Journal of Integer Sequences, Vol. 14 (2011),
Article 11.2.4

Combinatorial Remarks on the Cyclic Sum Formula for Multiple Zeta Values

Shingo Saito and Tatsushi Tanaka
Faculty of Mathematics
Kyushu University
744, Motooka, Nishi-ku
Fukuoka, 819–0395
Japan
ssaito@math.kyushu-u.ac.jp
t.tanaka@math.kyushu-u.ac.jp

Noriko Wakabayashi
Faculty of Engineering
Kyushu Sangyo University
3–1, Matsukadai 2-chome, Higashi-ku
Fukuoka, 813–8503
Japan
noriko@ip.kyusan-u.ac.jp

Abstract

The multiple zeta values are generalizations of the values of the Riemann zeta function at positive integers. They are known to satisfy a number of relations, among which are the cyclic sum formula. The cyclic sum formula can be stratified via linear operators defined by the second and third authors. We give the number of relations belonging to each stratum by combinatorial arguments.

1 Introduction

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is one of the most important functions in mathematics, and its values $\zeta(k)$ at $k \in \mathbb{Z}_{\geq 2}$ are among the most interesting real numbers. The *multiple zeta values* (MZVs for short) are defined as generalizations of the values $\zeta(k)$:

Definition 1. For $k_1, \ldots, k_l \in \mathbb{Z}_{\geq 1}$ with $k_1 \geq 2$, the *multiple zeta value* $\zeta(k_1, \ldots, k_l)$ is a real number defined by

$$\zeta(k_1, \ldots, k_l) = \sum_{n_1 > \cdots > n_l \geq 1} \frac{1}{n_1^{k_1} \cdots n_l^{k_l}}.$$

Little is known about the irrationality or transcendentality of MZVs; we however know that MZVs abound with relations among themselves, the simplest example being $\zeta(2, 1) = \zeta(3)$.

The *cyclic sum formula* (CSF for short), described in Section 3, is a class of $\mathbb{Q}$-linear relations among MZVs, established by Hoffman-Ohno [3]. Their proof appealed to partial fraction decomposition, whereas another proof given by the second and third authors [5] of the present article proceeded by showing that the CSF is included in Kawashima's relation, which is believed to be rich enough to yield all relations among MZVs.

This present paper is aimed at providing combinatorial arguments to find the ranks of linear operators defined in [5]. In order to facilitate access for both algebraists and combinatorists, we strive to make the exposition as self-contained as possible.

Sets of multi-indices

The study of MZVs inevitably requires frequent use of multi-indices. We here summarize the sets of multi-indices used in this paper. For $k, l \in \mathbb{Z}_{\geq 1}$, put

$$\begin{aligned} I^1_{k,l} &= \{(k_1, \ldots, k_l) \in \mathbb{Z}^l_{\geq 1} \mid k_1 + \cdots + k_l = k\}, \\ I^0_{k,l} &= \{(k_1, \ldots, k_l) \in I^1_{k,l} \mid k_1 \geq 2\}, \\ \check{I}^1_{k,l} &= \{(k_1, \ldots, k_l) \in I^1_{k,l} \mid \text{not all of } k_1, \ldots, k_l \text{ are } 1\}; \end{aligned}$$

for $k \in \mathbb{Z}_{\geq 1}$, put

$$I^1_k = \bigcup_{l=1}^{\infty} I^1_{k,l}, \qquad I^0_k = \bigcup_{l=1}^{\infty} I^0_{k,l}, \qquad \check{I}^1_k = \bigcup_{l=1}^{\infty} \check{I}^1_{k,l};$$

put

$$I^1 = \bigcup_{k,l=1}^{\infty} I^1_{k,l}, \qquad I^0 = \bigcup_{k,l=1}^{\infty} I^0_{k,l}, \qquad \check{I}^1 = \bigcup_{k,l=1}^{\infty} \check{I}^1_{k,l}.$$

The elements of $I^1_{k,l}$ are said to have *weight* k and *depth* l. For each $\boldsymbol{k} = (k_1, \ldots, k_l) \in I^0$ the MZV $\zeta(\boldsymbol{k}) = \zeta(k_1, \ldots, k_l)$ is defined.

2 Hoffman's algebra

In the discussion of MZVs, it is convenient to use the algebra introduced by Hoffman. Let $\mathfrak{H} = \mathbb{Q}\langle x, y \rangle$ be the noncommutative polynomial algebra over $\mathbb{Q}$ in two indeterminates x and

y, and put $\mathfrak{H}^0 = \mathbb{Q} + x\mathfrak{H}y$, which is a subalgebra of $\mathfrak{H}$. Write $z_k = x^{k-1}y \in \mathfrak{H}$ for $k \in \mathbb{Z}_{\geq 1}$, so that

$$\{1\} \cup \{z_{k_1} \cdots z_{k_l} \mid \boldsymbol{k} = (k_1, \ldots, k_l) \in I^0\}$$

is a $\mathbb{Q}$-vector space basis for $\mathfrak{H}^0$. It follows that we may define a $\mathbb{Q}$-linear map $Z\colon \mathfrak{H}^0 \to \mathbb{R}$ by setting $Z(1) = 1$ and $Z(z_{k_1} \cdots z_{k_l}) = \zeta(\boldsymbol{k})$ for $\boldsymbol{k} = (k_1, \ldots, k_l) \in I^0$.

The MZVs are known to fulfill a number of $\mathbb{Q}$-linear relations, each of which corresponds to an element of $\operatorname{Ker} Z \subset \mathfrak{H}^0$. Since Goncharov [1] conjectured that the MZVs $\zeta(\boldsymbol{k})$ with $\boldsymbol{k}$ having different weights are $\mathbb{Q}$-linearly independent, we look at the $\mathbb{Q}$-vector subspaces defined by

$$\begin{aligned} \mathfrak{H}^0_k &= \operatorname{span}_{\mathbb{Q}}\{z_{k_1} \cdots z_{k_l} \mid \boldsymbol{k} = (k_1, \ldots, k_l) \in I^0_k\} \\ &= \{w \in \mathfrak{H}^0 \mid w \text{ is a homogeneous polynomial of degree } k\} \cup \{0\} \subset \mathfrak{H}^0, \\ \mathcal{Z}_k &= Z(\mathfrak{H}^0_k) = \operatorname{span}_{\mathbb{Q}}\{\zeta(\boldsymbol{k}) \mid \boldsymbol{k} \in I^0_k\} \subset \mathbb{R} \end{aligned}$$

for each $k \in \mathbb{Z}_{\geq 1}$.

Let $(d_k)_{k\geq 1}$ be the Padovan sequence (A000931) defined by $d_1 = 0$, $d_2 = d_3 = 1$, and $d_k = d_{k-2} + d_{k-3}$ for $k \geq 4$. Zagier [7] conjectured the following:

Conjecture 2 (Zagier [7])**.** We have $\dim \mathcal{Z}_k = d_k$ for all $k \in \mathbb{Z}_{\geq 1}$.

Let $k \in \mathbb{Z}_{\geq 2}$. In light of the fact that $\dim \mathfrak{H}^0_k = \# I^0_k = 2^{k-2}$, Conjecture 2 means that the MZVs must satisfy plenty of $\mathbb{Q}$-linear relations. Note that Conjecture 2 is equivalent to saying that the restriction $Z|_{\mathfrak{H}^0_k}\colon \mathfrak{H}^0_k \to \mathbb{R}$ of Z to $\mathfrak{H}^0_k$ has rank d_k, which is also equivalent to

$$\dim \operatorname{Ker} Z|_{\mathfrak{H}^0_k} = \dim(\operatorname{Ker} Z \cap \mathfrak{H}^0_k) = 2^{k-2} - d_k.$$

Table 1: Dimensions concerning $Z|_{\mathfrak{H}^0_k}$

k	2	3	4	5	6	7	8	9	10	Sequence Number
$2^{k-2}(= \dim \mathfrak{H}^0_k)$	1	2	4	8	16	32	64	128	256	A000079
$d_k(\overset{?}{=} \operatorname{rank} Z|_{\mathfrak{H}^0_k})$	1	1	1	2	2	3	4	5	7	A000931
$2^{k-2} - d_k(\overset{?}{=} \dim \operatorname{Ker} Z|_{\mathfrak{H}^0_k})$	0	1	3	6	14	29	60	123	249	A038360

Goncharov [2] and Terasoma [6] partially proved Conjecture 2:

Theorem 3 (Goncharov [2], Terasoma [6])**.** *We have* $\dim \mathcal{Z}_k \leq d_k$ *for all* $k \in \mathbb{Z}_{\geq 1}$.

Since their proofs of Theorem 3 resort to algebraic geometry and fails to give concrete relations among MZVs, it still lies at the heart of research to find sufficiently many $\mathbb{Q}$-linear relations. Also, the converse inequality is far from being solved.

3 Cyclic sum formula

Numerous concrete $\mathbb{Q}$-linear relations among MZVs have been obtained so far, and our focus is on the following *cyclic sum formula* (CSF for short), first proved by Hoffman-Ohno [3]:

Theorem 4 (Cyclic sum formula)**.** *If* $(k_1,\ldots,k_l)\in\check{I}^1$*, then*

$$\sum_{j=1}^{l}\sum_{i=1}^{k_j-1}\zeta(k_j-i+1,k_{j+1},\ldots,k_l,k_1,\ldots,k_{j-1},i)=\sum_{j=1}^{l}\zeta(k_j+1,k_{j+1},\ldots,k_l,k_1,\ldots,k_{j-1}).$$

Example 5. The cyclic sum formula for $l=1$ and $k_1=2$ gives $\zeta(2,1)=\zeta(3)$.

In dealing with the CSF, it is convenient to extend the indices of k_j to all $j\in\mathbb{Z}$ by declaring $k_j=k_{j'}$ whenever $j\equiv j' \pmod{l}$. Then the CSF can simply be written as

$$\sum_{j=1}^{l}\sum_{i=1}^{k_j-1}\zeta(k_j-i+1,k_{j+1},\ldots,k_{j+l-1},i)=\sum_{j=1}^{l}\zeta(k_j+1,k_{j+1},\ldots,k_{j+l-1}).$$

This convention will be used tacitly throughout the paper.

In order to describe the CSF in terms of Hoffman's algebra, we write

$$\begin{aligned}\check{\mathfrak{H}}^1&=\operatorname{span}_{\mathbb{Q}}\{z_{k_1}\cdots z_{k_l}\mid \boldsymbol{k}=(k_1,\ldots,k_l)\in\check{I}^1\}\\&=\operatorname{span}_{\mathbb{Q}}\{w\in\mathfrak{H}\mid w\text{ is a monomial ending with }y\text{ but not a power of }y\}\subset\mathfrak{H}\end{aligned}$$

and define a $\mathbb{Q}$-linear map $\rho\colon\check{\mathfrak{H}}^1\to x\mathfrak{H}y\subset\mathfrak{H}^0$ by setting

$$\rho(z_{k_1}\cdots z_{k_l})=\sum_{j=1}^{l}\sum_{i=1}^{k_j-1}z_{k_j-i+1}z_{k_{j+1}}\cdots z_{k_{j+l-1}}z_i-\sum_{j=1}^{l}z_{k_j+1}z_{k_{j+1}}\cdots z_{k_{j+l-1}}$$

for $\boldsymbol{k}=(k_1,\ldots,k_l)\in\check{I}^1$. Then the CSF is equivalent to saying that $\operatorname{Im}\rho\subset\operatorname{Ker}Z$.

For each $k\in\mathbb{Z}_{\geq1}$, if we put

$$\begin{aligned}\check{\mathfrak{H}}^1_k&=\operatorname{span}_{\mathbb{Q}}\{z_{k_1}\cdots z_{k_l}\mid (k_1,\ldots,k_l)\in\check{I}^1_k\}\\&=\{w\in\check{\mathfrak{H}}^1\mid w\text{ is a homogeneous polynomial of degree }k\}\cup\{0\},\end{aligned}$$

then ρ satisfies that $\rho(\check{\mathfrak{H}}^1_k)\subset\mathfrak{H}^0_{k+1}$. Therefore, it follows from Theorem 4 that

$$\rho(\check{\mathfrak{H}}^1_{k-1})\subset\operatorname{Ker}Z\cap\mathfrak{H}^0_k$$

for all $k\in\mathbb{Z}_{\geq2}$.

In view of Conjecture 2, which is equivalent to $\dim(\operatorname{Ker}Z\cap\mathfrak{H}^0_k)=2^{k-2}-d_k$ for $k\in\mathbb{Z}_{\geq2}$, it is natural to ask for $\dim\rho(\check{\mathfrak{H}}^1_{k-1})$ because it can be regarded as the number of relations given by the CSF. The following theorem is known, though the authors have been unable to find a specific reference:

Theorem 6. *For $k \in \mathbb{Z}_{\geq 2}$, we have*

$$\dim \rho(\check{\mathfrak{H}}^1_{k-1}) = \frac{1}{k-1} \sum_{m \mid k-1} \varphi\left(\frac{k-1}{m}\right) 2^m - 2,$$

where φ denotes Euler's totient function.

Although this theorem can be proved rather easily, we omit its proof because it is a special case of our main theorem (Theorem 15).

Table 2: Dimensions concerning the CSF

k	2	3	4	5	6	7	8	9	10	Sequence Number
$2^{k-2} - d_k (\overset{?}{=} \dim(\operatorname{Ker} Z \cap \mathfrak{H}^0_k))$	0	1	3	6	14	29	60	123	249	A038360
$\dim \rho(\check{\mathfrak{H}}^1_{k-1})$	0	1	2	4	6	12	18	34	58	A052823

4 The operator ρ_n and the statement of our main theorem

4.1 The operator ρ_n

The second and third authors [5] of the present article defined linear maps $\rho_n \colon \mathfrak{H} \to \mathfrak{H}$ with the aim of giving an algebraic proof of the CSF by reducing it to Kawashima's relation. We will not elaborate on their proof here, but focus on the stratification of the CSF provided by ρ_n. Note that our usage of indices is different from that in [5]: what we mean by ρ_n is denoted by ρ_{n+1} in [5].

Let $n \in \mathbb{Z}_{\geq 0}$ and consider the $(n+2)$nd tensor power $\mathfrak{H}^{\otimes(n+2)}$ of $\mathfrak{H}$ over $\mathbb{Q}$. We first make $\mathfrak{H}^{\otimes(n+2)}$ an $\mathfrak{H}$-bimodule by setting

$$a \diamond (w_1 \otimes \cdots \otimes w_{n+2}) \diamond b = w_1 b \otimes w_2 \otimes \cdots \otimes w_{n+1} \otimes a w_{n+2}$$

for $a, b, w_1, \ldots, w_{n+2} \in \mathfrak{H}$. Writing $z = x + y$, we define a $\mathbb{Q}$-linear map $\mathcal{C}_n \colon \mathfrak{H} \to \mathfrak{H}^{\otimes(n+2)}$ by setting $\mathcal{C}_n(1) = 0$, $\mathcal{C}_n(x) = x \otimes z^{\otimes n} \otimes y$, $\mathcal{C}_n(y) = -x \otimes z^{\otimes n} \otimes y$, and

$$\mathcal{C}_n(ww') = \mathcal{C}_n(w) \diamond w' + w \diamond \mathcal{C}_n(w').$$

We next define a $\mathbb{Q}$-linear map $M_n \colon \mathfrak{H}^{\otimes(n+2)} \to \mathfrak{H}$ by setting

$$M_n(w_1 \otimes \cdots \otimes w_{n+2}) = w_1 \cdots w_{n+2}.$$

Finally, set $\rho_n = M_n \circ \mathcal{C}_n \colon \mathfrak{H} \to \mathfrak{H}$.

Remark 7. The recurrence relation in the definition of $\mathcal{C}_n$ shows that

$$\mathcal{C}_n(w_1 \cdots w_k) = \sum_{j=1}^{k} \bigl(w_1 \cdots w_{j-1} \diamond \mathcal{C}_n(w_j) \diamond w_{j+1} \cdots w_k\bigr)$$

for $w_1, \ldots, w_k \in \mathfrak{H}$.

Proposition 8. *We have $\rho_0(w) = \rho(w)$ for all $w \in \check{\mathfrak{H}}^1$.*

Proof. We may assume that $w = z_{k_1} \cdots z_{k_l}$ for some $\boldsymbol{k} = (k_1, \ldots, k_l) \in \check{I}^1$. For $k \in \mathbb{Z}_{\geq 1}$, we have

$$\begin{aligned}
\mathcal{C}_0(z_k) = \mathcal{C}_0(x^{k-1}y) &= \sum_{i=1}^{k-1} \bigl(x^{i-1} \diamond \mathcal{C}_0(x) \diamond x^{k-i-1}y\bigr) + x^{k-1} \diamond \mathcal{C}_0(y) \\
&= \sum_{i=1}^{k-1} \bigl(x^{i-1} \diamond (x \otimes y) \diamond x^{k-i-1}y\bigr) + x^{k-1} \diamond (-x \otimes y) \\
&= \sum_{i=1}^{k-1} (x^{k-i}y \otimes x^{i-1}y) - x \otimes x^{k-1}y = \sum_{i=1}^{k-1} (z_{k-i+1} \otimes z_i) - x \otimes z_k.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{C}_0(w) = \mathcal{C}_0(z_{k_1} \cdots z_{k_l}) &= \sum_{j=1}^{l} \bigl(z_{k_1} \cdots z_{k_{j-1}} \diamond \mathcal{C}_0(z_{k_j}) \diamond z_{k_{j+1}} \cdots z_{k_l}\bigr) \\
&= \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} \bigl(z_{k_1} \cdots z_{k_{j-1}} \diamond (z_{k_j-i+1} \otimes z_i) \diamond z_{k_{j+1}} \cdots z_{k_l}\bigr) \\
&\quad - \sum_{j=1}^{l} \bigl(z_{k_1} \cdots z_{k_{j-1}} \diamond (x \otimes z_{k_j}) \diamond z_{k_{j+1}} \cdots z_{k_l}\bigr) \\
&= \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} (z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} \otimes z_{k_1} \cdots z_{k_{j-1}} z_i) - \sum_{j=1}^{l} (x z_{k_{j+1}} \cdots z_{k_l} \otimes z_{k_1} \cdots z_{k_j}),
\end{aligned}$$

and so

$$\begin{aligned}
\rho_0(w) &= M_0\bigl(\mathcal{C}_0(w)\bigr) \\
&= \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} z_{k_1} \cdots z_{k_{j-1}} z_i - \sum_{j=1}^{l} x z_{k_{j+1}} \cdots z_{k_l} z_{k_1} \cdots z_{k_j} \\
&= \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_{j+l-1}} z_i - \sum_{j=1}^{l} x z_{k_{j+1}} \cdots z_{k_{j+l}},
\end{aligned}$$

which is equal to $\rho(w)$ because

$$\sum_{j=1}^{l} x z_{k_{j+1}} \cdots z_{k_{j+l}} = \sum_{j=1}^{l} x z_{k_j} \cdots z_{k_{j+l-1}} = \sum_{j=1}^{l} z_{k_j+1} z_{k_{j+1}} \cdots z_{k_{j+l-1}}.$$

$\square$

4.2 Properties of ρ_n

Definition 9. Define $\operatorname{sgn}\colon \{x,y,z\} \to \{1,-1,0\}$ by

$$\operatorname{sgn}(u) = \begin{cases} 1 & \text{if } u = x; \\ -1 & \text{if } u = y; \\ 0 & \text{if } u = z, \end{cases}$$

so that $\mathcal{C}_n(u) = \operatorname{sgn}(u)(x \otimes z^{\otimes n} \otimes y)$ for $u \in \{x,y,z\}$.

Lemma 10. *If $w = u_1 \cdots u_k$, where $u_1, \ldots, u_k \in \{x,y,z\}$, then*

$$\rho_n(w) = \sum_{j=1}^{k} \operatorname{sgn}(u_j) x u_{j+1} \cdots u_k z^n u_1 \cdots u_{j-1} y.$$

Proof. We have

$$\begin{aligned}
\mathcal{C}_n(w) = \mathcal{C}_n(u_1 \cdots u_k) &= \sum_{j=1}^{k} \bigl(u_1 \cdots u_{j-1} \diamond \mathcal{C}_n(u_j) \diamond u_{j+1} \cdots u_k\bigr) \\
&= \sum_{j=1}^{k} \operatorname{sgn}(u_j) \bigl(u_1 \cdots u_{j-1} \diamond (x \otimes z^{\otimes n} \otimes y) \diamond u_{j+1} \cdots u_k\bigr) \\
&= \sum_{j=1}^{k} \operatorname{sgn}(u_j)(x u_{j+1} \cdots u_k \otimes z^{\otimes n} \otimes u_1 \cdots u_{j-1} y),
\end{aligned}$$

and so

$$\rho_n(w) = M_n\bigl(\mathcal{C}_n(w)\bigr) = \sum_{j=1}^{k} \operatorname{sgn}(u_j) x u_{j+1} \cdots u_k z^n u_1 \cdots u_{j-1} y. \qquad \square$$

Proposition 11. *We have*

$$\rho_{n+1}(w) = \rho_n(zw)$$

for all $w \in \mathfrak{H}$.

Proof. We may assume that $w = u_1 \cdots u_k$ for some $u_1, \ldots, u_k \in \{x,y\}$. Then Lemma 10 shows that

$$\begin{aligned}
\rho_n(zw) &= \operatorname{sgn}(z) x u_1 \cdots u_k z^n y + \sum_{j=1}^{k} \operatorname{sgn}(u_j) x u_{j+1} \cdots u_k z^n z u_1 \cdots u_{j-1} y \\
&= \sum_{j=1}^{k} \operatorname{sgn}(u_j) x u_{j+1} \cdots u_k z^{n+1} u_1 \cdots u_{j-1} y = \rho_{n+1}(w).
\end{aligned} \qquad \square$$

Corollary 12. *We have*

$$\{0\} = \rho_{k-2}(\check{\mathfrak{H}}^1_1) \subset \rho_{k-3}(\check{\mathfrak{H}}^1_2) \subset \cdots \subset \rho_0(\check{\mathfrak{H}}^1_{k-1}) = \rho(\check{\mathfrak{H}}^1_{k-1}).$$

for $k \in \mathbb{Z}_{\geq 2}$.

Proof. For each $n \in \{0, \ldots, k-3\}$, we have $\rho_{n+1}(\check{\mathfrak{H}}^1_{k-n-2}) \subset \rho_n(\check{\mathfrak{H}}^1_{k-n-1})$ by Proposition 11 and by the fact that if $w \in \check{\mathfrak{H}}^1_{k-n-2}$, then $zw \in \check{\mathfrak{H}}^1_{k-n-1}$. $\square$

4.3 Statement of our main theorem

Corollary 12 can be interpreted as stratifying the $\mathbb{Q}$-linear relations provided by the CSF. Since Theorem 6 tells us the dimension of the whole space, we may well wish to find the dimensions of the subspaces $\rho_n(\check{\mathfrak{H}}_k^1)$ in general. Our main theorem (Theorem 15) provides a complete solution to this problem, and it uses the following generalization of the Lucas sequence:

Definition 13. For $n \in \mathbb{Z}_{\geq 1}$, the *n-step Lucas sequence* $(L_m^n)_{m\geq 1}$ is defined by

$$L_m^n = \begin{cases} 2^m - 1, & \text{for } m = 1, \dots, n; \\ L_{m-1}^n + \dots + L_{m-n}^n, & \text{for } m \geq n+1. \end{cases}$$

We adopt the convention that $L_m^0 = 0$ for all $m \in \mathbb{Z}_{\geq 1}$.

Table 3: n-step Lucas sequences

n	1	2	3	4	5	6	7	Sequence Number
L_m^1	1	1	1	1	1	1	1	A000012
L_m^2	1	3	4	7	11	18	29	A000032, A000204
L_m^3	1	3	7	11	21	39	71	A001644
L_m^4	1	3	7	15	26	51	99	A073817, A001648
L_m^5	1	3	7	15	31	57	113	A074048, A023424

Lemma 14. *Let $n \in \mathbb{Z}_{\geq 0}$.*

1. *We have $L_{n+1}^n = 2^{n+1} - n - 2$.*
2. *We have $L_m^n = 2L_{m-1}^n - L_{m-n-1}^n$ for $m \geq n+2$.*

Proof. Since both equations are obvious for $n = 0$, we assume that $n \geq 1$.

1. We have $L_{n+1}^n = \sum_{m=1}^n L_m^n = \sum_{m=1}^n (2^m - 1) = 2^{n+1} - n - 2$.
2. If $m \geq n+2$, then the recurrence relation shows that

$$\begin{aligned} L_m^n &= L_{m-1}^n + \dots + L_{m-n}^n = L_{m-1}^n + (L_{m-2}^n + \dots + L_{m-n-1}^n) - L_{m-n-1}^n \\ &= 2L_{m-1}^n - L_{m-n-1}^n. \end{aligned}$$

□

Theorem 15 (Main Theorem)**.** *If $n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 1}$, then*

$$\dim \rho_n(\check{\mathfrak{H}}_k^1) = \frac{1}{n+k} \sum_{m \mid n+k} \varphi\left(\frac{n+k}{m}\right)(2^m - L_m^n) - 2.$$

Remark 16. We may easily see that this theorem is a generalization of Theorem 6.

5 Proof of our main theorem

5.1 Cyclic equivalence and ρ_0

Definition 17. Let $l \in \mathbb{Z}_{\geq 1}$. Set $X_l = \{y, z\}^l$, and let the cyclic group $\mathbb{Z}/l\mathbb{Z}$ of order l act on X_l by cyclic shifts, i.e.,

$$j(u_1, \ldots, u_l) = (u_{j+1}, \ldots, u_{j+l})$$

for $j \in \mathbb{Z}/l\mathbb{Z}$ and $(u_1, \ldots, u_l) \in X_l$. The equivalence relation on X_l induced by the action is called the *cyclic equivalence* and denoted by $\sim$. Put $Y_l = X_l/{\sim}$.

Proposition 18. *For each $l \in \mathbb{Z}_{\geq 1}$, we may define $\tilde{\rho}_0 \colon Y_l \to \mathfrak{H}$ by setting*

$$\tilde{\rho}_0\big([(u_1, \ldots, u_l)]\big) = \rho_0(u_1 \cdots u_l)$$

for $(u_1, \ldots, u_l) \in X_l$.

Proof. Lemma 10 shows that if $(u_1, \ldots, u_l) \in X_l$, then

$$\begin{aligned}
\rho_0(u_1 \cdots u_l) &= \sum_{j=1}^{l} \operatorname{sgn}(u_j) x u_{j+1} \cdots u_l u_1 \cdots u_{j-1} y \\
&= \sum_{\substack{j=1,\ldots,l \\ (v_1,\ldots,v_l)=j(u_1,\ldots,u_l)}} \operatorname{sgn}(v_1) x v_2 \cdots v_l y.
\end{aligned}$$

It follows that $\rho_0(u_1 \cdots u_l)$ depends only on the equivalence class $[(u_1, \ldots, u_l)] \in Y_l$, as required. ∎

Example 19. For $l = 4$, we have

$$\begin{aligned}
\tilde{\rho}_0\big([(z, z, z, z)]\big) &= 0, \\
\tilde{\rho}_0\big([(z, z, z, y)]\big) &= -xzzzy, \\
\tilde{\rho}_0\big([(z, z, y, y)]\big) &= -x(yzz + zzy)y, \\
\tilde{\rho}_0\big([(z, y, z, y)]\big) &= -x(zyz + zyz)y, \\
\tilde{\rho}_0\big([(z, y, y, y)]\big) &= -x(yyz + yzy + zyy)y, \\
\tilde{\rho}_0\big([(y, y, y, y)]\big) &= -x(yyy + yyy + yyy + yyy)y.
\end{aligned}$$

Proposition 20. *The family $\big\{\tilde{\rho}_0(U) \bigm| U \in Y_l \setminus \{[(\underbrace{z, \ldots, z}_{l})]\}\big\}$ is $\mathbb{Q}$-linearly independent.*

Proof. This is because the values $\tilde{\rho}_0(U)$ for different equivalence classes $U \in Y_l \setminus \{[(z, \ldots, z)]\}$ are nonzero and consist of different monomials. ∎

5.2 Relationship between ρ_n and ρ_0

Definition 21. For $l \in \mathbb{Z}_{\geq 1}$ and $n \in \{0, \ldots, l\}$, we write $X_{l,n}$ for the subset of X_l consisting of all $(u_1, \ldots, u_l) \in X_l$ that contain at least n consecutive z's when written cyclically.

Example 22. For $l = 4$, we have

$$\begin{aligned}
X_{4,0} = \{&(z,z,z,z),\\
&(z,z,z,y),(z,z,y,z),(z,y,z,z),(y,z,z,z),\\
&(z,z,y,y),(z,y,y,z),(y,y,z,z),(y,z,z,y),\\
&(z,y,z,y),(y,z,y,z),\\
&(z,y,y,y),(y,y,y,z),(y,y,z,y),(y,z,y,y),\\
&(y,y,y,y)\}\\
= {}&[(z,z,z,z)] \cup [(z,z,z,y)] \cup [(z,z,y,y)] \cup [(z,y,z,y)] \cup [(z,y,y,y)] \cup [(y,y,y,y)]\\
= {}&X_4,\\
X_{4,1} = \{&(z,z,z,z),\\
&(z,z,z,y),(z,z,y,z),(z,y,z,z),(y,z,z,z),\\
&(z,z,y,y),(z,y,y,z),(y,y,z,z),(y,z,z,y),\\
&(z,y,z,y),(y,z,y,z),\\
&(z,y,y,y),(y,y,y,z),(y,y,z,y),(y,z,y,y)\}\\
= {}&[(z,z,z,z)] \cup [(z,z,z,y)] \cup [(z,z,y,y)] \cup [(z,y,z,y)] \cup [(z,y,y,y)],\\
X_{4,2} = \{&(z,z,z,z),\\
&(z,z,z,y),(z,z,y,z),(z,y,z,z),(y,z,z,z),\\
&(z,z,y,y),(z,y,y,z),(y,y,z,z),(y,z,z,y)\}\\
= {}&[(z,z,z,z)] \cup [(z,z,z,y)] \cup [(z,z,y,y)],\\
X_{4,3} = \{&(z,z,z,z),\\
&(z,z,z,y),(z,z,y,z),(z,y,z,z),(y,z,z,z)\}\\
= {}&[(z,z,z,z)] \cup [(z,z,z,y)],\\
X_{4,4} = \{&(z,z,z,z)\}\\
= {}&[(z,z,z,z)].
\end{aligned}$$

Remark 23. Each $X_{l,n}$ is invariant under the action of $\mathbb{Z}/l\mathbb{Z}$, which allows us to make the following definition.

Definition 24. For $l \in \mathbb{Z}_{\geq 1}$ and $n \in \{0, \ldots, l\}$, we write

$$Y_{l,n} = X_{l,n}/\!\sim\, = \big\{[(\underbrace{z, \ldots, z}_{n}, u_{n+1}, \ldots, u_l)] \in Y_l \;\big|\; u_{n+1}, \ldots, u_l \in \{y, z\}\big\}.$$

Proposition 25. *For $n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 1}$, we have*

$$\dim \rho_n(\check{\mathfrak{H}}^1_k) = \#Y_{n+k,n} - 2.$$

Proof. Since

$$\begin{aligned}
\check{\mathfrak{H}}^1_k &= \operatorname{span}_{\mathbb{Q}}\{z_{k_1}\cdots z_{k_l} \mid (k_1,\ldots,k_l) \in \check{I}^1_k\} \\
&= \operatorname{span}_{\mathbb{Q}}\bigl\{u_1\cdots u_{k-1}y \bigm| (u_1,\ldots,u_{k-1}) \in \{x,y\}^{k-1} \setminus \{(\underbrace{y,\ldots,y}_{k-1})\}\bigr\} \\
&= \operatorname{span}_{\mathbb{Q}}\bigl\{u_1\cdots u_{k-1}y - y^k \bigm| (u_1,\ldots,u_{k-1}) \in \{y,z\}^{k-1}\bigr\},
\end{aligned}$$

Proposition 11 shows that

$$\begin{aligned}
\rho_n(\check{\mathfrak{H}}^1_k) &= \operatorname{span}_{\mathbb{Q}}\bigl\{\rho_n(u_1\cdots u_{k-1}y) - \rho_n(y^k) \bigm| (u_1,\ldots,u_{k-1}) \in \{y,z\}^{k-1}\bigr\} \\
&= \operatorname{span}_{\mathbb{Q}}\bigl\{\rho_0(z^n u_1\cdots u_{k-1}y) - \rho_0(z^n y^k) \bigm| (u_1,\ldots,u_{k-1}) \in \{y,z\}^{k-1}\bigr\} \\
&= \operatorname{span}_{\mathbb{Q}}\bigl\{\tilde{\rho}_0(U) - \tilde{\rho}_0\bigl([(\underbrace{z,\ldots,z}_{n},\underbrace{y,\ldots,y}_{k})]\bigr) \bigm| U \in Y_{n+k,n} \setminus \{[(\underbrace{z,\ldots,z}_{n+k})]\}\bigr\},
\end{aligned}$$

which implies that $\dim \rho_n(\check{\mathfrak{H}}^1_k) = \#Y_{n+k,n} - 2$ because of Proposition 20. □

5.3 Calculation of $\#Y_{l,n}$

Proposition 25 reduces our main theorem (Theorem 15) to the following proposition:

Proposition 26. *For $l \in \mathbb{Z}_{\geq 1}$ and $n \in \{0,\ldots,l\}$, we have*

$$\#Y_{l,n} = \frac{1}{l}\sum_{m\mid l}\varphi\left(\frac{l}{m}\right)(2^m - L^n_m).$$

We first invoke the Cauchy-Frobenius lemma:

Proposition 27 (Cauchy-Frobenius lemma)**.** *If a finite group G acts on a finite set X, then we have*

$$\#(X/G) = \frac{1}{\#G}\sum_{g\in G}\#\{x \in X \mid gx = x\}.$$

Lemma 28. *For $l \in \mathbb{Z}_{\geq 1}$ and $n \in \{0,\ldots,l\}$, we have*

$$\#Y_{l,n} = \frac{1}{l}\sum_{m\mid l}\varphi\left(\frac{l}{m}\right)\#\{\boldsymbol{u} \in X_{l,n} \mid m\boldsymbol{u} = \boldsymbol{u}\}.$$

Proof. Applying the Cauchy-Frobenius lemma with $G = \mathbb{Z}/l\mathbb{Z}$ and $X = X_{l,n}$ gives

$$\begin{aligned}
\#Y_{l,n} &= \frac{1}{l}\sum_{j\in\mathbb{Z}/l\mathbb{Z}}\#\{\boldsymbol{u} \in X_{l,n} \mid j\boldsymbol{u} = \boldsymbol{u}\} \\
&= \frac{1}{l}\sum_{j\in\mathbb{Z}/l\mathbb{Z}}\#\{\boldsymbol{u} \in X_{l,n} \mid \gcd(j,l)\boldsymbol{u} = \boldsymbol{u}\} \\
&= \frac{1}{l}\sum_{m\mid l}\bigl(\#\{j \in \mathbb{Z}/l\mathbb{Z} \mid \gcd(j,l) = m\}\cdot\#\{\boldsymbol{u} \in X_{l,n} \mid m\boldsymbol{u} = \boldsymbol{u}\}\bigr) \\
&= \frac{1}{l}\sum_{m\mid l}\varphi\left(\frac{l}{m}\right)\#\{\boldsymbol{u} \in X_{l,n} \mid m\boldsymbol{u} = \boldsymbol{u}\}.
\end{aligned}$$

□

Lemma 29. *If $l \in \mathbb{Z}_{\geq 1}$, $n \in \{0, \ldots, l\}$, and m is a positive divisor of l, then*

$$\#\{\boldsymbol{u} \in X_{l,n} \mid m\boldsymbol{u} = \boldsymbol{u}\} = 2^m - L_m^n.$$

Proof. The map $f_{m,l}\colon X_m \to X_l$ defined by

$$f_{m,l}(\boldsymbol{v}) = (\underbrace{\boldsymbol{v}, \ldots, \boldsymbol{v}}_{l/m})$$

is injective and has image $\{\boldsymbol{u} \in X_l \mid m\boldsymbol{u} = \boldsymbol{u}\}$. Therefore it suffices to show that

$$\# f_{m,l}^{-1}(X_{l,n}) = 2^m - L_m^n.$$

Observe that for each $(m, n) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$, the set $f_{m,l}^{-1}(X_{l,n})$ is the same for all multiples l of m with $l \geq n$. Therefore, we may put $Z_{m,n} = X_m \setminus f_{m,l}^{-1}(X_{l,n}) = f_{m,l}^{-1}(X_{l,n}^c)$, aiming to show that $\#Z_{m,n} = L_m^n$ for all $m \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$.

If $m \leq n$, then

$$\#Z_{m,n} = \#\{\boldsymbol{v} \in X_m \mid \text{at least one component of } \boldsymbol{v} \text{ is } y\} = 2^m - 1 = L_m^n.$$

If $m = n + 1$, then

$$\begin{aligned} \#Z_{m,n} &= \#\{\boldsymbol{v} \in X_m \mid \text{at least two components of } \boldsymbol{v} \text{ are } y\} \\ &= 2^m - m - 1 = 2^{n+1} - n - 2 = L_m^n \end{aligned}$$

by Lemma 14 (1). Suppose that $m \geq n + 2$. If we put

$$Z_{m,n,i} = \{(v_1, \ldots, v_m) \in Z_{m,n} \mid v_1 = \cdots = v_{i-1} = z,\ v_i = y\}$$

for $i = 1, \ldots, n$, then $\{Z_{m,n,i} \mid i = 1, \ldots, n\}$ is a partition of $Z_{m,n}$. Set

$$\begin{aligned} Z_{m,n,i}^y &= \{(v_1, \ldots, v_m) \in Z_{m,n,i} \mid v_{i+1} = y\}, \\ Z_{m,n,i}^z &= \{(v_1, \ldots, v_m) \in Z_{m,n,i} \mid v_{i+1} = z\}. \end{aligned}$$

Then removing the $(i+1)$st component gives a bijection from $Z_{m,n,i}^y$ to $Z_{m-1,n,i}$ and an injection from $Z_{m,n,i}^z$ to $Z_{m-1,n,i}$ with image $Z_{m-1,n,i} \setminus Z_{m-1,n,i}^{z\cdots zy}$, where

$$Z_{m-1,n,i}^{z\cdots zy} = \{(v_1, \ldots, v_{m-1}) \in Z_{m-1,n,i} \mid v_{i+1} = \cdots = v_{i+n-1} = z,\ v_{i+n} = y\}.$$

Moreover removing components from the $(i+1)$st to the $(i+n)$th gives a bijection from $Z_{m-1,n,i}^{z\cdots zy}$ to $Z_{m-n-1,n,i}$. It follows that

$$\begin{aligned} \#Z_{m,n,i} &= \#Z_{m,n,i}^y + \#Z_{m,n,i}^z = \#Z_{m-1,n,i} + \#(Z_{m-1,n,i} \setminus Z_{m-1,n,i}^{z\cdots zy}) \\ &= 2\#Z_{m-1,n,i} - \#Z_{m-n-1,n,i}. \end{aligned}$$

Summing up for $i = 1, \ldots, n$ gives $\#Z_{m,n} = 2\#Z_{m-1,n} - \#Z_{m-n-1,n}$. Hence the lemma follows from Lemma 14 (2). $\square$

Lemmas 28 and 29 imply Proposition 26, thereby establishing our main theorem.

6 Multiple zeta-star values

6.1 Multiple zeta-star values and Hoffman's algebra

The *multiple zeta-star values* (MZSVs for short) are defined as the MZVs with equality allowed in the index of summation:

Definition 30. For $\boldsymbol{k} = (k_1, \ldots, k_l) \in I^0$, the *multiple zeta-star value* $\zeta^\star(k_1, \ldots, k_l)$ is a real number defined by

$$\zeta^\star(\boldsymbol{k}) = \zeta^\star(k_1, \ldots, k_l) = \sum_{n_1 \geq \cdots \geq n_l \geq 1} \frac{1}{n_1^{k_1} \cdots n_l^{k_l}}.$$

Example 31. We have

$$\begin{aligned}
\zeta^\star(4,2,1) &= \sum_{n_1 \geq n_2 \geq n_3 \geq 1} \frac{1}{n_1^4 n_2^2 n_3} \\
&= \sum_{n_1 > n_2 > n_3 \geq 1} \frac{1}{n_1^4 n_2^2 n_3} + \sum_{n_1 = n_2 > n_3 \geq 1} \frac{1}{n_1^4 n_2^2 n_3} + \sum_{n_1 > n_2 = n_3 \geq 1} \frac{1}{n_1^4 n_2^2 n_3} + \sum_{n_1 = n_2 = n_3 \geq 1} \frac{1}{n_1^4 n_2^2 n_3} \\
&= \sum_{n_1 > n_2 > n_3 \geq 1} \frac{1}{n_1^4 n_2^2 n_3} + \sum_{n_1 > n_3 \geq 1} \frac{1}{n_1^6 n_3} + \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^4 n_2^3} + \sum_{n_1 \geq 1} \frac{1}{n_1^7} \\
&= \zeta(4,2,1) + \zeta(6,1) + \zeta(4,3) + \zeta(7).
\end{aligned}$$

As the above example indicates, each MZSV can be expressed as a $\mathbb{Z}$-linear combination of MZVs. Hoffman's algebra is useful for describing this relationship between MZVs and MZSVs.

Definition 32. Define a $\mathbb{Q}$-linear map $\bar{Z}\colon \mathfrak{H}^0 \to \mathbb{R}$ by setting $\bar{Z}(1) = 1$ and $\bar{Z}(z_{k_1} \cdots z_{k_l}) = \zeta^\star(\boldsymbol{k})$ for $\boldsymbol{k} = (k_1, \ldots, k_l) \in I^0$.

Definition 33. Let γ denote the algebra automorphism on $\mathfrak{H}$ satisfying $\gamma(x) = x$ and $\gamma(y) = z$. Define a $\mathbb{Q}$-linear transformation d on $\mathbb{Q} + \mathfrak{H}y$ by setting $d(1) = 1$ and $d(wy) = \gamma(w)y$ for $w \in \mathfrak{H}$.

Proposition 34. *We have $\bar{Z} = Z \circ d\colon \mathfrak{H}^0 \to \mathbb{R}$.*

Proof. Easy and well known. □

6.2 Cyclic sum formula for multiple zeta-star values

Ohno and the third author [4] proved the following analog of the CSF for MZSVs:

Theorem 35 (Cyclic sum formula for multiple zeta-star values)**.** *If $(k_1, \ldots, k_l) \in \check{I}^1$, then*

$$\sum_{j=1}^{l} \sum_{i=1}^{k_j - 1} \zeta^\star(k_j - i + 1, k_{j+1}, \ldots, k_{j+l-1}, i) = (k_1 + \cdots + k_l)\zeta(k_1 + \cdots + k_l + 1).$$

The second and third authors [5] defined operators $\bar{\rho}_n$ for MZSVs as well as ρ_n for MZVs:

Definition 36. Let $n \in \mathbb{Z}_{\geq 0}$. Define a $\mathbb{Q}$-linear map $\bar{\mathcal{C}}_n \colon \mathfrak{H} \to \mathfrak{H}^{\otimes(n+2)}$ by setting $\bar{\mathcal{C}}_n(1) = 0$, $\bar{\mathcal{C}}_n(x) = x \otimes y^{\otimes(n+1)}$, $\bar{\mathcal{C}}_n(y) = -x \otimes y^{\otimes(n+1)}$, and

$$\bar{\mathcal{C}}_n(ww') = \bar{\mathcal{C}}_n(w) \diamond \gamma^{-1}(w') + \gamma^{-1}(w) \diamond \bar{\mathcal{C}}_n(w').$$

Write $\bar{\rho}_n = M_n \circ \bar{\mathcal{C}}_n$.

Proposition 37. *We have the following commutative diagram:*

$$\begin{tikzcd}
 & x\mathfrak{H} \otimes \mathfrak{H}^{\otimes n} \otimes \mathfrak{H}y \arrow[r, "M_n"'] \arrow[dd, "\gamma\otimes\gamma^{\otimes n}\otimes d", "\cong"'] & \mathfrak{H}^0 \arrow[dd, "d"', "\cong"] \arrow[dr, "\bar{Z}"] & \\
\mathfrak{H} \arrow[ur, "\bar{\mathcal{C}}_n"'] \arrow[dr, "\mathcal{C}_n"] \arrow[urr, bend left, "\bar{\rho}_n"] \arrow[drr, bend right, "\rho_n"'] & & & \mathbb{R}. \\
 & x\mathfrak{H} \otimes \mathfrak{H}^{\otimes n} \otimes \mathfrak{H}y \arrow[r, "M_n"] & \mathfrak{H}^0 \arrow[ur, "Z"'] &
\end{tikzcd}$$

Proof. Straightforward. $\square$

Proposition 38. *If $k_1, \ldots, k_l \in \mathbb{Z}_{\geq 1}$, then*

$$\sum_{j=1}^{l} (\bar{\rho}_0 \circ \alpha \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}) = \sum_{j=1}^{l} \sum_{i=1}^{k_j - 1} z_{k_j - i + 1} z_{k_{j+1}} \cdots z_{k_{j+l-1}} z_i - (k_1 + \cdots + k_l) z_{k_1 + \cdots + k_l + 1}.$$

Here $\alpha \colon \mathfrak{H}y \to \mathfrak{H}y$ denotes the $\mathbb{Q}$-linear map representing the division by depth, i.e.,

$$\alpha(z_{k'_1} \cdots z_{k'_{l'}}) = \frac{1}{l'} z_{k'_1} \cdots z_{k'_{l'}}$$

for all monomials $z_{k'_1} \cdots z_{k'_{l'}} \in \mathfrak{H}y$.

Proof. For simplicity, let A and B respectively denote the left- and right-hand sides of the desired identity. By the injectivity of d, it suffices to show that $d(A) = d(B)$; setting $A' = d(A)$ and $B' = d(B)$, we have

$$A' = \sum_{j=1}^{l} (d \circ \bar{\rho}_0 \circ \alpha \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}) = \sum_{j=1}^{l} (\rho_0 \circ \alpha \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}),$$

$$B' = \sum_{j=1}^{l} \sum_{i=1}^{k_j - 1} d(z_{k_j - i + 1} z_{k_{j+1}} \cdots z_{k_{j+l-1}} z_i) - (k_1 + \cdots + k_l) z_{k_1 + \cdots + k_l + 1}.$$

For $a \in \mathbb{Z}_{\geq 1}$, let $\delta_a \colon \mathfrak{H}y \to \mathfrak{H}y$ be the $\mathbb{Q}$-linear map that extracts the depth-a part, i.e.,

$$\delta_a(z_{k'_1} \cdots z_{k'_{l'}}) = \begin{cases} z_{k'_1} \cdots z_{k'_{l'}} & \text{if } l' = a; \\ 0 & \text{otherwise} \end{cases}$$

for all monomials $z_{k'_1} \cdots z_{k'_{l'}} \in \mathfrak{H}y$. Then it is enough to prove that $\delta_a(A') = \delta_a(B')$ for $a = 1, \dots, l+1$. Note that

$$\begin{aligned} (\delta_a \circ \rho_0)(w) &= \begin{cases} (\delta_a \circ \rho_0 \circ \delta_a)(w) + (\delta_a \circ \rho_0 \circ \delta_{a-1})(w) & \text{if } a \geq 2; \\ (\delta_a \circ \rho_0 \circ \delta_a)(w) & \text{if } a = 1, \end{cases} \\ (\delta_a \circ \alpha)(w) &= \frac{1}{a}\delta_a(w) \end{aligned}$$

for all $w \in \mathfrak{H}y$.

For $a = 1$, we have

$$\begin{aligned} \delta_1(A') &= \sum_{j=1}^{l}(\delta_1 \circ \rho_0 \circ \alpha \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}) = \sum_{j=1}^{l}(\delta_1 \circ \rho_0 \circ \delta_1 \circ \alpha \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}) \\ &= \sum_{j=1}^{l}(\delta_1 \circ \rho_0 \circ \delta_1 \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}) = \sum_{j=1}^{l}(\delta_1 \circ \rho_0)(z_{k_j + \cdots + k_{j+l-1}}) \\ &= \sum_{j=1}^{l}(\delta_1 \circ \rho_0)(z_{k_1 + \cdots + k_l}) = \sum_{j=1}^{l}(-z_{k_1 + \cdots + k_l + 1}) = -l z_{k_1 + \cdots + k_l + 1}, \\ \delta_1(B') &= \sum_{j=1}^{l}\sum_{i=1}^{k_j - 1}(\delta_1 \circ d)(z_{k_j - i + 1} z_{k_{j+1}} \cdots z_{k_{j+l-1}} z_i) - (k_1 + \cdots + k_l) z_{k_1 + \cdots + k_l + 1} \\ &= \sum_{j=1}^{l}\sum_{i=1}^{k_j - 1} z_{(k_j - i + 1) + k_{j+1} + \cdots + k_{j+l-1} + i} - (k_1 + \cdots + k_l) z_{k_1 + \cdots + k_l + 1} \\ &= \sum_{j=1}^{l}(k_j - 1) z_{k_1 + \cdots + k_l + 1} - (k_1 + \cdots + k_l) z_{k_1 + \cdots + k_l + 1} \\ &= -l z_{k_1 + \cdots + k_l + 1}, \end{aligned}$$

as required.

For $a = l+1$, we have

$$\begin{aligned}
\delta_{l+1}(A') &= \sum_{j=1}^{l} (\delta_{l+1} \circ \rho_0 \circ \alpha \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}) \\
&= \sum_{j=1}^{l} (\delta_{l+1} \circ \rho_0 \circ \delta_{l+1} \circ \alpha \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}) + \sum_{j=1}^{l} (\delta_{l+1} \circ \rho_0 \circ \delta_l \circ \alpha \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}) \\
&= \frac{1}{l+1} \sum_{j=1}^{l} (\delta_{l+1} \circ \rho_0 \circ \delta_{l+1} \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}) + \frac{1}{l} \sum_{j=1}^{l} (\delta_{l+1} \circ \rho_0 \circ \delta_l \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}) \\
&= \frac{1}{l} \sum_{j=1}^{l} (\delta_{l+1} \circ \rho_0)(z_{k_j} \cdots z_{k_{j+l-1}}) = \frac{1}{l} \sum_{j=1}^{l} \sum_{j'=1}^{l} \sum_{i=1}^{k_{j+j'-1}-1} z_{k_{j+j'-1}-i+1} z_{k_{j+j'}} \cdots z_{k_{j+j'+l-2}} z_i \\
&= \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_{j+l-1}} z_i = \delta_{l+1}(B'),
\end{aligned}$$

as required.

Now let $2 \le a \le l$. We first compute $\delta_a(A')$. It is easy to see that $\delta_a(A') = P + Q$, where

$$\begin{aligned}
P &= \frac{1}{a} \sum_{j=1}^{l} (\delta_a \circ \rho_0 \circ \delta_a \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}), \\
Q &= \frac{1}{a-1} \sum_{j=1}^{l} (\delta_a \circ \rho_0 \circ \delta_{a-1} \circ d)(z_{k_j} \cdots z_{k_{j+l-1}}).
\end{aligned}$$

For $b \in \mathbb{Z}_{\ge 1}$, set

$$M_b = I^1_{l,b} = \{\boldsymbol{m} = (m_1, \ldots, m_b) \in \mathbb{Z}^b_{\ge 1} \mid m_1 + \cdots + m_b = l\}.$$

For each $\boldsymbol{m} = (m_1, \ldots, m_b) \in M_b$, we extend the indices of m_p to all $p \in \mathbb{Z}$ by declaring $m_p = m_{p'}$ whenever $p \equiv p' \pmod{b}$. For $j = 1, \ldots, l$, $\boldsymbol{m} = (m_1, \ldots, m_b) \in M_b$, and $p = 1, \ldots, b$, write

$$k_{j,\boldsymbol{m},p} = \sum_{i=j+m_1+\cdots+m_{p-1}}^{j+m_1+\cdots+m_p-1} k_i = k_{j+m_1+\cdots+m_{p-1}} + \cdots + k_{j+m_1+\cdots+m_p-1}.$$

Then we have

$$\begin{aligned}
P &= \frac{1}{a} \sum_{j=1}^{l} (\delta_a \circ \rho_0) \left(\sum_{\boldsymbol{m} \in M_a} z_{k_{j,\boldsymbol{m},1}} \cdots z_{k_{j,\boldsymbol{m},a}} \right) \\
&= -\frac{1}{a} \sum_{\substack{j=1,\ldots,l \\ \boldsymbol{m} \in M_a}} \sum_{p=1}^{a} z_{k_{j,\boldsymbol{m},p}+1} z_{k_{j,\boldsymbol{m},p+1}} \cdots z_{k_{j,\boldsymbol{m},p+a-1}}.
\end{aligned}$$

Note here that

$$\sum_{\substack{j=1,\dots,l\\ \boldsymbol{m}\in M_a}} z_{k_{j,\boldsymbol{m},p}+1} z_{k_{j,\boldsymbol{m},p+1}} \cdots z_{k_{j,\boldsymbol{m},p+a-1}}$$

does not depend on $p = 1, \dots, a$, because the bijection from $\{1, \dots, l\} \times M_a$ to itself defined by

$$(j, \boldsymbol{m}) = \big(j, (m_1, \dots, m_a)\big) \mapsto (j', \boldsymbol{m}') = \big(j + m_1 + \cdots + m_{p-1}, (m_p, \dots, m_{p+a-1})\big)$$

has the property that

$$z_{k_{j,\boldsymbol{m},p}+1} z_{k_{j,\boldsymbol{m},p+1}} \cdots z_{k_{j,\boldsymbol{m},p+a-1}} = z_{k_{j',\boldsymbol{m}',1}+1} z_{k_{j',\boldsymbol{m}',2}} \cdots z_{k_{j',\boldsymbol{m}',a}}.$$

It follows that

$$P = -\sum_{\substack{j=1,\dots,l\\ \boldsymbol{m}\in M_a}} z_{k_{j,\boldsymbol{m},a}+1} z_{k_{j,\boldsymbol{m},1}} \cdots z_{k_{j,\boldsymbol{m},a-1}}.$$

Similar reasoning shows that

$$\begin{aligned}
Q &= \frac{1}{a-1} \sum_{j=1}^{l} (\delta_a \circ \rho_0) \left(\sum_{\boldsymbol{m}\in M_{a-1}} z_{k_{j,\boldsymbol{m},1}} \cdots z_{k_{j,\boldsymbol{m},a-1}} \right) \\
&= \frac{1}{a-1} \sum_{\substack{j=1,\dots,l\\ \boldsymbol{m}\in M_{a-1}}} \sum_{p=1}^{a-1} \sum_{i=1}^{k_{j,\boldsymbol{m},p}-1} z_{k_{j,\boldsymbol{m},p}-i+1} z_{k_{j,\boldsymbol{m},p+1}} \cdots z_{k_{j,\boldsymbol{m},p+a-2}} z_i \\
&= \sum_{\substack{j=1,\dots,l\\ \boldsymbol{m}\in M_{a-1}}} \sum_{i=1}^{k_{j,\boldsymbol{m},a-1}-1} z_{k_{j,\boldsymbol{m},a-1}-i+1} z_{k_{j,\boldsymbol{m},1}} \cdots z_{k_{j,\boldsymbol{m},a-2}} z_i.
\end{aligned}$$

We next compute $\delta_a(B')$. Observe that $\delta_a(B') = R + S$, where

$$R = \sum_{\substack{j=1,\dots,l\\ \boldsymbol{m}\in M_{a-1}}} \sum_{i=1}^{k_j-1} z_{k_{j,\boldsymbol{m},1}-i+1} z_{k_{j,\boldsymbol{m},2}} \cdots z_{k_{j,\boldsymbol{m},a-1}} z_i,$$

$$S = \sum_{\substack{j=1,\dots,l\\ \boldsymbol{m}\in M_{a}}} \sum_{i=1}^{k_j-1} z_{k_{j,\boldsymbol{m},1}-i+1} z_{k_{j,\boldsymbol{m},2}} \cdots z_{k_{j,\boldsymbol{m},a-1}} z_{k_{j,\boldsymbol{m},a}+i}.$$

Since the bijection from $\{1, \dots, l\} \times M_{a-1}$ to itself defined by

$$(j, \boldsymbol{m}) = \big(j, (m_1, \dots, m_{a-1})\big) \mapsto (j', \boldsymbol{m}') = \big(j + m_1, (m_2, \dots, m_{a-1}, m_1)\big)$$

has the property that

$$\sum_{i=1}^{k_j-1} z_{k_{j,\boldsymbol{m},1}-i+1} z_{k_{j,\boldsymbol{m},2}} \cdots z_{k_{j,\boldsymbol{m},a-1}} z_i = \sum_{i=1}^{k_{j'+m'_1+\cdots+m'_{a-2}}-1} z_{k_{j',\boldsymbol{m}',a-1}-i+1} z_{k_{j',\boldsymbol{m}',1}} \cdots z_{k_{j',\boldsymbol{m}',a-2}} z_i,$$

we have

$$R = \sum_{\substack{j=1,\ldots,l \\ \boldsymbol{m}=(m_1,\ldots,m_{a-1})\in M_{a-1}}} \sum_{i=1}^{k_{j+m_1+\cdots+m_{a-2}}-1} z_{k_{j,\boldsymbol{m},a-1}-i+1} z_{k_{j,\boldsymbol{m},1}} \cdots z_{k_{j,\boldsymbol{m},a-2}} z_i.$$

Similar reasoning shows that

$$S = \sum_{\substack{j=1,\ldots,l \\ \boldsymbol{m}=(m_1,\ldots,m_a)\in M_a}} \sum_{i=1}^{k_{j+m_1+\cdots+m_{a-1}}-1} z_{k_{j,\boldsymbol{m},a}-i+1} z_{k_{j,\boldsymbol{m},1}} \cdots z_{k_{j,\boldsymbol{m},a-2}} z_{k_{j,\boldsymbol{m},a-1}+i}.$$

What needs to be shown is that $P+Q = R+S$. Note that

$$\begin{aligned} S - P &= \sum_{\substack{j=1,\ldots,l \\ \boldsymbol{m}=(m_1,\ldots,m_a)\in M_a}} \sum_{i=0}^{k_{j+m_1+\cdots+m_{a-1}}-1} z_{k_{j,\boldsymbol{m},a}-i+1} z_{k_{j,\boldsymbol{m},1}} \cdots z_{k_{j,\boldsymbol{m},a-2}} z_{k_{j,\boldsymbol{m},a-1}+i} \\ &= \sum_{\substack{j=1,\ldots,l \\ \boldsymbol{m}=(m_1,\ldots,m_a)\in M_a}} \sum_{i=k_{j,\boldsymbol{m},a-1}}^{k_{j,\boldsymbol{m},a-1}+k_{j+m_1+\cdots+m_{a-1}}-1} z_{k_{j,\boldsymbol{m},a-1}+k_{j,\boldsymbol{m},a}-i+1} z_{k_{j,\boldsymbol{m},1}} \cdots z_{k_{j,\boldsymbol{m},a-2}} z_i. \end{aligned}$$

Fix $j = 1, \ldots, l$ and consider the map $\psi\colon M_a \to M_{a-1}$ defined by

$$\psi(m_1, \ldots, m_a) = (m_1, \ldots, m_{a-2}, m_{a-1} + m_a).$$

If $\psi(\boldsymbol{m}) = \boldsymbol{m}'$, then $k_{j,\boldsymbol{m},a-1} + k_{j,\boldsymbol{m},a} = k_{j,\boldsymbol{m}',a-1}$ and $k_{j,\boldsymbol{m},p} = k_{j,\boldsymbol{m}',p}$ for $p = 1, \ldots, a-2$. Moreover, for each $\boldsymbol{m}' = (m'_1, \ldots, m'_{a-1}) \in M_{a-1}$, the sets

$$\{i \in \mathbb{Z} \mid k_{j,\boldsymbol{m},a-1} \le i \le k_{j,\boldsymbol{m},a-1} + k_{j+m_1+\cdots+m_{a-1}} - 1\}$$

for $\boldsymbol{m} = (m_1, \ldots, m_a) \in \psi^{-1}(\boldsymbol{m}')$ are disjoint with union

$$\{i \in \mathbb{Z} \mid k_{j+m'_1+\cdots+m'_{a-2}} \le i \le k_{j,\boldsymbol{m}',a-1} - 1\}.$$

It follows that

$$S - P = \sum_{\substack{j=1,\ldots,l \\ \boldsymbol{m}'=(m'_1,\ldots,m'_{a-1})\in M_{a-1}}} \sum_{i=k_{j+m'_1+\cdots+m'_{a-2}}}^{k_{j,\boldsymbol{m}',a-1}-1} z_{k_{j,\boldsymbol{m}',a-1}-i+1} z_{k_{j,\boldsymbol{m}',1}} \cdots z_{k_{j,\boldsymbol{m}',a-2}} z_i.$$

We therefore conclude that

$$S - P + R = \sum_{\substack{j=1,\ldots,l \\ \boldsymbol{m}\in M_{a-1}}} \sum_{i=1}^{k_{j,\boldsymbol{m},a-1}-1} z_{k_{j,\boldsymbol{m},a-1}-i+1} z_{k_{j,\boldsymbol{m},1}} \cdots z_{k_{j,\boldsymbol{m},a-2}} z_i = Q,$$

as required. □

Proposition 39. *We have* $\bar{\rho}_0(\check{\mathfrak{H}}^1) \subset \operatorname{Ker} \bar{Z}$ *and* $\bar{\rho}_0(\check{\mathfrak{H}}^1_{k-1}) \subset \operatorname{Ker} \bar{Z} \cap \mathfrak{H}^0_k$ *for* $k \in \mathbb{Z}_{\geq 2}$*.*

Proof. If $w \in \check{\mathfrak{H}}^1$, then Proposition 37, Proposition 8, and Theorem 4 show that

$$\bar{Z}\big(\bar{\rho}_0(w)\big) = Z\big(\rho_0(w)\big) = Z\big(\rho(w)\big) = 0,$$

from which the first assertion follows. The second assertion is now obvious. □

Proof of Theorem 35. Immediate from Propositions 38 and 39. □

Proposition 40. *We have*

$$\{0\} = \bar{\rho}_{k-2}(\check{\mathfrak{H}}^1_1) \subset \bar{\rho}_{k-3}(\check{\mathfrak{H}}^1_2) \subset \cdots \subset \bar{\rho}_0(\check{\mathfrak{H}}^1_{k-1}) \subset \operatorname{Ker} \bar{Z} \cap \mathfrak{H}^0_k$$

for all $k \in \mathbb{Z}_{\geq 2}$*.*

Proof. Observe that $\bar{\rho}_{n+1}(w) = \bar{\rho}_n(zw)$ for all $n \in \mathbb{Z}_{\geq 0}$ and $w \in \mathfrak{H}$; indeed, we have

$$\bar{\rho}_{n+1}(w) = d^{-1}\big(\rho_{n+1}(w)\big) = d^{-1}\big(\rho_n(zw)\big) = \bar{\rho}_n(zw).$$

This finishes the proof. □

Theorem 41. *If* $n \in \mathbb{Z}_{\geq 0}$ *and* $k \in \mathbb{Z}_{\geq 1}$*, then*

$$\dim \bar{\rho}_n(\check{\mathfrak{H}}^1_k) = \dim \rho_n(\check{\mathfrak{H}}^1_k) = \frac{1}{n+k} \sum_{m \mid n+k} \varphi\left(\frac{n+k}{m}\right)(2^m - L^n_m) - 2.$$

Proof. The $\mathbb{Q}$-vector spaces $\bar{\rho}_n(\check{\mathfrak{H}}^1_k)$ and $\rho_n(\check{\mathfrak{H}}^1_k)$ are isomorphic as given by d, and the theorem follows from Theorem 15. □

7 Acknowledgments

The authors are grateful to Professor Yasuo Ohno for valuable comments that inspired this work, and to Mr. Hiroki Kondo for carefully reading the manuscript.

References

[1] A. B. Goncharov, The double logarithm and Manin's complex for modular curves. *Math. Res. Lett.* **4** (1997), 617–636.

[2] A. B. Goncharov, Periods and mixed motives, http://arxiv.org/abs/math/0202154, 2002.

[3] M. E. Hoffman and Y. Ohno, Relations of multiple zeta values and their algebraic expression, *J. Algebra* **262** (2003), 332–347.

[4] Y. Ohno and N. Wakabayashi, Cyclic sum of multiple zeta values, *Acta Arith.* **123** (2006), 289–295.

[5] T. Tanaka and N. Wakabayashi, An algebraic proof of the cyclic sum formula for multiple zeta values, *J. Algebra* **323** (2010), 766–778.

[6] T. Terasoma, Mixed Tate motives and multiple zeta values, *Invent. Math.* **149** (2002), 339–369.

[7] D. Zagier, Values of zeta functions and their applications, *First European Congress of Mathematics*, Vol. II (Paris, 1992), 497–512, Progr. Math., **120** (1994).

2010 *Mathematics Subject Classification*: Primary 11M32; Secondary 05A05.
Keywords: multiple zeta values, multiple zeta-star values, cyclic sum formula, Lucas numbers.

(Concerned with sequences A000012, A000032, A000079, A000204, A000931, A001644, A001648, A023424, A038360, A052823, A073817, and A074048.)

Received June 20 2010; revised version received February 2 2011. Published in *Journal of Integer Sequences*, February 20 2011.

Return to Journal of Integer Sequences home page.