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Four-term Recurrences, Orthogonal Polynomials and Riordan Arrays

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Abstract

We study constant coefficient four term recurrences for polynomials, in analogy to the three-term recurrences that are associated with orthogonal polynomials. We show that for a family of polynomials obeying such a four-term recurrence, the coefficient array is an ordinary Riordan array of a special type, and vice versa. In certain cases, it is possible to transform these polynomials into related orthogonal polynomials. We characterize the form of the production matrices of the inverse coefficient arrays.

1 Introduction

In this note we shall be concerned with families of monic polynomials $P_n(x)$ where $P_n(x)$ is of degree n. Thus we will have

$$P_n(x) = \sum_{k=0}^{n} a_{n,k} x^k, \quad a_{n,n} = 1.$$

The matrix of elements $a_{n,k}$ is then a lower-triangular matrix with 1's on the diagonal, and hence invertible. We shall call this matrix the coefficient array of the polynomial family. The inverse of this matrix will be called the inverse coefficient array. In the case of orthogonal polynomials, the elements of the first column of the inverse coefficient array are the moments of the family [20]. We shall refer to the elements of the first column of a general inverse coefficient array as being generalized (formal) moments.

We recall that an ordinary Riordan array [15, 18] is defined by a pair (g, f) of power series, where $g(x) = 1 + g_1x + g_2x^2 + \cdots$, $f(x) = x + f_2x^2 + \cdots$, and is associated to the matrix $(t_{n,k})_{0\le n,k\le\infty}$ where $t_{n,k} = [x]^n g(x)f(x)^k$. (These are proper Riordan arrays). The multiplication law for these pairs, which form a group, is given by

$$(g, f) \cdot (h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}(x) = \text{Rev}_x f(x)$ is the compositional inverse of f. For an invertible matrix L, we define the production matrix P_L to be the matrix $P_L = L^{-1}\hat{L}$, where $\hat{L}$ is the matrix L with its first row removed [5, 6]. When L is a Riordan array (g, f), the first column of P_L is generated by $Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right)$, while the k-th column of P_L is generated by $x^{k-1}A(x)$ (taking the first column to be indexed by 0), where $A(x) = \frac{x}{\bar{f}(x)}$. For a sequence a_n, the sequence of determinants $h_n = |a_{i+j}|_{0\le i,j\le n}$ is called the Hankel transform [11, 12, 13] of a_n. Sequences will be referred to by their OEIS [16, 17] numbers. For instance, the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n} = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ are sequence A000108.

It is a classical result that a family of monic polynomials $P_n(x)$ is a family of orthogonal polynomials [4, 7, 19] if and only if they satisfy a three-term recurrence of the form

$$P_n(x) = (x - \alpha_n)P_{n-1} - \beta_n P_{n-2},$$

with appropriate initial conditions. A more recent result [1, 2] is that a Riordan array A is the coefficient array of a family of orthogonal polynomials if and only if the production matrix of A^{-1} is tridiagonal (a result based on previous work [14, 21]). In the case of ordinary Riordan arrays, the coefficients α_n and β_n are necessarily independent of n. The most general form of ordinary Riordan array that coincides with the coefficient array of a family of orthogonal polynomials is then given by

$$\left(\frac{1 - cx - dx^2}{1 + ax + bx^2}, \frac{x}{1 + ax + bx^2}\right),$$

for appropriate values of a, b, c, d. In this note, we ask ourselves the question: what can be said about families of polynomials $P_n(x)$, that satisfy a four-term recurrence of the form:

$$P_n(x) = (x - \alpha)P_{n-1}(x) - (x - \beta)P_{n-2}(x) - \gamma P_{n-3}(x).$$

Note that in this note we are considering the simplest case of constant coefficients.

2 Main results

Our results are encapsulated in the following theorem.

Theorem 1. *A family of monic polynomials $P_n(x)$ where $P_n(x)$ is of degree n, satisfies the four-term recurrence*

$$P_n(x) = (x-\alpha+1)P_{n-1}(x) - (x+\beta)P_{n-2}(x) - \gamma P_{n-3},$$

with $P_n(x) = 0$ for $n < 0$, $P_0(x) = 1$ and $P_1(x) = x - \delta$, if and only if the coefficient array of the family $P_n(x)$ is given by the Riordan array

$$\left(\frac{1+(\alpha-\delta-1)x}{1+(\alpha-1)x+\beta x^2+\gamma x^3}, \frac{x(1-x)}{1+(\alpha-1)x+\beta x^2+\gamma x^3}\right).$$

In addition, for such a family, if $\gamma = 0$ (but $\alpha+\beta \neq 0$ and $\beta+\delta \neq 0$), then the family of polynomials

$$Q_n(x) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n-1}{n-k}P_k(x+1)$$

is a family of orthogonal polynomials that satisfies the three-term recurrence

$$Q_n(x) = (x-\alpha)Q_{n-1}(x) - (\alpha+\beta)Q_{n-2}(x),$$

with $Q_0(x) = 1$, and $Q_1(x) = x - \delta + 1$.

Proof. We let $p_n(x)$ be the family of polynomials whose coefficient array is given by the Riordan array

$$\left(\frac{1+(\alpha-\delta-1)x}{1+(\alpha-1)x+\beta x^2+\gamma x^3}, \frac{x(1-x)}{1+(\alpha-1)x+\beta x^2+\gamma x^3}\right).$$

Since this array is lower triangular with 1's on the diagonal, each polynomial $p_n(x)$ is monic of degree n. We let

$$L = \left(\frac{1+(\alpha-\delta-1)x}{1+(\alpha-1)x+\beta x^2+\gamma x^3}, \frac{x(1-x)}{1+(\alpha-1)x+\beta x^2+\gamma x^3}\right)^{-1}.$$

Then standard Riordan array techniques show that the production matrix P_L of L is given by

$$P_L = \begin{pmatrix} \delta & 1 & 0 & 0 & 0 & 0 & \cdots \\ \beta+\delta & \alpha & 1 & 0 & 0 & 0 & \cdots \\ \beta+\delta+\gamma & \alpha+\beta & \alpha & 1 & 0 & 0 & \cdots \\ \beta+\delta+\gamma & \alpha+\beta+\gamma & \alpha+\beta & \alpha & 1 & 0 & \cdots \\ \beta+\delta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta & \alpha & 1 & \cdots \\ \beta+\delta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta & \alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} \delta & 1 & 0 & 0 & 0 & 0 & \cdots \\ \beta+\delta & \alpha & 1 & 0 & 0 & 0 & \cdots \\ \beta+\delta+\gamma & \alpha+\beta & \alpha & 1 & 0 & 0 & \cdots \\ \beta+\delta+\gamma & \alpha+\beta+\gamma & \alpha+\beta & \alpha & 1 & 0 & \cdots \\ \beta+\delta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta & \alpha & 1 & \cdots \\ \beta+\delta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta & \alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} xp_0(x) \\ xp_1(x) \\ xp_2(x) \\ xp_3(x) \\ \vdots \end{pmatrix}.$$

We deduce the following.

$$p_1(x) + \delta p_0(x) = xp_0(x) \implies p_1(x) = x - \delta.$$

$$p_2(x) + \alpha p_1(x) + (\beta+\delta)p_0(x) = xp_1(x) \implies p_2(x) = (x-\alpha)p_1(x) - (\beta+\delta)$$

which further implies that

$$p_2(x) = (x-\alpha+1)p_1(x) - p_1(x) - (\beta+\delta) = (x-\alpha+1)p_1(x) - (x+\beta)p_0(x).$$

We have

$$p_3(x) + \alpha p_2(x) + (\alpha+\beta)p_1(x) + (\beta+\delta+\gamma)p_0(x) = xp_2(x)$$

which implies that

$$\begin{aligned}
p_3(x) &= (x-\alpha)p_2(x) - (\alpha+\beta)p_1(x) - (\beta+\delta+\gamma) \\
&= (x-\alpha+1)p_2(x) - p_2(x) - (\alpha+\beta)p_1(x) - (\beta+\delta+\gamma) \\
&= (x-\alpha+1)p_2(x) - (x-\alpha)p_1(x) + (\beta+\delta) - (\alpha+\beta)p_1(x) - (\beta+\delta+\gamma) \\
&= (x-\alpha+1)p_2(x) - (x+\beta)p_1(x) - \gamma p_0(x).
\end{aligned}$$

In similar fashion we can show that

$$p_4(x) = (x-\alpha+1)p_3(x) - (x+\beta)p_2(x) - \gamma p_1(x).$$

Now assume that

$$p_{n-1}(x) = (x-\alpha+1)p_{n-2}(x) - (x+\beta)p_{n-3}(x) - \gamma p_{n-4}(x).$$

We have

$$p_n = (x-\alpha)p_{n-1} - (\alpha+\beta)p_{n-2} - (\alpha+\beta+\gamma)p_{n-3} - (\alpha+\beta+\gamma)p_{n-4} - \cdots - (\alpha+\beta+\gamma)p_1 - (\beta+\delta+\gamma).$$

Then

$$p_n = (x-\alpha+1)p_{n-1} - p_{n-1} - (\alpha+\beta)p_{n-2} - (\alpha+\beta+\gamma)p_{n-3} - (\alpha+\beta+\gamma)p_{n-4} - \cdots,$$

or

$$\begin{aligned}
p_n = (x-\alpha+1)p_{n-1} - (x-\alpha+1)p_{n-2} + (x+\beta)p_{n-3} + \gamma p_{n-4} - \\
(\alpha+\beta)p_{n-2} - (\alpha+\beta+\gamma)p_{n-3} - \cdots.
\end{aligned}$$

Thus

$$\begin{aligned}
p_n &= (x-\alpha+1)p_{n-1} - (x+\beta+1)p_{n-2} - (-x+\alpha+\gamma)p_{n-3} - (\alpha+\beta)p_{n-4} - \cdots \\
&= (x-\alpha+1)p_{n-1} - (x+\beta)p_{n-2} - p_{n-2} - (-x+\alpha+\gamma)p_{n-3} - (\alpha+\beta)p_{n-4} - \cdots \\
&= (x-\alpha+1)p_{n-1} - (x+\beta)p_{n-2} - (x-\alpha+1)p_{n-3} + (x+\beta)p_{n-4} + \gamma p_{n-5} - \cdots \\
&= (x-\alpha+1)p_{n-1} - (x+\beta)p_{n-2} - \gamma p_{n-3} - p_{n-3} + (x-\alpha)p_{n-4} + \gamma p_{n-5} - \cdots.
\end{aligned}$$

We must now show that

$$-p_{n-3}+(x-\alpha)p_{n-4}-(\alpha+\beta)p_{n-5}-(\alpha+\beta+\gamma)p_{n-6}-\cdots-(\alpha+\beta+\gamma)p_1-(\beta+\delta+\gamma)=0.$$

Since

$$p_{n-3}=(x-\alpha+1)p_{n-4}-(x+\beta)p_{n-5}-\gamma p_{n-6}$$

this reduces to showing that

$$-p_{n-4}+(x-\alpha)p_{n-5}-(\alpha+\beta)p_{n-6}-\cdots-(\alpha+\beta+\gamma)p_1-(\beta+\delta+\gamma)=0.$$

Iterating on decreasing n we get

$$0=0.$$

Thus the family of polynomials $p_n(x)$ satisfy the four-term recurrence with the appropriate initial conditions.

Conversely, if we start with a family of monic polynomials $P_n(x)$ of degree n that satisfy the four-term recurrence

$$P_n(x)=(x-\alpha+1)P_{n-1}(x)-(x+\beta)P_{n-2}(x)-\gamma P_{n-3},$$

with $P_n(x)=0$ for $n<0$, $P_0(x)=1$ and $P_1(x)=x-\delta$, then the Riordan array

$$\left(\frac{1+(\alpha-\delta-1)x}{1+(\alpha-1)x+\beta x^2+\gamma x^3},\frac{x(1-x)}{1+(\alpha-1)x+\beta x^2+\gamma x^3}\right)$$

will coincide with the coefficient array of $P_n(x)$, since clearly $p_n(x)=P_n(x)$ for all n as they have the same initial values and obey the same recurrence.

We now form the matrix

$$\left(1,\frac{x}{1+x}\right)\cdot\left(\frac{1+(\alpha-\delta-1)x}{1+(\alpha-1)x+\beta x^2+\gamma x^3},\frac{x(1-x)}{1+(\alpha-1)x+\beta x^2+\gamma x^3}\right)\cdot\left(\frac{1}{1-x},\frac{x}{1-x}\right).$$

This evaluates to the Riordan array

$$\left(\frac{(1+(\alpha-\delta)x)(1+x)^2}{1+(\alpha+1)x+(2\alpha+\beta)x^2+(\alpha+\beta+\gamma)x^3},\frac{x(1+x)}{1+(\alpha+1)x+(2\alpha+\beta)x^2+(\alpha+\beta+\gamma)x^3}\right).$$

Letting $\gamma=0$, this simplifies to

$$\left(\frac{(1+(\alpha-\delta)x)(1+x)}{1+\alpha x+(\alpha+\beta)x^2},\frac{x}{1+\alpha x+(\alpha+\beta)x^2}\right).$$

But this is the coefficient array of a family of orthogonal polynomials $q_n(x)$ that satisfy

$$q_n(x)=(x-\alpha)q_{n-1}-(\alpha+\beta)q_{n-2}.$$

Now the general term of the Riordan array $\left(1,\frac{x}{1-x}\right)$ is $\binom{n-1}{n-k}$, while that of $\left(\frac{1}{1-x},\frac{x}{1+x}\right)$ is $\binom{n}{k}$. Noting that $(x+1)^n=\sum_{k=0}^n\binom{n}{k}x^k$ (binomial theorem) completes the proof. □

In general, the production matrix of the triple product

$$\left(1, \frac{x}{1+x}\right) \cdot \left(\frac{1+(\alpha-\delta-1)x}{1+(\alpha-1)x+\beta x^2+\gamma x^3}, \frac{x(1-x)}{1+(\alpha-1)x+\beta x^2+\gamma x^3}\right) \cdot \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$$

is equal to

$$\begin{pmatrix} \delta-1 & 1 & 0 & 0 & 0 & 0 & \dots \\ \beta+\delta & \alpha & 1 & 0 & 0 & 0 & \dots \\ \gamma & \alpha+\beta & \alpha & 1 & 0 & 0 & \dots \\ -\gamma & \gamma & \alpha+\beta & \alpha & 1 & 0 & \dots \\ \gamma & -\gamma & \gamma & \alpha+\beta & \alpha & 1 & \dots \\ -\gamma & \gamma & -\gamma & \gamma & \alpha+\beta & \alpha & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that the intermediate product

$$\left(1, \frac{x}{1+x}\right) \cdot \left(\frac{1+(\alpha-\delta-1)x}{1+(\alpha-1)x+\beta x^2+\gamma x^3}, \frac{x(1-x)}{1+(\alpha-1)x+\beta x^2+\gamma x^3}\right)$$

is equal to

$$\left(\frac{(1+(\alpha-\delta)x)(1+x)^2}{1+(\alpha+2)x+(2\alpha+\beta+1)x^2+(\alpha+\beta+\gamma)x^3}, \frac{x(1+x)}{1+(\alpha+2)x+(2\alpha+\beta+1)x^2+(\alpha+\beta+\gamma)x^3}\right).$$

The inverse of this array has production matrix given by

$$\begin{pmatrix} \delta & 1 & 0 & 0 & 0 & 0 & \dots \\ \beta+\delta & \alpha+1 & 1 & 0 & 0 & 0 & \dots \\ \gamma & \alpha+\beta & \alpha+1 & 1 & 0 & 0 & \dots \\ -\gamma & \gamma & \alpha+\beta & \alpha+1 & 1 & 0 & \dots \\ \gamma & -\gamma & \gamma & \alpha+\beta & \alpha+1 & 1 & \dots \\ -\gamma & \gamma & -\gamma & \gamma & \alpha+\beta & \alpha+1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Corollary 2. *When $\gamma = 0$, the polynomials*

$$R_n(x) = \sum_{k=0}^{n} \binom{n-1}{n-k} P_k(x)$$

form the family of orthogonal polynomials satisfying

$$R_n(x) = (x-(\alpha+1))R_{n-1} - (\alpha+\beta)R_{n-2}(x),$$

with $R_0(x) = 1$, $R_1(x) = x - \delta$.

3 Examples and further results

Example 3. We take the Riordan array $(1-x, x(1-x))$ which corresponds to $\alpha=1$, $\delta=1$, $\beta=\gamma=0$. This is the coefficient array for the polynomials $P_n(x)=\sum_{k=0}^n \binom{k+1}{n-k}(-1)^{n-k}x^k$, which satisfy

$$P_n(x)=xP_{n-1}(x)-xP_{n-2},$$

with $P_0(x)=1$, $P_1(x)=x-1$. The inverse of the coefficient array $(1-x, x(1-x))$ is the Catalan array $(c(x), xc(x))$ A033184 which has production matrix

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The first column of $(c(x), xc(x))$ is given by the Catalan numbers A000108, and hence we can regard the Catalan numbers as the generalized moments of the family $P_n(x)$. Now forming the triple product

$$\left(1, \frac{x}{1+x}\right)\cdot(1-x, x(1-x))\cdot\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$$

we obtain the orthogonal polynomial coefficient array given by

$$\left(\frac{1+x}{1+x+x^2}, \frac{x}{1+x+x^2}\right).$$

This is the coefficient array of the orthogonal polynomials $Q_n(x)$ for which

$$Q_n(x)=(x-1)Q_{n-1}(x)-Q_{n-2}(x),$$

with $Q_0(x)=1$, $Q_1(x)=x$. The moments of these polynomials are the so-called "Motzkin sums", A005043. These appear as the elements of the first column of the inverse coefficient array, which has production matrix

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The inverse coefficient array is A089942. We note that the Catalan numbers and the Motzkin sums have the same Hankel transform. This is because the Motzkin sums are the inverse binomial transform of the Catalan numbers.

Proposition 4. *The moments of the orthogonal polynomials $Q_n(x)$ above are the inverse binomial transforms of the generalized moments of the polynomials $P_n(x)$. In particular, both sets of moments have the same Hankel transform.*

Proof. Consider the triple product

$$\left(1, \frac{1}{1+x}\right) \cdot A \cdot \left(\frac{1}{1-x}, \frac{x}{1-x}\right),$$

where A is the coefficient array of $P_n(x)$. The (generalized) moments of $P_n(x)$ are the elements of the first column of A^{-1}. Now the moments of $Q_n(x)$ will be given by the elements of the first column of

$$\left(\left(1, \frac{1}{1+x}\right) \cdot A \cdot \left(\frac{1}{1-x}, \frac{x}{1-x}\right)\right)^{-1} = \left(\frac{1}{1+x}, \frac{x}{1+x}\right) \cdot A^{-1} \cdot \left(1, \frac{x}{1-x}\right).$$

Thus if $A^{-1} = (g, f)$, where g is the generating function of the moments of $P_n(x)$, then the moments of $Q_n(x)$ have a generating function given by

$$\left(\frac{1}{1+x}, \frac{x}{1+x}\right) \cdot g(x) = \frac{1}{1+x} g\left(\frac{x}{1+x}\right),$$

since the first member of $\left(1, \frac{x}{1-x}\right)$ is 1. This proves the first assertion. The Hankel transforms are equal because one sequence is related to another by a binomial transform [10, 13]. $\square$

Note that the above proof actually proves more, since the case of orthogonal $Q_n(x)$ only arises when $\gamma = 0$.

Example 5. We look at the case of the Riordan array

$$\left(\frac{1-x}{1+3x+3x^2+x^3}, \frac{x(1-x)}{1+3x+3x^2+x^3}\right) = \left(\frac{1-x}{(1+x)^3}, \frac{x(1-x)}{(1+x)^3}\right).$$

This is the coefficient array of the polynomials $P_n(x)$ that satisfy

$$P_n(x) = (x-3)P_{n-1}(x) - (x+3)P_{n-2}(x) - P_{n-3}(x),$$

with $P_0(x) = 1$, $P_1(x) = x - 4$. The moments of this family are given by A007297, which begins

$$1, 4, 23, 156, 1162, 9192, 75819, 644908, 5616182, \ldots$$

They count the number of connected graphs on n nodes on a circle without crossing edges. Their g.f. is equal to

$$\frac{1}{x} \mathrm{Rev}_x \frac{x(1-x)}{(1+x)^3}.$$

We remark that $\frac{x(1-x)}{(1+x)^3}$ is the generating function of the signed squares

$$0, 1, -4, 9, -16, 25, \ldots$$

The production matrix of the inverse coefficient array is

$$\begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 7 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 8 & 7 & 4 & 1 & 0 & 0 & \ldots \\ 8 & 8 & 7 & 4 & 1 & 0 & \ldots \\ 8 & 8 & 8 & 7 & 4 & 1 & \ldots \\ 8 & 8 & 8 & 8 & 7 & 4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Passing to the triple product we obtain the Riordan array

$$\left(\frac{(1+x)^2}{1+5x+11x^2+8x^3}, \frac{x(1+x)}{1+5x+11x^2+8x^3}\right).$$

The production matrix of the inverse coefficient array in this case is given by

$$\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 7 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 7 & 4 & 1 & 0 & 0 & \ldots \\ -1 & 1 & 7 & 4 & 1 & 0 & \ldots \\ 1 & -1 & 1 & 7 & 4 & 1 & \ldots \\ -1 & 1 & -1 & 1 & 7 & 4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The moments for this coefficient array begin

$$1, 3, 16, 98, 661, 4731, 35299, \ldots,$$

which are the inverse binomial transform of A007297.

It is instructive to look at the Riordan array

$$\left(\frac{1-x}{1+3x+3x^2}, \frac{x(1-x)}{1+3x+3x^2}\right) = \left(\frac{1-x}{(1+x)^3-x^3}, \frac{x(1-x)}{(1+x)^3-x^3}\right).$$

In this case, $\gamma = 0$. The triple product is then equal to

$$\left(\frac{1+x}{1+4x+7x^2}, \frac{x}{1+4x+7x^2}\right),$$

whose inverse has production matrix

$$\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 7 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 7 & 4 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 7 & 4 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 7 & 4 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 7 & 4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus the triple product is the coefficient array of the orthogonal polynomials $Q_n(x)$ that satisfy

$$Q_n(x) = (x-4)Q_{n-1}(x) - 7Q_{n-2},$$

with $Q_0(x) = 1$, $Q_1(x) = x - 3$. The moments of this family of orthogonal polynomials, which begin

$$1, 3, 16, 97, 648, 4590, 33888, 257925, 2009464, \ldots$$

then have g.f. given by

$$\cfrac{1}{1-3x-\cfrac{7x^2}{1-4x-\cfrac{7x^2}{1-4x-\cfrac{7x^2}{1-4x-\cdots}}}}.$$

The polynomials $P_n(x)$ whose coefficient array is

$$\left(\frac{1-x}{1+3x+3x^2}, \frac{x(1-x)}{1+3x+3x^2}\right)$$

can now be recovered through the formula

$$P_n(x) = \sum_{k=0}^{n} \binom{n-1}{n-k} Q_k(x-1).$$

The moments of the family $P_n(x)$, which begin

$$1, 4, 23, 155, 1145, 8976, 73347, \ldots,$$

and which are the binomial transform of the moments of Q_n, have g.f. given by

$$\cfrac{1}{1-4x-\cfrac{7x^2}{1-5x-\cfrac{7x^2}{1-5x-\cfrac{7x^2}{1-5x-\cdots}}}}.$$

Note that the production matrix of the inverse coefficient array to $\left(\frac{1-x}{1+3x+3x^2}, \frac{x(1-x)}{1+3x+3x^2}\right)$ is given by

$$\left(\begin{array}{ccccccc} 4 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 7 & 4 & 1 & 0 & 0 & 0 & \ldots \\ 7 & 7 & 4 & 1 & 0 & 0 & \ldots \\ 7 & 7 & 7 & 4 & 1 & 0 & \ldots \\ 7 & 7 & 7 & 7 & 4 & 1 & \ldots \\ 7 & 7 & 7 & 7 & 7 & 4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Example 6. The case of $\gamma = 0$ and $\beta + \delta = 0$ is illustrated by the Riordan array

$$\left(\frac{1}{1-x+x^2}, \frac{x(1-x)}{1-x+x^2}\right),$$

where $\alpha = 0$ too. The moments of the polynomials defined by this array are given by $(-1)^n$, with corresponding production matrix

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 1 & 1 & 0 & 1 & \dots \\ 0 & 1 & 1 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The triple product

$$\left(1, \frac{x}{1+x}\right) \cdot \left(\frac{1}{1-x+x^2}, \frac{x(1-x)}{1-x+x^2}\right) \cdot \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$$

then evaluates to

$$\left(\frac{(1+x)^2}{1+x^2}, \frac{x}{1+x^2}\right).$$

The moments of the polynomials defined by this array are given by $(-2)^n$, and the corresponding production matrix is given by

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The zero in the $(2,1)$ position shows that this family is not orthogonal.

Example 7. We illustrate the exceptional case $\gamma = 0$ and $\alpha + \beta = 0$ by taking $\alpha = -1$, $\beta = 1$ and $\delta = -1$. We obtain the Riordan array

$$\left(\frac{1-x}{1-2x+x^2}, \frac{x(1-x)}{1-2x+x^2}\right) = \left(\frac{1}{1-x}, \frac{x}{1-x}\right),$$

which is the Binomial matrix (Pascal's triangle, A007318). The polynomials $P_n(x)$ are given by $P_n(x) = (1+x)^n$, and the moments are $(-1)^n$. The production matrix of the inverse coefficient array (which is the inverse binomial matrix) is given by

$$\left(\begin{array}{ccccccc} -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & -1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & -1 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & -1 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 0 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Forming the triple product we get the Riordan array

$$\left(\frac{1+x}{1-x}, \frac{x}{1-x}\right).$$

The moments of the corresponding polynomials $Q_n(x)$ are then $(-2)^n$, and the associated production array is

$$\left(\begin{array}{ccccccc} -2 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & -1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & -1 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & -1 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 0 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Clearly the polynomials $Q_n(x)$ are not orthogonal. We have $Q_n(x) = (1+x)^{n-1}(x+2)$ for $n > 0$, and $Q_0(x) = 1$.

Since all matrices above are invertible, it is possible to reverse the above triple product process. A case of particular interest is the following.

Proposition 8. *Let $Q_n(x)$ be a family of monic orthogonal polynomials given by*

$$Q_n(x) = (x-\alpha)Q_{n-1} - \beta Q_{n-2}(x),$$

with $Q_0(x) = 1$, $Q_1(x) = x - \alpha + 1$. Then the polynomials

$$P_n(x) = \sum_{k=0}^{n} \binom{n-1}{n-k} Q_k(x-1)$$

satisfy the recurrence

$$P_n(x) = (x-\alpha+1)P_{n-1}(x) - (x+\beta-\alpha)P_{n-2},$$

with $P_0(x) = 1$, $P_1(x) = x - \alpha$.

The coefficient array of the polynomials $Q_n(x)$ in the above proposition is given by the Riordan array

$$\left(\frac{1+x}{1+\alpha x+\beta x^2}, \frac{x}{1+\alpha x+\beta x^2}\right).$$

Forming the triple product

$$\left(1, \frac{x}{1-x}\right) \cdot \left(\frac{1+x}{1+\alpha x+\beta x^2}, \frac{x}{1+\alpha x+\beta x^2}\right) \cdot \left(\frac{1}{1+x}, \frac{x}{1+x}\right)$$

we obtain the coefficient array of the polynomials $P_n(x)$. This is the Riordan array

$$\left(\frac{1-x}{1+(\alpha-1)x+(\beta-\alpha)x^2}, \frac{x(1-x)}{1+(\alpha-1)x+(\beta-\alpha)x^2}\right).$$

The production matrix of the inverse of this array is given by

$$\left(\begin{array}{ccccccc} \alpha & 1 & 0 & 0 & 0 & 0 & \ldots \\ \beta & \alpha & 1 & 0 & 0 & 0 & \ldots \\ \beta & \beta & \alpha & 1 & 0 & 0 & \ldots \\ \beta & \beta & \beta & \alpha & 1 & 0 & \ldots \\ \beta & \beta & \beta & \beta & \alpha & 1 & \ldots \\ \beta & \beta & \beta & \beta & \beta & \alpha & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The production matrix for the inverse of the coefficient array of the orthogonal polynomials $Q_n(x)$ in the proposition is given by

$$\left(\begin{array}{ccccccc} \alpha-1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ \beta & \alpha & 1 & 0 & 0 & 0 & \ldots \\ 0 & \beta & \alpha & 1 & 0 & 0 & \ldots \\ 0 & 0 & \beta & \alpha & 1 & 0 & \ldots \\ 0 & 0 & 0 & \beta & \alpha & 1 & \ldots \\ 0 & 0 & 0 & 0 & \beta & \alpha & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Corollary 9. *The (generalized) moments associated to the polynomials $P_n(x)$ whose coefficient array is given by*

$$\left(\frac{1-x}{1+(\alpha-1)x+(\beta-\alpha)x^2}, \frac{x(1-x)}{1+(\alpha-1)x+(\beta-\alpha)x^2}\right)$$

have generating function expressible as

$$\cfrac{1}{1-\alpha x-\cfrac{\beta x^2}{1-(\alpha+1)x-\cfrac{\beta x^2}{1-(\alpha+1)x-\cdots}}}.$$

Proof. They are the inverse binomial transform of the moments of $Q_n(x)$, which have g.f. given by

$$\cfrac{1}{1-(\alpha-1)x-\cfrac{\beta x^2}{1-\alpha x-\cfrac{\beta x^2}{1-\alpha x-\cdots}}}.$$

The result follows since the inverse binomial transform increases the x coefficient by 1 [3]. $\square$

In fact, we could have proved this result independently of the foregoing.

Proposition 10. *The (generalized) moments of the Riordan array*

$$\left(\frac{1-x}{1+rx+sx^2}, \frac{x(1-x)}{1+rs+sx^2}\right)$$

have g.f.

$$g(x)=\frac{1-rx-\sqrt{1-2x(r+2)+x^2(r^2-4s)}}{2x(1+sx)},$$

which can be expressed as

$$g(x)=\cfrac{1}{1-(r+1)x-\cfrac{(r+s+1)x^2}{1-(r+2)x-\cfrac{(r+s+1)x^2}{1-(r+2)x-\cdots}}}.$$

The production matrix of the inverse array is given by

$$\left(\begin{array}{ccccccc} r+1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ r+s+1 & r+1 & 1 & 0 & 0 & 0 & \ldots \\ r+s+1 & r+s+1 & r+1 & 1 & 0 & 0 & \ldots \\ r+s+1 & r+s+1 & r+s+1 & r+1 & 1 & 0 & \ldots \\ r+s+1 & r+s+1 & r+s+1 & r+s+1 & r+1 & 1 & \ldots \\ r+s+1 & r+s+1 & r+s+1 & r+s+1 & r+s+1 & r+1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The Hankel transform of the moments is $h_n=(r+s+1)^{\binom{n+1}{2}}$.

Proof. We have

$$g(x)=\frac{1}{x}\mathrm{Rev}_x\frac{x(1-x)}{1+rx+sx^2}=\frac{1-rx-\sqrt{1-2x(r+2)+x^2(r^2-4s)}}{2x(1+sx)}.$$

If we let $g_1(x)$ represent the continued fraction, then we have

$$g_1(x)=\frac{1}{1-(r+1)x-(r+s+1)x^2u} \quad \text{where} \quad u=\frac{1}{1-(r+2)x-(r+s+1)x^2u}.$$

Solving for u and then $g_1(x)$ we find that $g_1(x)=g(x)$. The production matrix is calculated using standard Riordan array techniques. The expression for the Hankel transform follows directly from [11]. $\square$

Example 11. We consider the generalized Chebyshev polynomials with coefficient array

$$\left(\frac{1+x}{1+x^2}, \frac{x}{1+x^2}\right).$$

This is the family of orthogonal polynomials $Q_n(x)$ which satisfy

$$Q_n(x) = xQ_{n-1} - Q_{n-2},$$

with $Q_0(x) = 1$, $Q_1(x) = x + 1$. The moments of this family of orthogonal polynomials are given by $u_n = (-1)^n\binom{n}{\lfloor \frac{n}{2} \rfloor}$, the alternating sign version of the central binomial coefficients A001405. We form the polynomials

$$P_n(x) = \sum_{k=0}^{n} \binom{n-1}{n-k} Q_k(x-1).$$

We find that the coefficient array of these transformed polynomials is given by

$$\left(\frac{1-x}{1-x+x^2}, \frac{x(1-x)}{1-x+x^2}\right),$$

whose inverse array has production matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & 1 & \dots \\ 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The first column of the inverse array (generalized moments) is given by the so-called "Motzkin sums", A005043. Again, both moment sequences have the same Hankel transform. This is because both are related by a binomial transform. In this case the polynomials $P_n(x)$ satisfy the recurrence

$$P_n(x) = (x+1)P_{n-1} - (x+1)P_{n-2},$$

with $P_0(x) = 1$, $P_1(x) = x$.

Example 12. We consider the Riordan array

$$\left(\frac{1+x}{1+2x+3x^2}, \frac{x}{1+2x+3x^2}\right).$$

The moments u_n of the orthogonal polynomials $Q_n(x)$ generated by this array A129147 begin

$$1, 1, 4, 13, 52, 214, 928, \dots$$

They can be expressed in closed form as

$$u_n = \sum_{k=0}^{n} \binom{k}{n-k} 2^{n-k} C_k,$$

and have Hankel transform $3^{\binom{n+1}{2}}$. The generating function of this sequence can be expressed as the continued fraction

$$\cfrac{1}{1-x-\cfrac{3x^2}{1-2x-\cfrac{3x^2}{1-2x-\cdots}}}.$$

The corresponding family of polynomials $P_n(x)$ will have coefficient array

$$\left(\frac{1-x}{1+x+x^2}, \frac{x(1-x)}{1+x+x^2}\right).$$

The moments of this family of polynomials is the sequence A064641, which counts certain Łukasiewicz paths [9]. This sequence has a generating function that can be expressed as

$$\cfrac{1}{1-2x-\cfrac{3x^2}{1-3x-\cfrac{3x^2}{1-3x-\cdots}}}.$$

The production matrix of the inverse coefficient array is given by

$$\left(\begin{array}{ccccccc} 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 3 & 2 & 1 & 0 & 0 & 0 & \dots \\ 3 & 3 & 2 & 1 & 0 & 0 & \dots \\ 3 & 3 & 3 & 2 & 1 & 0 & \dots \\ 3 & 3 & 3 & 3 & 2 & 1 & \dots \\ 3 & 3 & 3 & 3 & 3 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Reverting to the initial four-term recurrence, we see that the case of

$$\alpha + \beta + \gamma = 0$$

is worthy of attention. In this case, $\beta + \gamma = -\alpha$, and the original production matrix P_L simplifies to

$$P_L = \left(\begin{array}{ccccccc} \delta & 1 & 0 & 0 & 0 & 0 & \dots \\ \beta+\delta & \alpha & 1 & 0 & 0 & 0 & \dots \\ -\alpha+\delta & \alpha+\beta & \alpha & 1 & 0 & 0 & \dots \\ -\alpha+\delta & 0 & \alpha+\beta & \alpha & 1 & 0 & \dots \\ -\alpha+\delta & 0 & 0 & \alpha+\beta & \alpha & 1 & \dots \\ -\alpha+\delta & 0 & 0 & 0 & \alpha+\beta & \alpha & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Thus in this case we obtain a family of polynomials that are "almost orthogonal". They are defined by

$$P_n(x) = (x-\alpha)P_{n-1}(x) - (\alpha+\beta)P_{n-2} + (\alpha-\delta),$$

with $P_0(x) = 1$, and $P_1(x) = x - \delta$. In the special case $\alpha+\beta+\gamma = 0$ and $\alpha = \delta$, we have

$$\frac{1+(\alpha-\delta-1)x}{1+(\alpha-1)x+\beta x^2+\gamma x^3} = \frac{1-x}{1+(\alpha-1)x+\beta x^2-(\alpha+\beta)x^3} = \frac{1}{1+\alpha x+(\alpha+\beta)x^2}.$$

Hence we obtain the family of orthogonal polynomials

$$P_n(x) = (x-\alpha)P_{n-1}(x) - (\alpha+\beta)P_{n-2},$$

with $P_0(x) = 1$, and $P_1(x) = x - \alpha$. The transformed polynomials

$$Q_n(x) = \sum_{k=0}^{n} (-1)^{n-k} P_k(x+1)$$

then correspond to the Riordan array

$$\left(\frac{1+x}{1+\alpha x}, \frac{x}{1+\alpha x}\right),$$

and can be expressed as

$$Q_n(x) = (x-\alpha)Q_{n-1}(x),$$

with $Q_0(x) = 1$, $Q_1(x) = x - \alpha + 1$. The production matrix of the inverse to the coefficient array $\left(\frac{1+x}{1+\alpha x}, \frac{x}{1+\alpha x}\right)$ is given by

$$P_L = \begin{pmatrix} \alpha-1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & \alpha & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & \alpha & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \alpha & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \alpha & 1 & \ldots \\ 0 & 0 & 0 & 0 & 0 & \alpha & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can express $Q_n(x)$ in closed form as

$$Q_n(x) = \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{1}{n-j} \binom{j}{k} (-\alpha)^{j-k} x^k.$$

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