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Binomial Coefficient Predictors

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Abstract

For a prime p and nonnegative integers n, k, consider the set $A^{(p)}_{n,k} = \{x \in [0, 1, ..., n] : p^k || \binom{n}{x}\}$. Let the expansion of $n+1$ in base p be $n+1 = \alpha_0 p^\nu + \alpha_1 p^{\nu-1} + \cdots + \alpha_\nu$, where $0 \le \alpha_i \le p-1, i = 0, \ldots, \nu$. Then n is called a *binomial coefficient predictor in base* p(p-BCP), if $|A^{(p)}_{n,k}| = \alpha_k p^{\nu-k}, k = 0, 1, \ldots, \nu$. We give a full description of the p-BCP's in every base p.

1 Introduction

Let p be a prime. For nonnegative integers n, k, consider the set

$$A^{(p)}_{n,k} = \{x \in [0, 1, \ldots, n] : p^k || \binom{n}{x}\}. \tag{1}$$

A natural question that arises in connection with the set (1) is the following.

Question 1. How, knowing n, can we find the finite sequence

$$|A^{(p)}_{n,0}|, |A^{(p)}_{n,1}|, \ldots$$

(of course, without direct calculation)?

In 1899, Glaisher [5] found a simple formula for $|A^{(2)}_{n,0}|$. It was the first result on a thorny path of solution of this difficult problem. Unfortunately, this topic was forgotten for almost a half-century. Only in 1947, Fine [3] obtained a generalization of Glaisher's result for $|A^{(p)}_{n,0}|$. 20 years later, Carlitz [1] solved a difficult problem for evaluation of $|A^{(p)}_{n,1}|$ and by this made, after the first Glaisher-Fine results, the first step on a way of creating a *theory* of the question.

Now only four years later Hovard [8] found a full solution of the question in case $p = 2$, by discovering a formula for $|A^{(2)}_{n,k}|$. Since then it became certain that a general solution of the question is possible. Besides, Hovard [9] found a solution for $|A^{(p)}_{n,2}|$. After these results of Hovard there appear many other works. We indicate only papers by Granville [7] Huard, Spearman and Williams [10, 11, 12].

Finally in 2008, Everett [2], using his origin matrix method, obtained a full solution of the question, although, due to its generality, the Everett formula is complicated.

The aim of this paper is to show that there are cases for which Question 1 is solved practically immediately. An essential resource for the discovery of such cases is the famous Kummer theorem on carries which could possibly lead in the future to more general results with simpler formulas. In this paper we select cases for which a solution of Question 1 for an arbitrary p and a fixed n immediately follows from the expansion of $n + 1$ in base p. Conversely, in the limits of these cases, knowing the sequence $\{|A^{(p)}_{n,k}|\}$, we can "predict" the expansion of $n + 1$ in base p. In the connection with this, we introduce the following notion.

Definition 2. Let p be prime and n be a nonnegative integer. A nonnegative integer n is called a *binomial coefficient predictor in base* p(p-BCP) if the numbers $|A^{(p)}_{n,k}|$ are summands of the expansion of $n + 1$ in base p; more exactly, $|A^{(p)}_{n,k}| = \alpha_k p^{\nu-k}, k = 0, 1, \ldots, \nu$, such that the expansion of $n + 1$ in base p is: $n + 1 = \alpha_0 p^\nu + \alpha_1 p^{\nu-1} + \cdots + \alpha_\nu$, where $0 \leq \alpha_i \leq p - 1, i = 0, \ldots, \nu$.

Consider some examples.

Example 3. It is easy to see that, for every p, $n = 0$ is p-BCP.

Indeed, $|A^{(p)}_{0,0}| = 1$, that is the expansion of 1 in every base p.

Example 4. Let $p = 2, n = 11$.

Then $n + 1 = 8 + 4$. The row of the binomial coefficients $\{\binom{n}{x}, x = 0, 1, \ldots, 11\}$ is

$$1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1.$$

Here $|A^{(2)}_{11,0}| = 8, |A^{(2)}_{11,1}| = 4, |A^{(2)}_{11,k}| = 0, k \geq 2$. Thus, by the definition, $n = 11$ is 2-BCP.

Example 5. Let $p = 3, n = 23$. Then $n + 1 = 2 \cdot 3^2 + 2 \cdot 3$.

Here $|A^{(3)}_{23,0}| = 18, |A^{(3)}_{23,1}| = 6, |A^{(3)}_{23,k}| = 0, k \geq 2$. Thus $n = 23$ is 3-BCP.

Below we use the symbol $\vee$ for concatenation.

In this paper we prove the following.

Theorem 6. *A number $n \geq 0$ is a p-BCP if and only if one of the three following conditions holds:*

(a) $n = 0$;

(b) $p \geq 3$ *and the expansion of* n *in base* p *is of the form*

$$i \vee (p - 1) \vee (p - 1) \vee \cdots \vee (p - 1),$$

where $1 \leq i \leq p - 1$;

(c) the expansion of n *in base* p *is of the form*

$$(p - 1) \vee \cdots \vee (p - 1) \vee (p - 2) \vee (p - 1) \vee \cdots \vee (p - 1).$$

Definition 7. A nonnegative integer n is called a Zumkeller number (see sequence A089633 in Sloane's *Encyclopedia* [19]) , *if either it is 0 or its binary expansion has all digits* 1, *except, maybe, one.*

Corollary 8. $n \geq 0$ *is* 2*-BCP if and only if* n *is a Zumkeller number.*

2 Some classical results on binomial coefficients

The binomial coefficients play a very important role in numerous questions of number theory. For example, there is a well-known proof of the beautiful Chebyshev theorem, using the binomial coefficients (see, for example, a version of the proof due to Finsler in [20]). A connection between some questions of divisibility of the binomial coefficients and the old conjecture of the infinity of twin primes was given by the author in [17].

The first important contributions to the theory of binomial coefficients are due to Legendre (1830), Kummer (1852) and Lucas (1878). Let p be a prime and $a_p(n)$ be the exponent such that

$$p^{a_p(n)}||n. \tag{2}$$

Let, furthermore,

$$n = n_0p^m + n_1p^{m-1} + \cdots + n_m, 0 \leq n_i \leq p-1, \tag{3}$$

be the expansion of n in base p. Denote

$$s_p(n) = n_0 + n_1 + \cdots + n_m. \tag{4}$$

A.-M. Legendre (see [14], p.12) proved that (in our notation)

$$a_p(n!) = (n - s_p(n))/(p-1). \tag{5}$$

For another proof, see, e.g., [16]. From (5) we immediately obtain:

$$a_p(\binom{n}{x}) = a_p(n!) - a_p(x!) - a_p((n-x)!) =$$

$$((n - s_p(n)) - (x - s_p(x)) - (n - x - s_p(n-x))/(p-1) =$$

$$(s_p(x) + s_p(n-x) - s_p(n))/(p-1). \tag{6}$$

Example 9.

Since $s_2(2n) = s_2(n)$, then for the central binomial coefficients we find

$$a_2(\binom{2n}{n}) = s_2(n);\ a_2(\binom{2n+1}{n}) = s_2(n+1) - 1. \tag{7}$$

Notice that in [18] was posed a question which remains open up to now.

Question 10. Does the diophantine equation $s_p(n) = s_q(n)$, where $p \neq q$ are fixed primes, have infinitely many solutions?

It easy to see that, by (6), the following equalities are equivalent:

$$s_p((p-1)n) = s_q((q-1)m),$$

$$(p-1)a_p(\binom{pn}{p}) = (q-1)a_q(\binom{qm}{q}).$$

Furthermore, note that a trivial inequality holds (it formally follows from (6))

$$s_p(x+y) \leq s_p(x) + s_p(y).$$

Moreover, using (6), we conclude that the equality holds if and only if $\binom{x+y}{x}$ is not multiple of p.

Now we give another treatment of Question 1. Consider the equation

$$s_p(x) + s_p(n-x) - s_p(n) = k(p-1), x \in [0, 1, \dots, n]. \tag{8}$$

For $k \geq 0$, denote $\lambda_p^{(k)}(n)$ the number of solutions of (8). Thus we see that

$$|A_{n,k}^{(p)}| = \lambda_p^{(k)}(n), k = 0, 1, \dots \tag{9}$$

In 1852, Kummer [13] made an important observation (a proof one can find, e.g., in [4]): *$a_p(\binom{n}{x})$ is the number of carries which appear in adding x and $n-x$ in base p.*

Another important result was obtained by Lucas [15]. He proved that if together with (3)

$$t = t_0 p^m + t_1 p^{m-1} + \dots + t_m, 0 \leq t_i \leq p-1. \tag{10}$$

then

$$\binom{n}{t} \equiv \prod_{i=0}^{m} \binom{n_i}{t_i} \pmod{p}. \tag{11}$$

From this we immediately obtain the next corollary ([3]).

Corollary 11.

$$\lambda_p^{(0)}(n) = (n_0+1)(n_1+1)\cdots(n_m+1). \tag{12}$$

Proof. Indeed, (12) gives the number of all nonzero products in (11), when $0 \leq t_i \leq n_i, i = 0, \dots, m$. Since $t_i, n_i \in [0, \dots, p-1]$, then none of the considered products is divisible by p. $\square$

Note that in the binary case in (12) it is sufficient to consider factors with $n_i = 1$ only. Therefore, we have

$$\lambda_2^{(0)}(n) = 2^{s_2(n)}. \tag{13}$$

The latter is the 1899 result of J. Glaisher. Note that in [6] A. Granville gives an elegant proof of (13), using unexpected new arguments.

3 Proof of necessity

We easily derive a proof of necessity by using *only* the first condition of Definition 2, i.e., from the equality $|A^{(p)}_{n,0}| = \alpha_0 p^\nu$. In connection with (3), note that $\nu \neq m$ only when $n_0 = n_1 = \cdots = n_m = p-1$. In other cases we consider the equality $|A^{(p)}_{n,0}| = \alpha_0 p^m$. In turn, $\alpha_0 \neq n_0 \leq p-2$ only when $n_1 = \cdots = n_m = p-1$. Let now n be p-BCP when $n_0 = p-1$, but not all n_i equal $p-1$. Thus, we have

$$|A^{(p)}_{n,0}| = (p-1)p^m. \tag{14}$$

From this equality and Corollary 11 we conclude that exactly m from $m+1$ brackets in product (12) equal p, while one bracket equals $p-1$. This means that exactly $m-1$ digits from $n_1, \ldots, n_m$ equal to $p-1$, while one digit equals $p-2$. $\square$

4 Proof of sufficiency

Here we use the Kummer theorem in the following equivalent form: $a_p(\binom{n}{x})$ is the number of carries which appear in subtracting x from n in base p.

Let $n \geq 0$ satisfy the conditions of Theorem 6. Evidently, in the trivial case when $n = p^{m+1} - 1 = \underbrace{(p-1) \vee (p-1) \vee \cdots \vee (p-1)}_{m+1}$ we have a p-BCP. Indeed, here $n+1 = p^{m+1}$ and, by (9) and (12), $|A^{(p)}_{n,0}| = p^{m+1}$. On the other hand, for every $x \leq n$, in subtracting x from n in base p we have no carries, that is, $|A^{(p)}_{n,k}| = 0, k \geq 1$. Analogously, we have a p-BCP in case $n = n_0 \vee \underbrace{(p-1) \vee (p-1) \vee \cdots \vee (p-1)}_{m}, n_0 \leq p-2$. Indeed, here $n+1 = (n_0+1)p^m$ and, by (9) and (12), $|A^{(p)}_{n,0}| = (n_0+1)p^m$ and before and again we have no carries in subtracting x from n in base p. Let now n have a unique digit $p-1$ in its expansion in base p. Consider such an n of the general form:

$$n = \underbrace{(p-1) \vee \cdots \vee (p-1)}_{t} \vee (p-2) \vee \underbrace{(p-1) \vee \cdots \vee (p-1)}_{m-t}. \tag{15}$$

Then

$$n+1 = \underbrace{(p-1) \vee \cdots \vee (p-1)}_{t+1} \vee \underbrace{0 \vee \cdots \vee 0}_{m-t} =$$

$$(p-1)(p^m + p^{m-1} + \cdots + p^{m-t}). \tag{16}$$

Let x have the form

$$x = x_0 \vee \cdots \vee x_{t-1} \vee x_t \vee x_{t+1} \vee \cdots \vee x_m \tag{17}$$

in base p.

If $x_t \leq p-2$, then in subtracting x from n in base p we have no carries. Evidently, the number of such x's is

$$|A^{(p)}_{n,0}| = p^t(p-1)p^{m-t} = (p-1)p^m. \tag{18}$$

If $x_t = p-1$, such that also the equality

$$x_t = x_{t-1} = x_{t-2} = \cdots = x_r = p-1, \tag{19}$$

holds, then, in view of $x \le n$, the length of this chain is not more than t, that is,

$$1 \le r \le t. \tag{20}$$

In this case the number of carries is equal to the length of the chain, i.e., $t-r+1$, and, by Kummer's theorem, we have

$$x \in A^{(p)}_{n,t-r+1}. \tag{21}$$

Putting

$$t-r+1 = k, \tag{22}$$

we easily calculate $|A^{(p)}_{n,k}|$.

$$|A^{(p)}_{n,k}| = p^{m+1-k} \cdot \frac{p-1}{p} = (p-1)p^{m-k}, k = 1, \ldots, t, \tag{23}$$

where the factor $\frac{p-1}{p}$ corresponds to the digit x_{r-1}, i.e., the place after the end of chain (19). Comparison of (18) and (23) with (16) shows that (p,n) is a p-BCP. This completes the proof of Theorem 6. $\square$

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