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On the Sums of Reciprocal Hyperfibonacci Numbers and Hyperlucas Numbers

Rui Liu[1] and Feng-Zhen Zhao
Department of Mathematics
Dalian University of Technology
Dalian, Liaoning 116024
P. R. China
liurui1515@gmail.com
fengzhenzhao@yahoo.com.cn

Abstract

In this paper, we discuss the properties of hyperfibonacci numbers and hyperlucas numbers. We investigate the sums of reciprocal hyperfibonacci numbers and hyperlucas numbers. In addition, we establish some identities related to reciprocal hyperfibonacci numbers and hyperlucas numbers.

1 Introduction

Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$ have fascinated both amateurs and professional mathematicians for centuries. They are generalized to many forms. Dil and Mezö [4] introduced the definition of "hyperfibonacci" numbers $F_n^{(r)}$ and "hyperlucas" numbers $L_n^{(r)}$:

$$\begin{aligned} F_n^{(r)} &= \sum_{j=0}^{n} F_j^{(r-1)}, \quad with \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1, \\ L_n^{(r)} &= \sum_{j=0}^{n} L_j^{(r-1)}, \quad with \quad L_n^{(0)} = L_n, \quad L_0^{(r)} = 2, \quad L_1^{(r)} = 2r+1, \end{aligned}$$

where r is a positive integer. It is well known that the Binet forms of $\{F_n\}$ and $\{L_n\}$ are

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}}, \quad L_n = \alpha^n + (-1)^n \alpha^{-n}, \tag{1}$$

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where $\alpha = (1+\sqrt{5})/2$. The sequences $\{F_n\}$ and $\{L_n\}$ satisfy the linear recurrence relation

$$W_n = W_{n-1} + W_{n-2}, \quad n \geq 2. \tag{2}$$

It is clear that

$$F_n^{(1)} = F_{n+2} - 1, \quad L_n^{(1)} = L_{n+2} - 1. \tag{3}$$

Some values of $\{F_n^{(1)}\}$ and $\{L_n^{(1)}\}$ are given below.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$F_n^{(1)}$	0	1	2	4	7	12	20	33	54	88	143	232	376	609	986
$L_n^{(1)}$	2	3	6	10	17	28	46	75	122	198	321	520	842	1363	2206

These are sequences A000071 and A001610 in Sloane's *Encyclopedia* [11]. Some properties of $\{F_n^{(r)}\}$ and $\{L_n^{(r)}\}$ are studied in the paper of Ning-Ning Cao and Feng-Zhen Zhao [3]. In this paper, we investigate the sums of reciprocal hyperfibonacci numbers and hyperlucas numbers.

Now we recall some definitions involved in this paper. The Fibonacci and Lucas zeta functions are defined by

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \quad \text{and} \quad \zeta_L(s) = \sum_{n=1}^{\infty} \frac{1}{L_n^s},$$

where $\{F_n\}$ and $\{L_n\}$ are the Fibonacci and Lucas sequences, respectively. Recently, properties of $\zeta_F(s)$ and $\zeta_L(s)$ are investigated in several different ways, see for instance [5, 6, 7, 9]. In [5], the partial infinite sums of reciprocal Fibonacci numbers were studied by Ohtsuka and Nakamura, [10]. They proved that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2} - 1, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \tag{4}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. In [8], Holliday and Komatsu generalize (4) to the generalized Fibonacci sequence. They showed that

$$\left\lfloor \left(\sum_{k=1000n}^{\infty} \frac{1}{G_k} \right)^{-1} \right\rfloor = \begin{cases} G_n - G_{n-1}, & \text{if } n \text{ is even and } n \geq 2; \\ G_n - G_{n-1} - 1, & \text{if } n \text{ is odd and } n \geq 1, \end{cases}$$

where $\{G_n\}$ is generalized Fibonacci sequence defined by $G_{k+2} = aG_{k+1} + G_k (k \geq 0)$ with $G_0 = 0, G_1 = 1$, and a is a positive integer. In this paper, we discuss the partial infinite sums of reciprocal hyperfibonacci numbers and hyperlucas numbers. In the next section, we investigate the sums of the following forms

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^{(1)}} \right)^{-1} \right\rfloor, \quad \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k^{(1)}} \right)^{-1} \right\rfloor.$$

In addition, we establish some identities related to reciprocal hyperfibonacci numbers and hyperlucas numbers.

2 The partial infinite sums of reciprocal hyperfibonacci numbers and hyperlucas numbers

In this section, we discuss the partial infinite sums of reciprocal hyperfibonacci numbers and hyperlucas numbers.

Lemma 1. *For $\{F_n\}$ and $\{L_n\}$, the following formulas hold:*

$$\begin{aligned} F_{n+1}F_{n+3} - F_nF_{n+4} &= 2(-1)^n, & (5)\\ L_{n+1}L_{n+3} - L_nL_{n+4} &= 10(-1)^{n+1}, & (6)\\ F_{n+2}^2 - F_{n+1}F_{n+3} &= (-1)^{n+1}, & (7)\\ L_{n+2}^2 - L_{n+1}L_{n+3} &= 5(-1)^n. & (8)\end{aligned}$$

Proof. From (1), we can verify that (5)-(8) hold. □

Theorem 2. *For $\{F_n^{(1)}\}$ and $\{L_n^{(1)}\}$ $(n \geq 3)$, we have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^{(1)}} \right)^{-1} \right\rfloor = F_n - 1, \tag{9}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_k^{(1)}} \right)^{-1} \right\rfloor = L_n - 1, \quad n \geq 4. \tag{10}$$

Proof. By using (2)–(3) and (5), we get

$$\begin{aligned} \frac{1}{F_n^{(1)} - F_{n-1}^{(1)}} - \frac{1}{F_n^{(1)}} - \frac{1}{F_{n+1}^{(1)}} - \frac{1}{F_{n+2}^{(1)} - F_{n+1}^{(1)}} &= \frac{1}{F_n} - \frac{1}{F_n^{(1)}} - \frac{1}{F_{n+1}^{(1)}} - \frac{1}{F_{n+2}} \\ &= \frac{F_n^{(1)}(F_{n+1}F_{n+3} - F_nF_{n+2} - F_{n+1}) - F_nF_{n+2}F_{n+1}^{(1)}}{F_nF_{n+2}F_n^{(1)}F_{n+1}^{(1)}} \\ &= \frac{F_n^{(1)}[2(-1)^n + F_nF_{n+3} - F_{n+1}] - F_nF_{n+2}F_{n+1}^{(1)}}{F_nF_{n+2}F_n^{(1)}F_{n+1}^{(1)}} \\ &= \frac{F_n^{(1)}[2(-1)^n - F_{n+1}] + F_n[F_n^{(1)}F_{n+3} - F_{n+2}F_{n+1}^{(1)}]}{F_nF_{n+2}F_n^{(1)}F_{n+1}^{(1)}} \\ &= \frac{F_n^{(1)}[2(-1)^n - F_{n+1}] - F_nF_{n+1}}{F_nF_{n+2}F_n^{(1)}F_{n+1}^{(1)}}, \quad n \geq 2, \end{aligned}$$

and

$$\frac{1}{F_n^{(1)} - F_{n-1}^{(1)} - 1} - \frac{1}{F_n^{(1)}} - \frac{1}{F_{n+1}^{(1)}} - \frac{1}{F_{n+2}^{(1)} - F_{n+1}^{(1)} - 1} = \frac{F_{n+1}F_{n+1}^{(1)} - (F_n - 1)F_n^{(1)} - (F_n - 1)F_{n+1}^{(1)}}{(F_n - 1)F_n^{(1)}F_{n+1}^{(1)}}$$

$$
\begin{aligned}
&= \frac{F_{n+1}F_{n+3} - F_{n+1} - F_nF_{n+3} - F_{n+1} + 2F_n - F_nF_{n+2} + F_n^{(1)} + F_{n+1}^{(1)}}{(F_n-1)F_n^{(1)}F_{n+1}^{(1)}} \\
&= \frac{F_nF_{n+4} + 2(-1)^n - F_nF_{n+3} - F_nF_{n+2} - F_{n+1} + 2F_n + F_{n+2} + F_{n+3} - 2}{(F_n-1)F_n^{(1)}F_{n+1}^{(1)}} \\
&= \frac{2((-1)^n + F_n + F_{n+2} - 1)}{(F_n-1)F_n^{(1)}F_{n+1}^{(1)}}, \quad n \geq 3.
\end{aligned}
$$

By using (2)–(3) and (6), we get

$$
\frac{1}{L_n^{(1)} - L_{n-1}^{(1)}} - \frac{1}{L_n^{(1)}} - \frac{1}{L_{n+1}^{(1)}} - \frac{1}{L_{n+2}^{(1)} - L_{n+1}^{(1)}} = \frac{(10(-1)^{n+1} - L_{n+1})L_n^{(1)} - L_nL_{n+1}}{L_nL_{n+2}L_n^{(1)}L_{n+1}^{(1)}}
$$

for $n \geq 4$ and

$$
\frac{1}{L_n^{(1)} - L_{n-1}^{(1)} - 1} - \frac{1}{L_n^{(1)}} - \frac{1}{L_{n+1}^{(1)}} - \frac{1}{L_{n+2}^{(1)} - L_{n+1}^{(1)} - 1} = \frac{10(-1)^{n+1} + 2(L_n + L_{n+2} - 1)}{(L_n-1)L_n^{(1)}L_{n+1}^{(1)}},
$$

for $n \geq 2$.

From the inequalities

$$
\begin{aligned}
(2(-1)^n - F_{n+1})F_n^{(1)} - F_nF_{n+1} &< 0, \quad n \geq 2, \\
(10(-1)^{n+1} - L_{n+1})L_n^{(1)} - L_nL_{n+1} &< 0, \quad n \geq 4, \\
(-1)^n + F_n + F_{n+2} - 1 &> 0, \quad n \geq 3, \\
10(-1)^{n+1} + 2(L_n + L_{n+2} - 1) &> 0, \quad n \geq 2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{1}{F_n^{(1)} - F_{n-1}^{(1)}} < &\sum_{k=n}^{\infty} \frac{1}{F_k^{(1)}} < \frac{1}{F_n^{(1)} - F_{n-1}^{(1)} - 1}, \quad n \geq 3, \\
F_n - 1 < &\left(\sum_{k=n}^{\infty} \frac{1}{F_k^{(1)}} \right)^{-1} < F_n, \quad n \geq 3, \\
\frac{1}{L_n^{(1)} - L_{n-1}^{(1)}} < &\sum_{k=n}^{\infty} \frac{1}{L_k^{(1)}} < \frac{1}{L_n^{(1)} - L_{n-1}^{(1)} - 1}, \quad n \geq 4, \\
L_n - 1 < &\left(\sum_{k=n}^{\infty} \frac{1}{L_k^{(1)}} \right)^{-1} < L_n, \quad n \geq 4.
\end{aligned}
$$

Hence the relations (9)–(10) hold. □

Theorem 3. *For $\{F_n^{(1)}\}$ $(n \geq 2)$ we have*

$$
\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{(F_k^{(1)})^2} \right) \right\rfloor = \begin{cases} F_{n-1}^{(1)}F_n^{(1)} + F_{n-1}^{(1)} - 1, & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-1}^{(1)}F_n^{(1)} + F_{n-1}^{(1)}, & \text{if } n \text{ is odd and } n \geq 1. \end{cases} \tag{11}
$$

$$
\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{(L_k^{(1)})^2} \right) \right\rfloor = L_{n-1}^{(1)}L_n^{(1)} + L_{n-1}^{(1)} - 1, \qquad \text{if } n \text{ is odd and } n > 1. \tag{12}
$$

Proof. By applying (2)–(3) and (7), we get

$$\begin{aligned}
\frac{1}{F_{n-1}^{(1)}F_n^{(1)}+F_{n-1}^{(1)}-1}-\frac{1}{(F_n^{(1)})^2}-\frac{1}{F_n^{(1)}F_{n+1}^{(1)}+F_n^{(1)}-1} &= \frac{1}{F_{n-1}^{(1)}F_{n+2}-1}-\frac{1}{(F_n^{(1)})^2}-\frac{1}{F_n^{(1)}F_{n+3}-1}\\
&= \frac{(F_n^{(1)})^2(F_n^{(1)}F_{n+3}-F_{n-1}^{(1)}F_{n+2})-(F_{n-1}^{(1)}F_{n+2}-1)(F_n^{(1)}F_{n+3}-1)}{(F_{n-1}^{(1)}F_{n+2}-1)(F_n^{(1)})^2(F_n^{(1)}F_{n+3}-1)}\\
&= \frac{{F_n^{(1)}}^2(F_{n+2}^2-F_{n+1})-F_{n-1}^{(1)}F_n^{(1)}F_{n+2}F_{n+3}+F_{n-1}^{(1)}F_{n+2}+F_n^{(1)}F_{n+3}-1}{(F_{n-1}^{(1)}F_{n+2}-1)(F_n^{(1)})^2(F_n^{(1)}F_{n+3}-1)}\\
&= \frac{(F_{n+3}+(-1)^{n+1}F_{n+2})F_n^{(1)}+F_n^{(1)}F_{n+1}+F_{n-1}^{(1)}F_{n+2}-1}{(F_{n-1}^{(1)}F_{n+2}-1)(F_n^{(1)})^2(F_n^{(1)}F_{n+3}-1)}>0.
\end{aligned}$$

Thus, we have

$$\sum_{k=n}^{\infty}\frac{1}{(F_k^{(1)})^2}<\frac{1}{F_{n-1}^{(1)}F_n^{(1)}+F_{n-1}^{(1)}-1}.$$

Similarly, we can prove that

$$\sum_{k=n}^{\infty}\frac{1}{(F_k^{(1)})^2}>\frac{1}{F_{n-1}^{(1)}F_n^{(1)}+F_{n-1}^{(1)}+1}.$$

On the other hand, we have

$$\begin{aligned}
\frac{1}{F_{n-1}^{(1)}F_n^{(1)}+F_{n-1}^{(1)}}-\frac{1}{(F_n^{(1)})^2}-\frac{1}{F_n^{(1)}F_{n+1}^{(1)}+F_n^{(1)}} &= \frac{1}{F_{n-1}^{(1)}F_{n+2}}-\frac{1}{(F_n^{(1)})^2}-\frac{1}{F_n^{(1)}F_{n+3}}\\
&= \frac{(F_n^{(1)})^2F_{n+3}-F_{n-1}^{(1)}F_n^{(1)}F_{n+2}-F_{n-1}^{(1)}F_{n+2}F_{n+3}}{F_{n-1}^{(1)}(F_n^{(1)})^2F_{n+2}F_{n+3}}\\
&= \frac{F_n^{(1)}(F_n^{(1)}F_{n+3}-F_{n-1}^{(1)}F_{n+2})-F_{n-1}^{(1)}F_{n+2}F_{n+3}}{F_{n-1}^{(1)}(F_n^{(1)})^2F_{n+2}F_{n+3}}\\
&= \frac{(-1)^{n+1}F_{n+2}+F_{n+1}}{F_{n-1}^{(1)}(F_n^{(1)})^2F_{n+2}F_{n+3}}.
\end{aligned}$$

When $n\geq 2$ is even, we can verify that

$$\sum_{k=n}^{\infty}\frac{1}{(F_k^{(1)})^2}>\frac{1}{F_{n-1}^{(1)}F_n^{(1)}+F_{n-1}^{(1)}},$$

and when $n\geq 1$ is odd

$$\sum_{k=n}^{\infty}\frac{1}{(F_k^{(1)})^2}<\frac{1}{F_{n-1}^{(1)}F_n^{(1)}+F_{n-1}^{(1)}}.$$

Hence when n is even, we obtain

$$F_{n-1}^{(1)}F_n^{(1)}+F_{n-1}^{(1)}-1<\left(\sum_{k=n}^{\infty}\frac{1}{(F_k^{(1)})^2}\right)^{-1}<F_{n-1}^{(1)}F_n^{(1)}+F_{n-1}^{(1)},$$

and when n is odd, we have

$$F_{n-1}^{(1)}F_n^{(1)}+F_{n-1}^{(1)}<\left(\sum_{k=n}^{\infty}\frac{1}{(F_k^{(1)})^2}\right)^{-1}<F_{n-1}^{(1)}F_n^{(1)}+F_{n-1}^{(1)}+1.$$

Then (11) holds.

By applying (2)–(3) and (8), we get

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{(L_k^{(1)})^2} &< \frac{1}{L_{n-1}^{(1)}L_n^{(1)}+L_{n-1}^{(1)}-1},\\ \sum_{k=n}^{\infty}\frac{1}{(L_k^{(1)})^2} &> \frac{1}{L_{n-1}^{(1)}L_n^{(1)}+L_{n-1}^{(1))}}, \quad n \text{ is odd.}\end{aligned}$$

Then (12) holds. □

In the final part of this section, we consider the generalized hyperfibonacci numbers $\{U_n^{(r)}\}$:

$$U_n^{(r)} = \sum_{j=0}^{n}U_j^{(r-1)}, \quad with \quad U_n^{(0)}=U_n, \quad U_0^{(r)}=0, \quad U_1^{(r)}=1.$$

where

$$U_n=\frac{\tau^n-(-1)^n\tau^{-n}}{\sqrt{\Delta}}, \quad \tau=(p+\sqrt{\Delta})/2, \quad \Delta=p^2+4,$$

and p is a positive integer. It is evident that

$$U_n^{(1)}=\frac{U_n+U_{n+1}-1}{p}. \tag{13}$$

And $\{U_n\}$ satisfy that

$$W_n=pW_{n-1}+W_{n-2}, \quad n\geq 2. \tag{14}$$

When $p=1$, $U_n^{(1)}=F_n^{(1)}$.

Now we discuss the partial infinite sum of reciprocal generalized hyperfibonacci numbers.

Lemma 4. *For $\{U_n\}$, the following formulas hold:*

$$U_{n+1}^2-U_nU_{n+2} = (-1)^n, \tag{15}$$

From the definition of $\{U_n\}$, we can prove that (15) holds.

Theorem 5. *When $n \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k^{(1)}} \right)^{-1} \right\rfloor \;=\; U_n - 1, \tag{16}$$

Proof. It follows from (13) and (14)–(15) that

$$\begin{aligned}
\frac{1}{U_n^{(1)} - U_{n-1}^{(1)}} - \frac{1}{U_n^{(1)}} - \frac{1}{U_{n+1}^{(1)}} - \frac{1}{U_{n+2}^{(1)} - U_{n+1}^{(1)}} \;&=\; \frac{pU_{n+1}}{U_nU_{n+2}} - \frac{U_n^{(1)} + U_{n+1}^{(1)}}{U_n^{(1)}U_{n+1}^{(1)}} \\
&= \frac{U_{n+1}(U_n + U_{n+1} - 1)(U_{n+1} + U_{n+2} - 1) - U_nU_{n+2}(U_n + 2U_{n+1} + U_{n+2} - 2)}{pU_nU_{n+2}U_n^{(1)}U_{n+1}^{(1)}} \\
&= \frac{(U_n + U_{n+1} - 1)((-1)^n + U_{n+1}U_{n+2} - U_{n+1}) - U_nU_{n+2}(U_{n+1} + U_{n+2} - 1)}{pU_nU_{n+2}U_n^{(1)}U_{n+1}^{(1)}} \\
&= \frac{(U_n + U_{n+1} - 1)((-1)^n - U_{n+1}) + U_{n+2}(U_{n+1}^2 - U_{n+1} - U_nU_{n+2} + U_n)}{pU_nU_{n+2}U_n^{(1)}U_{n+1}^{(1)}} \\
&= \frac{(U_n + U_{n+1} - 1)((-1)^n - U_{n+1}) + U_{n+2}((-1)^n - (p-1)U_n - U_{n-1})}{pU_nU_{n+2}U_n^{(1)}U_{n+1}^{(1)}} \\
&< 0,
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{U_n^{(1)} - U_{n-1}^{(1)} - 1} - \frac{1}{U_n^{(1)}} - \frac{1}{U_{n+1}^{(1)}} - \frac{1}{U_{n+2}^{(1)} - U_{n+1}^{(1)} - 1} \;&=\; \frac{pU_{n+1}}{(U_n - 1)(U_{n+2} - 1)} - \frac{1}{U_n^{(1)}} - \frac{1}{U_{n+1}^{(1)}} \\
&= \frac{U_{n+1}^{(1)}(U_{n+2} + U_n - U_{n+1} - 1 + (-1)^n) + U_n^{(1)}(U_n + U_{n+2} - 1)}{(U_n - 1)(U_{n+2} - 1)U_n^{(1)}U_{n+1}^{(1)}} + \frac{U_n(U_{n+1}^{(1)}U_{n+1} - U_n^{(1)}U_{n+2})}{(U_n - 1)(U_{n+2} - 1)U_n^{(1)}U_{n+1}^{(1)}} \\
&= \frac{U_{n+1}^{(1)}(U_{n+2} + U_n - U_{n+1} - 1 + (-1)^n) + U_n^{(1)}(U_n + U_{n+2} - 1)}{(U_n - 1)(U_{n+2} - 1)U_n^{(1)}U_{n+1}^{(1)}} + \frac{U_n((-1)^n - U_{n+1} + U_{n+2})}{p(U_n - 1)(U_{n+2} - 1)U_n^{(1)}U_{n+1}^{(1)}} \\
&> 0.
\end{aligned}$$

Then we obtain

$$\frac{1}{U_n^{(1)} - U_{n-1}^{(1)}} \;<\; \sum_{k=n}^{\infty} \frac{1}{U_k^{(1)}} < \frac{1}{U_n^{(1)} - U_{n-1}^{(1)} - 1}.$$

Hence (16) holds. □

3 Some identities related to reciprocal hyperfibonacci numbers and hyperlucas numbers

In this section, we give some identities related to inverse of hyperfibonacci and hyperlucas numbers. There are some identities containing the reciprocals of Fibonacci and Lucas numbers (see [1, 2]):

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + F_s} = \frac{\sqrt{5}s}{2L_s}, \quad s \text{ odd},$$

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + L_s/\sqrt{5}} = \frac{s}{2F_s}, \quad s > 0 \text{ is even}.$$

For hyperfibonacci and hyperlucas numbers $F_n^{(1)}$ $L_n^{(1)}$, we have

Theorem 6. *Let m be a positive integer. For $F_n^{(1)}$ and $L_n^{(1)}$, we have*

$$\sum_{n=1}^{\infty} \frac{L_{n+2m+2}}{F_n^{(1)} F_{n+4m}^{(1)}} = \frac{1}{F_{2m}} \sum_{k=1}^{4m} \frac{1}{F_k^{(1)}}, \tag{17}$$

$$\sum_{n=1}^{\infty} \frac{F_{n+2m+2}}{L_n^{(1)} L_{n+4m}^{(1)}} = \frac{1}{5F_{2m}} \sum_{n=1}^{4m} \frac{1}{L_k^{(1)}}, \tag{18}$$

$$\sum_{n=1}^{\infty} \frac{F_{n+2m+3}}{F_n^{(1)} F_{n+4m+2}^{(1)}} = \frac{1}{L_{2m+1}} \sum_{k=1}^{4m+2} \frac{1}{F_k^{(1)}}, \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{L_{n+2m+3}}{L_n^{(1)} L_{n+4m+2}^{(1)}} = \frac{1}{L_{2m+1}} \sum_{k=1}^{4m+2} \frac{1}{L_k^{(1)}}. \tag{20}$$

Proof. It follows from (2) that

$$\begin{aligned}
\frac{1}{F_n^{(1)}} - \frac{1}{F_{n+4m}^{(1)}} &= \frac{F_{n+4m}^{(1)} - F_n^{(1)}}{F_n^{(1)} F_{n+4m}^{(1)}}, \\
\frac{1}{L_n^{(1)}} - \frac{1}{L_{n+4m}^{(1)}} &= \frac{L_{n+4m}^{(1)} - L_n^{(1)}}{L_n^{(1)} - L_{n+4m}^{(1)}}, \\
\frac{1}{F_n^{(1)}} - \frac{1}{F_{n+4m+2}^{(1)}} &= \frac{F_{n+4m+4} - F_{n+2}}{F_n^{(1)} F_{n+4m+2}^{(1)}}, \\
\frac{1}{L_n^{(1)}} - \frac{1}{L_{n+4m+2}^{(1)}} &= \frac{L_{n+4m+4} - L_{n+2}}{L_n^{(1)} L_{n+4m+2}^{(1)}}.
\end{aligned}$$

From

$$\begin{aligned}
F_{n+4m}^{(1)} - F_n^{(1)} &= F_{2m} L_{n+2m+2}, \\
L_{n+4m}^{(1)} - L_n^{(1)} &= 5F_{2m} F_{n+2m+2}, \\
F_{n+4m+4} - F_{n+2} &= F_{n+2m+3} L_{2m+1}, \\
L_{n+4m+4} - L_{n+2} &= L_{n+2m+3} L_{2m+1}.
\end{aligned}$$

we obtain the formula (17)–(20). $\square$

We can give other identities for $F_n^{(1)}$ and $L_n^{(1)}$. The following lemma will be used (see [12]).

Lemma 7. *Let t be a real number with $|t| > 1$, s and a be positive integers, and b be a nonnegative integer. Then one has that*

$$\sum_{n=0}^{\infty} \frac{1}{t^{2an+b} + t^{-2an-b} - (t^{as} + t^{-as})} = \frac{1}{t^{as} - t^{-as}} \sum_{n=0}^{s-1} \frac{1}{1 - t^{2an+b-as}} \tag{21}$$

Theorem 8. *Suppose that a, b and s are positive integers with $b > as$. For $F_n^{(1)}$ and $L_n^{(1)}$, we have:*
(i) when a, b and s are odd,

$$\sum_{n=0}^{\infty} \frac{1}{F_{2an+b-2}^{(1)} - F_{as-2}^{(1)}} = \frac{\sqrt{5}}{L_{as}} \sum_{n=0}^{s-1} \frac{1}{1 - \alpha^{2an+b-as}}, \tag{22}$$

(ii) when b and s are both even,

$$\sum_{n=0}^{\infty} \frac{1}{L_{2an+b-2}^{(1)} - L_{as-2}^{(1)}} = \frac{1}{\sqrt{5}F_{as}} \sum_{n=0}^{s-1} \frac{1}{1 - \alpha^{2an+b-as}}. \tag{23}$$

Proof. By means of (22) and $F_n^{(1)} = F_{n+2} - 1$, $L_n^{(1)} = L_{n+2} - 1$, we can easily prove that (22) and (23) hold. $\square$

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