 3 \cdot 7 \cdot 43 \cdot 3041 \cdot 4447.$$

- With 7 factors:

$$\begin{aligned}
\mathbf{432749205173838} &= 2 \cdot 3 \cdot 7 \cdot 59 \cdot 163 \cdot 1381 \cdot 775807,\\
\mathbf{14737133470010574} &= 2 \cdot 3 \cdot 7 \cdot 71 \cdot 103 \cdot 67213 \cdot 713863,\\
\mathbf{550843391309130318} &= 2 \cdot 3 \cdot 7 \cdot 71 \cdot 103 \cdot 61559 \cdot 29133437.
\end{aligned}$$

- With 8 factors:

$$\begin{aligned}
\mathbf{244197000982499715087866346} &= 2 \cdot 3 \cdot 11 \cdot 23 \cdot 31 \cdot 47137 \cdot 28282147 \cdot\\
&\ 3892535183,\\
\mathbf{554079914617070801288578559178} &= 2 \cdot 3 \cdot 11 \cdot 23 \cdot 31 \cdot 47059 \cdot 2259696349 \cdot\\
&\ 110725121051,\\
\mathbf{1910667181420507984555759916338506} &= 2 \cdot 3 \cdot 7 \cdot 43 \cdots 1831 \cdot 138683 \cdot 2861051 \cdot\\
&\ 1456230512169437.
\end{aligned}$$

There are no other Giuga numbers with fewer than 8 prime factors. There is another known Giuga number (found by Frederick Schneider in 2006) which has 10 prime factors, but it is not known if there is any Giuga number between this and the previous ones. This largest known Giuga number is the following:

$$\begin{aligned}
&\mathbf{420001794970774706203871150967065663240419575375163060922876441614255721158209843254519032347481 8}\\
\end{aligned}$$

$$\mathbf{4200017949707747062038711509670656632404195753751630609228764416}$$
$$\mathbf{1425572115820984325451903234748 18} = 2 \cdot 3 \cdot 11 \cdot 23 \cdot 47059 \cdot 2217342227 \cdot$$
$$1729101023519 \cdot 8491659218261819498490029296021 \cdot$$
$$658254480569119734123541298976556403.$$

Observe that all known Giuga numbers are even. If an odd Giuga number exists, it must be the product of at least 14 primes. It is not even known if there are infinitely many Giuga numbers.

In 2009, Paolo P. Lava, an active collaborator of the *On-Line Encyclopedia of Integer Sequences*, conjectured that Giuga numbers were exactly the solutions of the differential equation $n' = n + 1$, with n' being the arithmetic derivative of n. It is not a well-known conjecture (see the book by Lava [2] or comments about sequence A007850 in Sloane's *On-Line Encyclopedia of Integer Sequences*) but it seems known enough to appear in wikipedia's article on Giuga numbers [8].

2 Giuga numbers and the arithmetic derivative: bringing doubts on Lava's conjecture

It is surprising that Lava's conjecture has not been resolved. Certainly the thirteen known Giuga numbers satisfy $n' = n + 1$. Nevertheless, we can observe that this is exclusively due to the fact that these thirteen known Giuga numbers satisfy $\sum_{p|n} \frac{1}{p} - \prod_{p|n} \frac{1}{p} = 1$.

Let us introduce now a novel characterization of Giuga numbers in terms of the arithmetic derivative.

To do so we first need the following technical lemma [7, Corollary 2]..

Lemma 4. *If $n' = an + 1$, then n is square-free.*

Proof. If there is a prime p such that p^2 divides n, then p must divide n'. Since $n' = an + 1$ this yields a contradiction. $\square$

Proposition 5. *Let n be an integer. Then n is a Giuga number if and only if $n' = an + 1$ for some $a \in \mathbb{N}$.*

Proof. If n is a Giuga number then it is composite and square-free. Thus, we can put $n = p_1 p_2 \cdots p_k$ con $k > 1$ and since n is a Giuga number

$$\sum_{i=1}^{k} \frac{1}{p_i} - \prod_{i=1}^{k} \frac{1}{p_i} = \sum_{i=1}^{k} \frac{1}{p_i} - \frac{1}{n} = a \in \mathbb{Z}.$$

Consequently, $n \sum_{i=1}^{k} \frac{1}{p_i} - 1 = an$ and it follows by definition that $n' = an + 1$.

Conversely, assume that $n' = an + 1$ with $a \in \mathbb{N}$. Then n is square-free, i.e., $\prod_{p|n} \frac{1}{p} = \frac{1}{n}$. Thus, $an + 1 = n' = n \sum_{i=1}^{k} \frac{1}{p_i}$ implies that $\sum_{i=1}^{k} \frac{1}{p_i} - \frac{1}{n} = a \in \mathbb{N}$, i.e., n is a Giuga number, as claimed. $\square$

This result shows that Lava's conjecture is as close (or far away) to being refuted as the discovery of a Giuga number with

$$\sum_{i=1}^{k} \frac{1}{p_i} - \frac{1}{n} > 1.$$

This is not likely to happen in the near future, since it is known that such a number must have more than 59 prime factors [5]. In any case it has been pointed out that Lava's conjecture is not plausible and it could well be false.

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(Concerned with sequences A003415 and A007850.)

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