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Weighted Gcd-Sum Functions

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Abstract

We investigate weighted gcd-sum functions, including the alternating gcd-sum function and those having as weights the binomial coefficients and values of the Gamma function. We also consider the alternating lcm-sum function.

1 Introduction

The gcd-sum function, called also Pillai's arithmetical function (OEIS A018804) is defined by

$$P(n) := \sum_{k=1}^{n} \gcd(k,n) \qquad (n \in \mathbb{N} := \{1,2,\ldots\}). \tag{1}$$

The function P is multiplicative and its arithmetical and analytical properties are determined by the representation

$$P(n) = \sum_{d \mid n} d\, \phi(n/d) \qquad (n \in \mathbb{N}), \tag{2}$$

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where ϕ is Euler's function. See the survey paper [5]. Note that for every prime power p^a ($a \in \mathbb{N}$),

$$P(p^a) = (a+1)p^a - ap^{a-1}. \tag{3}$$

Now let

$$P_{\text{altern}}(n) := \sum_{k=1}^{n} (-1)^{k-1} \gcd(k, n) \qquad (n \in \mathbb{N}) \tag{4}$$

be the alternating gcd-sum function. As far as we know, the function (4) was not considered before.

Furthermore, let

$$P_{\text{binom}}(n) := \sum_{k=1}^{n} \binom{n}{k} \gcd(k, n) \qquad (n \in \mathbb{N}) \tag{5}$$

(OEIS A159068), where $\binom{n}{k}$ are the binomial coefficients. Every term of the sum (5) is a multiple of n, since $\gcd(k, n) = kx + ny$ with suitable integers x, y and $k\binom{n}{k} = n\binom{n-1}{k-1}$ ($1 \le k \le n$). Note also the symmetry $\binom{n}{k} \gcd(k, n) = \binom{n}{n-k} \gcd(n-k, n)$ ($1 \le k \le n-1$).

More generally, consider the weighted gcd-sum functions defined by

$$P_w(n) := \sum_{k=1}^{n} w(k, n) \gcd(k, n) \qquad (n \in \mathbb{N}), \tag{6}$$

where the weights are functions $w : \mathbb{N}^2 \to \mathbb{C}$.

In this paper we evaluate the alternating gcd-sum function $P_{\text{altern}}(n)$, deduce a formula for the function $P_{\text{binom}}(n)$ and investigate other special cases of (6). We also give a formula for the alternating lcm-sum function defined by

$$L_{\text{altern}}(n) := \sum_{k=1}^{n} (-1)^{k-1} \operatorname{lcm}[k, n] \qquad (n \in \mathbb{N}). \tag{7}$$

Similar results can be derived for the weighted versions of certain analogs and generalizations of the gcd-sum function, see [5], but we confine ourselves to the function (6).

2 General results

We first give the following simple result.

Proposition 1. *i) Let $w : \mathbb{N}^2 \to \mathbb{C}$ be an arbitrary function. Then*

$$P_w(n) = \sum_{d \mid n} \phi(d) \sum_{j=1}^{n/d} w(dj, n) \qquad (n \in \mathbb{N}). \tag{8}$$

ii) Assume that there is a function $g : (0, 1] \to \mathbb{C}$ such that $w(k, n) = g(k/n)$ ($1 \le k \le n$) and let $G(n) = \sum_{k=1}^{n} g(k/n)$ ($n \in \mathbb{N}$). Then

$$P_w(n) = \sum_{d \mid n} \phi(d) G(n/d) \qquad (n \in \mathbb{N}). \tag{9}$$

Proof. i) Using Gauss' formula $m = \sum_{d|m} \phi(d)$ for $m = \gcd(k,n)$, grouping the terms of (6) and denoting $k = dj$ we obtain at once

$$P_w(n) := \sum_{k=1}^{n} w(k,n) \sum_{d|\gcd(k,n)} \phi(d) = \sum_{d|n} \phi(d) \sum_{j=1}^{n/d} w(dj,n).$$

ii) Use (8) and that

$$\sum_{j=1}^{n/d} w(dj,n) = \sum_{j=1}^{n/d} g(dj/n) = \sum_{j=1}^{n/d} g(j/(n/d)) = G(n/d).$$

□

For $w(k,n) = 1$ we reobtain formula formula (2). In the next section we investigate other special cases, including those already mentioned in the Introduction.

Remark 2. Consider the function

$$R_w(n) := \sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} w(k,n) \qquad (n \in \mathbb{N}). \tag{10}$$

Then, similar to the proof of i), now with the Möbius μ function instead of ϕ,

$$R_w(n) = \sum_{k=1}^{n} w(k,n) \sum_{d|\gcd(k,n)} \mu(d) = \sum_{d|n} \mu(d) \sum_{j=1}^{n/d} w(dj,n). \tag{11}$$

If condition ii) is satisfied, then we have

$$R_w(n) = \sum_{d|n} \mu(d) G(n/d) \qquad (n \in \mathbb{N}). \tag{12}$$

We will also point out some special cases of (11) and (12).

3 Special cases

3.1 Alternating gcd-sum function

Let $w(k,n) = (-1)^{k-1}$ $(k,n \in \mathbb{N})$. Then we have the function $P_{\text{altern}}(n)$ defined by (4).

Proposition 3. *Let $n \in \mathbb{N}$ and write $n = 2^a m$, where $a \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ and $m \in \mathbb{N}$ is odd. Then*

$$P_{\text{altern}}(n) = \begin{cases} n, & \text{if } n \text{ is odd } (a=0); \\ -2^{a-1} a P(m) = -\frac{a}{a+2} P(n), & \text{if } n \text{ is even } (a \geq 1). \end{cases} \tag{13}$$

Proof. Use formula (8). Here

$$S_d(n) := \sum_{j=1}^{n/d} w(dj, n) = \sum_{j=1}^{n/d} (-1)^{dj-1} = -\sum_{j=1}^{n/d} (-1)^{dj}.$$

If n is odd, then every divisor d of n is also odd and obtain $S_d(n) = -\sum_{j=1}^{n/d}(-1)^j = 1$, where n/d is odd. Hence, $P_{\text{altern}}(n) = \sum_{d|n} \phi(d) = n$.

Now let n be even and let $d \mid n$. For d odd, $S_d(n) = -\sum_{j=1}^{n/d}(-1)^j = 0$, since n/d is even. For d even, $S_d(n) = -\sum_{j=1}^{n/d} 1 = -n/d$. We obtain that

$$P_{\text{altern}}(n) = -\sum_{\substack{d|n \\ d \text{ even}}} \phi(d)\frac{n}{d} = -\sum_{d|n} \phi(d)\frac{n}{d} + \sum_{\substack{d|n \\ d \text{ odd}}} \phi(d)\frac{n}{d},$$

where the first sum is $P(n)$ (cf. (2)), and the second one is

$$\sum_{d|m} \phi(d)\frac{2^a m}{d} = 2^a P(m).$$

Using (3), $P(n) = P(2^a)P(m) = 2^{a-1}(a+2)P(m)$ and deduce

$$\begin{aligned} P_{\text{altern}}(n) &= -P(n) + 2^a P(m) = P(m)(2^a - 2^{a-1}(a+2)) \\ &= -2^{a-1} a P(m) = -\frac{a}{a+2} P(n). \end{aligned}$$

□

Remark 4. More generally, consider the polynomial

$$f_n(x) := \sum_{k=1}^{n} \gcd(k, n) x^{k-1}, \tag{14}$$

i.e., put $w(k, n) = x^{k-1}$ (formally). Then $f_n(1) = P(n)$, $f_n(-1) = P_{\text{altern}}(n)$ and deduce from Proposition 1,

$$f_n(x) := (1 - x^n) \sum_{d|n} \frac{\phi(d) x^{d-1}}{1 - x^d}. \tag{15}$$

3.2 Logarithms as weights

Let

$$P_{\log}(n) := \sum_{k=1}^{n} (\log k) \gcd(k, n). \tag{16}$$

Proposition 5. *For every $n \in \mathbb{N}$,*

$$P_{\log}(n) = P(n) \log n + \sum_{d|n} \log(d!/d^d) \phi(n/d). \tag{17}$$

Proof. Apply formula (8). For $w(k,n) = \log k$,

$$\sum_{j=1}^{n/d} w(dj,n) = \sum_{j=1}^{n/d} \log(dj) = \frac{n}{d}\log d + \log\left(\frac{n}{d}\right)!,$$

hence

$$P_{\log}(n) = \sum_{d|n} \phi(d)\left(\frac{n}{d}\log d + \log\left(\frac{n}{d}\right)!\right),$$

and a short computation leads to (17). □

Remark 6. Writing the exponential form of (17),

$$\prod_{k=1}^{n} k^{\gcd(k,n)} = n^{P(n)} \prod_{d|n} \left(\frac{d!}{d^d}\right)^{\phi(n/d)}. \tag{18}$$

Compare this to the known formula

$$\prod_{\substack{k=1\\ \gcd(k,n)=1}}^{n} k = n^{\phi(n)} \prod_{d|n} \left(\frac{d!}{d^d}\right)^{\mu(n/d)}, \tag{19}$$

cf. [2, p. 197, Ex. 24] (OEIS A001783).

3.3 Discrete Fourier transform of the gcd's

Consider $w(k,n) = \exp(2\pi ikr/n)$ $(k,n \in \mathbb{N})$, where $r \in \mathbb{Z}$, and denote

$$P_{\mathrm{DFT}}^{(r)}(n) := \sum_{k=1}^{n} \exp(2\pi ikr/n)\gcd(k,n), \tag{20}$$

representing the discrete Fourier transform of the function $f(k) = \gcd(k,n)$ $(k \in \mathbb{N})$.

Proposition 7. *For every $n \in \mathbb{N}$, $r \in \mathbb{Z}$,*

$$P_{\mathrm{DFT}}^{(r)}(n) = \sum_{d|\gcd(n,r)} d\,\phi(n/d). \tag{21}$$

Proof. Here $\exp(2\pi ikr/n) = g(k/n)$ with $g(x) = \exp(2\pi irx)$. Using formula (9) and that

$$\sum_{k=1}^{n} \exp(2\pi irk/n) = \begin{cases} n, & \text{if } n \mid r; \\ 0, & \text{otherwise;} \end{cases}$$

we obtain

$$P_{\mathrm{DFT}}^{(r)}(n) = \sum_{d|n,n/d|r} \phi(d)\frac{n}{d} = \sum_{d|n,d|r} d\phi(n/d).$$

□

Remark 8. Formula (21) can be written in the form

$$P^{(r)}_{\mathrm{DFT}}(n) = \sum_{d\mid n} d c_{n/d}(r), \tag{22}$$

where $c_n(k)$ are the Ramanujan sums. Furthermore, (22) can be extended for r-even functions. See [4], [6, Prop. 2]. Note that in the present treatment we do not need properties of the Ramanujan sums and of r-even functions.

For $r = 0$ (more generally, in case $n \mid r$) we reobtain from (21) formula (2). For $r = 1$ we deduce

$$\sum_{k=1}^{n} \exp(2\pi i k/n) \gcd(k,n) = \phi(n) \qquad (n \in \mathbb{N}), \tag{23}$$

which gives by writing the real and the imaginary parts, respectively,

$$\sum_{k=1}^{n} \cos(2\pi k/n) \gcd(k,n) = \phi(n) \qquad (n \in \mathbb{N}), \tag{24}$$

$$\sum_{k=1}^{n} \sin(2\pi k/n) \gcd(k,n) = 0 \qquad (n \in \mathbb{N}), \tag{25}$$

similar relations being valid for $\gcd(n, r) = 1$.

Formulae (23), (24), (25) were pointed out in [4, Ex. 3].

3.4 Binomial coefficients as weights

Let $w(k,n) = \binom{n}{k}$ $(k, n \in \mathbb{N})$. Then we have the function $P_{\mathrm{binom}}(n)$ defined by (5).

Proposition 9. *For every $n \in \mathbb{N}$,*

$$P_{\mathrm{binom}}(n) = 2^n \sum_{d\mid n} \frac{\phi(d)}{d} \sum_{\ell=1}^{d} (-1)^{\ell} \cos^n(\ell\pi/d) - n. \tag{26}$$

Proof. Let $\varepsilon_r^j = \exp(2\pi i j/r)$ $(1 \le j \le r)$ denote the r-th roots of unity. Using the known identity

$$\sum_{k=0}^{\lfloor n/r \rfloor} \binom{n}{kr} = \frac{1}{r} \sum_{j=1}^{r} (1 + \varepsilon_r^j)^n = \frac{2^n}{r} \sum_{j=1}^{r} \cos^n(j\pi/r) \cos(nj\pi/r) \qquad (n, r \in \mathbb{N}), \tag{27}$$

cf. [1, p. 84], and applying (8) we deduce

$$P_{\mathrm{binom}}(n) = \sum_{d\mid n} \phi(d) \sum_{j=1}^{n/d} \binom{n}{dj} = \sum_{d\mid n} \phi(d) \left(\frac{2^n}{d} \sum_{\ell=1}^{d} \cos^n(\ell\pi/d) \cos(n\ell\pi/d) - 1 \right)$$

$$= 2^n \sum_{d\mid n} \frac{\phi(d)}{d} \sum_{\ell=1}^{d} (-1)^{\ell} \cos^n(\ell\pi/d) - \sum_{d\mid n} \phi(d),$$

giving (26). □

Note that (11) and (27) lead to the following formula for the sequence OEIS A056188:

$$R_{\text{binom}}(n) := \sum_{\substack{k=1\\ \gcd(k,n)=1}}^{n} \binom{n}{k} = 2^n \sum_{d\mid n} \frac{\mu(d)}{d} \sum_{\ell=1}^{d} (-1)^{\ell} \cos^n(\ell\pi/d) \qquad (n>1). \tag{28}$$

3.5 Weights concerning the Gamma function

Now let

$$P_{\text{Gamma}}(n) := \sum_{k=1}^{n} \log \Gamma\left(\frac{k}{n}\right) \gcd(k,n), \tag{29}$$

where Γ is the Gamma function.

Proposition 10. *For every $n \in \mathbb{N}$,*

$$P_{\text{Gamma}}(n) = \frac{\log 2\pi}{2}\left(P(n)-n\right) - \frac{1}{2} n \log n + \frac{1}{2}\sum_{d\mid n} \phi(d) \log d. \tag{30}$$

Proof. This follows by (9) and by

$$\prod_{k=1}^{n} \Gamma\left(\frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-1/2}, \qquad (n \in \mathbb{N}),$$

which is a consequence of Gauss' multiplication formula. □

Remark 11. (30) can be written also as

$$P_{\text{Gamma}}(n) = \frac{\log 2\pi}{2}\left(P(n)-n\right) - \frac{1}{2}(\phi * \log)(n), \tag{31}$$

where $*$ deotes the Dirichlet convolution. Note that $\phi * \log = \mu * \mathrm{id} * \log = \Lambda * \mathrm{id}$, where $\mathrm{id}(n) = n$ ($n \in \mathbb{N}$) and Λ is the von Mangoldt function.

Writing the exponential form,

$$\prod_{k=1}^{n} \left(\Gamma\left(\frac{k}{n}\right)\right)^{\gcd(k,n)} = (2\pi)^{(P(n)-n)/2} n^{-n/2} \prod_{d\mid n} d^{\phi(d)/2}. \tag{32}$$

Compare this to the following formula given in [3]:

$$\prod_{\substack{k=1\\ \gcd(k,n)=1}}^{n} \Gamma\left(\frac{k}{n}\right) = \frac{(2\pi)^{\phi(n)/2}}{\exp(\Lambda(n)/2)} = \begin{cases} (2\pi)^{\phi(n)/2}/\sqrt{p}, & n = p^a \text{ (a prime power);} \\ (2\pi)^{\phi(n)/2}, & \text{otherwise.} \end{cases} \tag{33}$$

3.6 Further special cases

It is possible to investigate other special cases, too. As examples we give the next ones with weights regarding, among others, the floor function $\lfloor \bullet \rfloor$, and the saw-tooth function ψ defined as $\psi(x) = x - \lfloor x \rfloor - 1/2$ for $x \in \mathbb{R} \setminus \mathbb{Z}$ and $\psi(x) = 0$ for $x \in \mathbb{Z}$.

Proposition 12. *For every $n \in \mathbb{N}$,*

$$P_{\mathrm{id}}(n) := \sum_{k=1}^{n} k \gcd(k, n) = \frac{n}{2}(P(n) + n). \tag{34}$$

Proposition 13. *For every $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$,*

$$P_{\mathrm{floor}}(n) := \sum_{k=1}^{n} \left\lfloor \alpha + \frac{k}{n} \right\rfloor \gcd(k, n) = \sum_{d \mid n} \phi(d) \left\lfloor \frac{n\alpha}{d} \right\rfloor. \tag{35}$$

Proposition 14. *For every $n, r \in \mathbb{N}$,*

$$P^{(r)}_{\text{saw-tooth}}(n) := \sum_{k=1}^{n} \psi(kr/n) \gcd(k, n) = 0. \tag{36}$$

Proposition 15. *For every $n \in \mathbb{N}, n > 1$,*

$$P_{\sin}(n) := \sum_{k=1}^{n-1} (\log \sin(k\pi/n)) \gcd(k, n) = (\phi * \log)(n) - (\log 2)(P(n) - n). \tag{37}$$

Proposition 16. *For every $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ with $\alpha + k/n \notin \mathbb{Z}$ $(1 \le k \le n)$,*

$$P_{\cot}(n) := \sum_{k=1}^{n} \cot \pi(\alpha + k/n) \gcd(k, n) = n \sum_{d \mid n} \frac{\phi(d)}{d} \cot(\pi n\alpha/d). \tag{38}$$

These follow from Proposition 1 using the following well-known formulae:

$$\sum_{k=1}^{n} \left\lfloor \alpha + \frac{k}{n} \right\rfloor = \lfloor n\alpha \rfloor, \qquad (n \in \mathbb{N}), \tag{39}$$

$$\sum_{k=1}^{n} \psi(kr/n) = 0 \qquad (n, r \in \mathbb{N}), \tag{40}$$

$$\prod_{k=1}^{n-1} \sin(k\pi/n) = \frac{n}{2^{n-1}} \qquad (n \in \mathbb{N}) \tag{41}$$

(for $n = 1$ the empty product is 1),

$$\sum_{k=1}^{n} \cot \pi(\alpha + k/n) = n \cot \pi n\alpha \qquad (n \in \mathbb{N}, \alpha \in \mathbb{R}, \alpha + k/n \notin \mathbb{Z}, 1 \le k \le n). \tag{42}$$

4 The alternating lcm-sum function

Some of the previous results have counterparts for the lcm-sum function (OEIS A051193)

$$L(n) := \sum_{k=1}^{n} \operatorname{lcm}[k,n] = \frac{n}{2}\left(1+\sum_{d\mid n} d\phi(d)\right) \qquad (n\in\mathbb{N}). \tag{43}$$

We consider here the alternating lcm-sum function defined by (7) and then the analog of (18).

Let $F(n) := \frac{1}{n}\sum_{d\mid n} d\phi(d)$. Note that $F(n) = \sum_{k=1}^{n}(\gcd(k,n))^{-1}$ representing the arithmetic mean of the orders of elements in the cyclic group of order n, cf. [5, p. 3]. Furthermore, let $\beta(n) := (\mathbf{1} * \mu\,\mathrm{id})(n) = \prod_{d\mid n}(1-p)$ and let $h(n) := \prod_{k=1}^{n} k^k$ be the sequence of hyperfactorials (OEIS A002109).

Proposition 17. *Let $n\in\mathbb{N}$. If n is odd, then*

$$L_{\text{altern}}(n) = \frac{n}{2}\left(1+\sum_{d\mid n} d\mu(d)\tau(n/d)\right) = \frac{n}{2}\left(1+\prod_{p^a\|n}(a(1-p)+1)\right), \tag{44}$$

where τ is the divisor function.

If n is even of the form $n=2^a m$, where $a\ge 1$ and $m\in\mathbb{N}$ is odd, then

$$L_{\text{altern}}(n) = 2^{a-1}m\left(\frac{2^{2a}-1}{3}mF(m)-1\right) = \frac{n}{2}\left(\frac{2^{2a}-1}{2^{2a+1}+1}nF(n)-1\right). \tag{45}$$

Proof. Let $id_{-1}(n)=n^{-1}$ and $\mathbf{1}(n)=1$ $(n\in\mathbb{N})$. We have

$$L_{\text{altern}}(n) = n\sum_{k=1}^{n}(-1)^{k-1}k\frac{1}{\gcd(k,n)} = n\sum_{k=1}^{n}(-1)^{k-1}k\sum_{d\mid\gcd(k,n)}(\mathrm{id}_{-1}*\mu)(d)$$

$$= n\sum_{d\mid n}\beta(d)\sum_{j=1}^{n/d}(-1)^{dj-1}j.$$

Now using that $\sum_{k=1}^{n}(-1)^{k-1}k = (-1)^{n-1}\lfloor (n+1)/2\rfloor$ $(n\in\mathbb{N})$ the given formulae are obtained along the same lines with the proof of Proposition 3. □

Proposition 18. *For every $n\in\mathbb{N}$,*

$$\left(\prod_{k=1}^{n} k^{\operatorname{lcm}[k,n]}\right)^{1/n} = \prod_{d\mid n} h(n/d)^{\beta(d)}\left(\prod_{d\mid n} d^{\beta(d)/d}\right)^{n/2}\left(\prod_{d\mid n} d^{\beta(d)/d^2}\right)^{n^2/2}. \tag{46}$$

Proof. Similar to the proofs of above,

$$\sum_{k=1}^{n} (\log k) \operatorname{lcm}[k,n] = n \sum_{k=1}^{n} (k \log k) \frac{1}{\gcd(k,n)}$$

$$= n \sum_{k=1}^{n} (k \log k) \sum_{d \mid \gcd(k,n)} (\mathrm{id}_{-1} * \mu)(d) = n \sum_{d \mid n} (\mathrm{id}_{-1} * \mu)(d) \sum_{j=1}^{n/d} jd \log(jd)$$

$$= n \sum_{d \mid n} \beta(d) \log h(n/d) + \frac{n^2}{2} \sum_{d \mid n} \beta(d) \frac{\log d}{d} + \frac{n^3}{2} \sum_{d \mid n} \beta(d) \frac{\log d}{d^2},$$

equivalent to (46). □

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