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Sums of Products of Poly-Bernoulli Numbers of Negative Index

Ken Kamano
Department of Mathematics
Osaka Institute of Technology
5-16-1, Omiya, Asahi, Osaka 535-8585
Japan
kamano@ge.oit.ac.jp

Abstract

We give a formula that expresses a sum of products of poly-Bernoulli numbers of negative index as a linear combination of poly-Bernoulli numbers. More generally, we show that if a two-variable formal power series satisfies a certain partial differential equation, then its coefficients satisfy this type of formula. As an appendix, we solve this partial differential equation.

1 Introduction and main results

Bernoulli numbers B_n $(n = 0, 1, 2, \ldots)$ are rational numbers defined by the following generating function:

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

These numbers are important, especially in number theory, because they are related to special values of the Riemann zeta function, class numbers of algebraic fields and so on. There are many relations among Bernoulli numbers. For example, the following identity is classically known:

$$\sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = nB_{n-1} - (n-1)B_n \quad (n \geq 1). \tag{1}$$

Many generalizations of (1) have been considered. Chen [5] gave formulas on sums of N products of Bernoulli polynomials, generalized Bernoulli numbers and Euler polynomials by using special values of certain zeta functions at non-positive integers. Agoh and Dilcher [1, 2] introduced certain lacunary sums of products of Bernoulli numbers and gave some formulas for them. For other types of sums of products, see [6, 7, 12, 13].

The author [9] studied the following type of sums whose products include one poly-Bernoulli number:

$$\sum_{i_1+\cdots+i_m=n} \frac{n!}{i_1!\cdots i_m!} b_{i_1}\cdots b_{i_{m-1}} B_{i_m}^{(k)},$$

where k is a fixed integer. Here $b_n = (-1)^n B_n$ (it should be noted that b_n is denoted by B_n in [9]) and $B_n^{(k)}$ are poly-Bernoulli numbers, which will be defined below. For $m = 2$ and 3, these sums were expressed explicitly in terms of poly-Bernoulli numbers [9, Theorem 3 and 4]. In this paper we deal with another type of sums running over both indices m and n of $B_n^{(-m)}$.

We first review poly-Bernoulli numbers. For any integer k, Kaneko [10] introduced the n-th poly-Bernoulli number of index k (denoted by $B_n^{(k)}$) as

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} \frac{B_n^{(k)}}{n!} t^n,$$

where Li_k is the k-th polylogarithm defined by $\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} x^n/n^k$. Since $\mathrm{Li}_1(x) = -\log(1-x)$, the number $B_n^{(1)}$ is nothing but the ordinary n-th Bernoulli number B_n. Poly-Bernoulli numbers of positive index are also rational numbers, and they have a connection with multiple zeta functions. For example, Arakawa-Kaneko [3] introduced a function $\xi_k(s)$ ($k \geq 1$), which can be expressed as a sum of certain multiple zeta functions. They proved that its special values at non-positive integers are expressed in terms of poly-Bernoulli numbers of index k [3, Theorem 6].

For a non-positive index case, it is known that $B_n^{(-m)}$ ($m \geq 0$) are positive integers and satisfy a duality relation:

$$B_n^{(-m)} = B_m^{(-n)} \qquad (m, n \geq 0)$$

[10, Theorem 2]. These numbers $B_n^{(-m)}$ have some combinatorial applications. Launois [11] proved that $B_n^{(-m)}$ is equal to the number of permutations $\sigma \in \mathfrak{S}_{m+n}$ such that $-m \leq i - \sigma(i) \leq n$ for all $i = 1, \ldots, m+n$. Recently Brewbaker [4] proved that $B_n^{(-m)}$ coincides with the number of certain $m \times n$ matrices called *lonesum matrix* (see [4, 14], for details).

One reason why $B_n^{(-m)}$ have these combinatorial applications is that their generating function can be written simply as

$$\sum_{m\geq 0,\, n\geq 0} B_n^{(-m)} \frac{x^m}{m!}\frac{y^n}{n!} = \frac{1}{e^{-x}+e^{-y}-1} \tag{2}$$

(see [10, §1]). We remark that the right-hand side of (2) satisfies the partial differential equation

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) F = F + F^2. \tag{3}$$

Now we consider a slightly more general situation. Let $\mathbb{C}$ be the complex number field and $F(x,y) \in \mathbb{C}[[x,y]]$ a two-variable formal power series:

$$F(x,y) = \sum_{m\geq 0,\, n\geq 0} \alpha(m,n) \frac{x^m}{m!}\frac{y^n}{n!}, \quad (\alpha(m,n) \in \mathbb{C}). \tag{4}$$

We define a differential operator $D := \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and assume that F satisfies a partial differential equation

$$DF = F + F^2. \tag{5}$$

Of course the right-hand side of (2) is such an example of F.

The following is the main theorem of this paper. This result means that a sum of products of $\alpha(i,j)$ can be expressed as a linear combination of themselves, like the classical identity (1) of Bernoulli numbers.

Theorem 1. *Assume that $F \in \mathbb{C}[[x,y]]$ satisfies (5). For $m,n \geq 0$ and $k \geq 1$, we have*

$$\begin{aligned}
&\sum_{\substack{i_1,\dots,i_k\geq 0\\ i_1+\cdots+i_k=m}} \sum_{\substack{j_1,\dots,j_k\geq 0\\ j_1+\cdots+j_k=n}} \frac{m!}{i_1!\cdots i_k!}\frac{n!}{j_1!\cdots j_k!}\alpha(i_1,j_1)\cdots\alpha(i_k,j_k)\\
&= \frac{1}{(k-1)!}\sum_{\substack{i,j\geq 0\\ i+j\leq k-1}} (-1)^{k-1+i+j} \begin{bmatrix} k \\ i+j+1 \end{bmatrix} \binom{i+j}{i} \alpha(m+i,n+j),
\end{aligned} \tag{6}$$

where $\left[{k \atop l}\right]$ are Stirling numbers of the first kind.

Since the generating function $(e^{-x}+e^{-y}-1)^{-1}$ of $B_n^{(-m)}$ satisfies (5), we obtain the following corollary.

Corollary 2. *For $m,n \geq 0$ and $k \geq 1$, we have*

$$\begin{aligned}
&\sum_{\substack{i_1,\dots,i_k\geq 0\\ i_1+\cdots+i_k=m}} \sum_{\substack{j_1,\dots,j_k\geq 0\\ j_1+\cdots+j_k=n}} \frac{m!}{i_1!\cdots i_k!}\frac{n!}{j_1!\cdots j_k!} B_{i_1}^{(-j_1)}\cdots B_{i_k}^{(-j_k)}\\
&= \frac{1}{(k-1)!}\sum_{\substack{i,j\geq 0\\ i+j\leq k-1}} (-1)^{k-1+i+j} \begin{bmatrix} k \\ i+j+1 \end{bmatrix} \binom{i+j}{i} B_{m+i}^{(-n-j)}.
\end{aligned}$$

2 Proof of the Main Theorem

We first recall two kinds of Stirling numbers. For integers $m \geq 1$ and l with $0 \leq l \leq m$, Stirling numbers of the first kind $\left[{m \atop l}\right]$ and the second kind $\left\{{m \atop l}\right\}$ are defined as follows:

$$x(x+1)\cdots(x+m-1) = \sum_{l=0}^{m} \begin{bmatrix} m \\ l \end{bmatrix} x^l,$$

$$\sum_{l=0}^{m} \begin{Bmatrix} m \\ l \end{Bmatrix} x(x-1)\cdots(x-m+1) = x^m.$$

These numbers play important roles in combinatorial theory and many relations among them have been known (e.g., [8, 6-1]). We note that Stirling numbers of the first kind appear in formulas for sums of products of the ordinary Bernoulli numbers and polynomials (e.g., [6, Theorem 3]).

Throughout this section, we assume that $F \in \mathbb{C}[[x,y]]$ satisfies a differential equation $DF = F + F^2$. Then it can be proved by induction on n that

$$DF^n = nF^n + nF^{n+1} \quad (n \geq 1). \tag{7}$$

By using this, we obtain the following lemma.

Lemma 3. *For $n \geq 1$, we have*

$$D^{n-1}F = \sum_{i=1}^{n} \left\{ {n \atop i} \right\} (i-1)! F^i. \tag{8}$$

This can be written as the following matrix representation:

$$\begin{pmatrix} D^0F \\ D^1F \\ D^2F \\ D^3F \\ \vdots \end{pmatrix} = \begin{pmatrix} \left\{ {1 \atop 1} \right\} & & & & \\ \left\{ {2 \atop 1} \right\} & \left\{ {2 \atop 2} \right\} & & & \\ \left\{ {3 \atop 1} \right\} & \left\{ {3 \atop 2} \right\} & \left\{ {3 \atop 3} \right\} & & \cdots \\ \left\{ {4 \atop 1} \right\} & \left\{ {4 \atop 2} \right\} & \left\{ {4 \atop 3} \right\} & \left\{ {4 \atop 4} \right\} & \\ & \vdots & & & \end{pmatrix} \begin{pmatrix} 0!F \\ 1!F^2 \\ 2!F^3 \\ 3!F^4 \\ \vdots \end{pmatrix} \tag{9}$$

Proof. We prove the lemma by induction on n. Since $\left\{ {1 \atop 1} \right\} = 1$, Eq. (8) obviously holds for $n = 1$. We assume that (8) holds for a certain n. By the inductive assumption, we have

$$\begin{aligned} D^nF &= DD^{n-1}F \\ &= \sum_{i=1}^{n} \left\{ {n \atop i} \right\} (i-1)! DF^i. \end{aligned}$$

By (7) and the well-known identity $\left\{ {n \atop i} \right\} i + \left\{ {n \atop i-1} \right\} = \left\{ {n+1 \atop i} \right\}$, we have

$$\begin{aligned} D^nF &= \sum_{i=1}^{n} \left\{ {n \atop i} \right\} (i-1)! (iF^i + iF^{i+1}) \\ &= \sum_{i=1}^{n+1} \left(\left\{ {n \atop i} \right\} i + \left\{ {n \atop i-1} \right\} \right) (i-1)! F^i \\ &= \sum_{i=1}^{n+1} \left\{ {n+1 \atop i} \right\} (i-1)! F^i, \end{aligned}$$

and this proves that (8) holds for $n+1$. It is obvious that (8) means the matrix representation (9). ∎

This lemma states that $D^{n-1}F$ can be expressed as a linear combination of F^i $(1 \le i \le n)$. Conversely, we can express F^k as a linear combination of D^lF $(1 \le l \le k)$ by multiplying the inverse matrix:

Lemma 4. *For $k \ge 1$, we have*

$$(k-1)!\, F^k = \sum_{l=1}^{k} (-1)^{k-l} \genfrac{[}{]}{0pt}{}{k}{l} D^{l-1}F. \tag{10}$$

This can be written as the following matrix representation:

$$\begin{pmatrix} \genfrac{[}{]}{0pt}{}{1}{1} & & & & \\ -\genfrac{[}{]}{0pt}{}{2}{1} & \genfrac{[}{]}{0pt}{}{2}{2} & & & \\ \genfrac{[}{]}{0pt}{}{3}{1} & -\genfrac{[}{]}{0pt}{}{3}{2} & \genfrac{[}{]}{0pt}{}{3}{3} & & \cdots \\ -\genfrac{[}{]}{0pt}{}{4}{1} & \genfrac{[}{]}{0pt}{}{4}{2} & -\genfrac{[}{]}{0pt}{}{4}{3} & \genfrac{[}{]}{0pt}{}{4}{4} & \\ & \vdots & & & \end{pmatrix} \begin{pmatrix} D^0F \\ D^1F \\ D^2F \\ D^3F \\ \vdots \end{pmatrix} = \begin{pmatrix} 0!F \\ 1!F^2 \\ 2!F^3 \\ 3!F^4 \\ \vdots \end{pmatrix}$$

Proof. By Lemma 3, the right-hand side of (10) is equal to

$$\sum_{l=1}^{k} (-1)^{k-l} \genfrac{[}{]}{0pt}{}{k}{l} \sum_{i=1}^{l} (-1)^{k-l} \genfrac{\{}{\}}{0pt}{}{l}{i} (i-1)!F^i = \sum_{1 \le i \le l \le k} (-1)^{k-l} \genfrac{[}{]}{0pt}{}{k}{l} \genfrac{\{}{\}}{0pt}{}{l}{i} (i-1)!F^i. \tag{11}$$

It is known that, for $1 \le i \le k$,

$$\sum_{l=i}^{k} (-1)^{k-l} \genfrac{[}{]}{0pt}{}{k}{l} \genfrac{\{}{\}}{0pt}{}{l}{i} = \delta_{ik},$$

where δ_{ik} is the Kronecker delta function (e.g., [8, 6-1]). Hence all terms in the sum (11) vanish except for $i = l = k$. Therefore this sum equals $(k-1)!F^k$ and this completes the proof. □

Now it is easy to prove our main result.

Proof of the Main Theorem. The left-hand side of (6) is equal to the coefficient of $x^m y^n / m!n!$ in F^k. Hence, by Lemma 4, it is equal to

$$\frac{1}{(k-1)!} \sum_{l=1}^{k} (-1)^{k-l} \genfrac{[}{]}{0pt}{}{k}{l} \sum_{i=0}^{l-1} \binom{l-1}{i} \alpha(m+i, n+l-1-i).$$

By rearranging the summation, we obtain (6). □

3 Solutions of the partial differential equation

In this section, we solve the partial differential equation

$$DF = F + F^2. \tag{12}$$

If a solution of (12) have a series expansion (4), then its coefficients satisfy a relation (6) in our main theorem.

Proposition 5. *Let $F(x, y)$ be a partially differentiable function satisfying (12). Then $F(x, y) = 0$, -1, or*

$$\left(\pm \exp\left(h(x-y) - \frac{x+y}{2}\right) - 1\right)^{-1}, \tag{13}$$

where h is an arbitrary differentiable function.

Proof. Obviously $F = 0$ and -1 satisfy the differential equation (12). We assume that $F \neq 0, -1$. By the transformation of variables

$$x = u + v, \quad y = u - v \qquad \left(\text{i.e., } u = \frac{x+y}{2}, \quad v = \frac{x-y}{2}\right),$$

Eq. (12) is rewritten as an ordinary differential equation with respect to u:

$$\frac{\partial F}{\partial u} = F + F^2.$$

By the method of separation of variables, we have

$$F = (\pm \exp(-g(v) - u) - 1)^{-1},$$

where g is an arbitrary function. By replacing $-g(v)$ with $h(2v)$, we obtain (13). Since F is partially differentiable, the function h have to be differentiable. $\square$

If the exponential part is positive and $h(s) = \log(e^{s/2} + e^{-s/2})$ in (13), then $F(x, y) = (e^{-x} + e^{-y} - 1)^{-1}$, which is a generating function of poly-Bernoulli numbers.

Remark 6. 1. We can regard the solution $F(x, y) = 0$ (resp. -1) as a general solution (13) for $h(s) = +\infty$ (resp. $-\infty$).

2. Not all solutions of (13) have a series expansion such as (4). In fact, we get a solution $F(x, y) = (e^{-\frac{x+y}{2}} - 1)^{-1}$ by setting $h(s) = 0$. In this case, however, the value $F(0, 0)$ does not exist and F can not have a series expansion.

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