Journal of Integer Sequences, Vol. 14 (2011),
Article 11.1.6

Circular Permutations Avoiding Runs of $i, i+1, i+2$ or $i, i-1, i-2$

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Abstract

In this paper we examine permutations that avoid increasing or decreasing runs and extend known results to the circular and modular cases, allowing us to calculate sequence A078628 in Sloane's *On-Line Encyclopedia of Integer Sequences*.

1 Introduction

The main purpose of this paper is to answer the following question: In how many ways can n cards, numbered 0 to $n-1$, be placed in a circle so that no three consecutive cards are labeled consecutively? For example, if $n = 8$, then both of the following permutations have consecutively labeled cards in positions 3, 4, and 5 (we use 0-origin): $(5, 1, 0, 2, 3, 4, 6, 7)$ and $(5, 1, 0, 4, 3, 2, 6, 7)$. If we consider $(n-1, 0, 1)$ and $(n-2, n-1, 0)$ as consecutively labeled cards, then this sequence of numbers is sequence A078628 in Sloane's Encyclopedia [14] which previously had only thirteen terms calculated. We actually consider four sequences and we begin with the following definition.

Definition 1. A permutation $\pi = (a_0, a_1, \ldots, a_{n-1})$ has a run if there is an i such that either $a_{i+1} = a_i + 1$ and $a_{i+2} = a_i + 2$ or $a_{i+1} = a_i - 1$ and $a_{i+2} = a_i - 2$. If the equalities are replaced by congruences modulo n, then the runs are modular runs. The set of n-long permutations without runs is denoted by $\mathcal{S}^*(n)$ and the set without modular runs is denoted by $\overline{\mathcal{S}^*}(n)$.

As is customary, we denote the size of $\mathcal{S}^*(n)$ by $s^*(n)$ and likewise for the other sets in this paper. A *circular permutation* $\pi = (a_0, \ldots, a_{n-1})$ is a permutation in which the indices are from the ring of integers modulo n and so there are $(n-1)!$ circular permutations on n objects. We let $\mathcal{C}^*(n)$ be the set of n-long circular permutations that have no runs and $\overline{\mathcal{C}^*}(n)$ be the set of circular permutations without a modular run. Note that $\overline{c^*}(n)$ is the sequence A078628 mentioned above and that for $n = 8$, neither $(0,1,5,4,6,2,3,7) = (7,0,1,5,4,6,2,3)$ nor $(0,7,5,4,6,2,3,1) = (1,0,7,5,4,6,2,3)$ are in $\overline{\mathcal{C}^*}(n)$ since both have modular runs $(7,0,1)$ and $(1,0,7)$ respectively, but $(1,0,6,4,5,7,2,3)$ is in $\overline{\mathcal{C}^*}(n)$.

In the following, we often need to create a regular permutation from a circular one. This process we call *flattening*. If $\pi = (a_0,\ldots,a_{n-1})$ is a circular permutation, then π^{f_a} is π flattened at a if i is the index for which $a_i = a$ and $\pi^{f_a} = (a_i, a_{i+1},\ldots,a_{n-1},a_0,\ldots,a_{i-1})$. If we flatten at 0, then we denote this by π^f. Changing a regular permutation into a circular one is straightforward. If $\pi = (a_0,\ldots,a_{n-1})$ is a regular permutation, then π *circularized* is the circular permutation $\pi^c = (a_0,\ldots,a_{n-1})$.

Note that if π has no runs, then π^{f_a} has no runs but it is possible to flatten a circular permutation in the middle of a run so that the run is not preserved in the flattened permutation. Also, a regular permutation π may not have a run but π^c might. For example, $\pi = (3,5,0,6,4,1,2)$ does not have a run but $\pi^c = (3,5,0,6,4,1,2) = (1,2,3,5,0,6,4)$ has the run $(1,2,3)$.

One other useful operation we call *prepending*. Given a straight permutation $\pi = (a_1,\ldots,a_n)$ of length n, prepending a_0 gives us a straight permutation $(a_0,a_1,\ldots,a_n)$ of length $n+1$. We must take care when prepending a_0 to π that we actually get a permutation. Likewise *appending* adds a digit to the end of a straight permutation.

If $\pi = (a_0,\ldots,a_{n-1})$ is a permutation (straight or circular), then $-\pi = (n-1-a_0,\ldots,n-1-a_{n-1})$ is also a permutation. Again, π has no runs if and only if $-\pi$ has no runs.

The context for this paper is the study of arithmetic progressions in permutations. There are at least two ways of defining an arithmetic progression in a permutation $\pi = (a_0,a_1,\ldots,a_{n-1})$. We follow Hegarty [5] in defining a progression of rise r and distance d in π as a sequence (a_i, a_{i+d}, a_{i+2d}) where $r = a_{i+d} - a_i = a_{i+2d} - a_{i+d}$. Note that a run is a progression with distance $d = 1$ and rise $r = \pm 1$.

Riordan [11] computed the number of permutations with x progressions of rise $r = 1$ and $d = 1$ and called them 3-sequences. He noted that these were generalizations of a topic Whitworth considered, see [16, pp. 103–107]. Charalambides [2, pp. 176–184] further generalized Whitworth's sequences. Jackson and Read [6] called progressions of distance 1 and rise ± 1 increasing runs if $r = 1$ and decreasing runs if $r = -1$ and they gave generating functions for the number of permutations without progressions with $d = 1$ and $r = \pm 1$ of length ℓ. Jackson and Reilly [7] derived a generating function for the number of permutations with exactly x increasing runs of length ℓ and showed that the computation of the number of such permutations is $O(n^3)$. Myers [9] called a progression of rise 1 and distance 1 a block and counted the number of permutations that contains m blocks. He also considered rigid patterns and noticed that difficulties occur if patterns can be interweaved.

The definition of an arithmetic progression in the permutation π that we are *not* using is a sequence (a_i, a_j, a_k) with $0 \le i < j < k < n$ where $a_j - a_i = a_k - a_j$. Progressions of this type were studied by several authors, [3, 4, 12] (see also [8, 10, 13, 15]).

2 The Main Results

In Table 1 we list values for the sequences, $s^*(n)$, $\overline{s^*}(n)$, $c^*(n)$, and $\overline{c^*}(n)$. Jackson and Read [6] computed the first column and Abramson and Moser [1] gave a formula for it (see Theorem 2). We will relate the other three columns of Table 1 to the first column giving a fairly easy way to compute each of these columns. Since there three summations in Theorem 2 and the only multiplications are computing factorials and binomial coefficients, $s^*(n)$ can be computed with at most $O(n^4)$ operations. Also, as we will show, the other columns can be computed from the first by $O(n)$ additions and so all these sequences can be computed in polynomial time. We believe that $\overline{c^*}(n)$, $n \geq 13$, is computed here for the first time and the first fifty values of $\overline{c^*}(n)$ are listed at the end of the paper. We first compute $c^*(n)$.

Theorem 2. *(Abramson, Moser, Jackson, Read) For $n \geq 3$,*

$$s^*(n) = \left(\sum_{k=1}^{n-2} (-1)^k \sum_{i=0}^{k-1} \binom{k-1}{i} (n-k-i-1)! \sum_{j=0}^{i} \binom{i}{j} \binom{n-i-k-1}{j+1} 2^{j+1} \right) + n!.$$

n	$s^*(n)$	$\overline{s^*}(n)$	$c^*(n)$	$\overline{c^*}(n)$
3	4	0	0	0
4	18	16	4	4
5	92	80	16	12
6	570	516	86	76
7	4082	3794	542	494
8	33292	31456	3932	3662
9	304490	290970	32330	30574
10	3086890	2974380	297438	284398
11	34357812	33311520	3028320	2918924
12	416526730	405773448	33814454	32791604
13	5463479106	5342413414	410954878	400400062
14	77094352076	75612301688	5400878692	5281683678

Table 1: Values of $s^*(n)$, $\overline{s^*}(n)$, $c^*(n)$, and $\overline{c^*}(n)$

Theorem 3. *For $n \geq 4$ and if $s^*(1) = 0$ and $s^*(2) = 1$, then*

$$c^*(n) = s^*(n-1) - 2 \sum_{i=1}^{\lfloor (n-2)/3 \rfloor} \Big(s^*(n-3i) - s^*(n-1-3i) \Big).$$

Proof. Let π be in $\mathcal{C}^*(n)$. If we flatten π at $n-1$, we get a permutation $\pi^{f_{n-1}} = (n-1, a_0, \ldots, a_{n-2})$. Note that $(a_0, \ldots, a_{n-2})$ is in $\mathcal{S}^*(n-1)$, but not all permutations in $\mathcal{S}^*(n-1)$

can have an $n-1$ prepended and then circularized to be in $\mathcal{C}^*(n)$. In fact, these are precisely the permutations for which either $a_0 = n-2$ and $a_1 = n-3$ or $a_{n-3} = n-3$ and $a_{n-2} = n-2$. In the first case, π had the run $(n-1, n-2, n-3)$ and in the second case, π had the run $(n-3, n-2, n-1)$. Thus we need to exclude those permutations in $\mathcal{S}^*(n-1)$ of the form $(n-2, n-3, a_2, \ldots, a_{n-2})$ or $(a_0, \ldots, a_{n-4}, n-3, n-2)$.

Since there are no runs in either $(a_2, \ldots, a_{n-2})$ or $(a_0, \ldots, a_{n-4})$, there are at most $2s^*(n-3)$ permutations of the type we need to exclude and so $c^*(n) \geq s^*(n-1) - 2s^*(n-3)$. The reason this inequality is not an equality is that if $a_2 = n-4$ or $a_{n-4} = n-4$ in a permutation in $\mathcal{S}^*(n-3)$, then neither $(n-2, n-3, n-4, a_3 \ldots, a_{n-2})$ nor $(a_0, \ldots, a_{n-5}, n-4, n-3, n-2)$ would be in $\mathcal{S}^*(n-1)$. Since both $(a_3, \ldots, a_{n-2})$ and $(a_0, \ldots, a_{n-5})$ must be in $\mathcal{S}^*(n-4)$, we have $c^*(n) \leq s^*(n-1) - 2s^*(n-3) + 2s^*(n-4)$.

As in the first exclusion, $(a_0, \ldots, a_{n-7}, n-6, n-5, n-4)$ and $(n-4, n-5, n-6, a_5, \ldots, a_{n-2})$ are not in $\mathcal{S}^*(n-3)$ but $(a_0, \ldots, a_{n-7}, n-6, n-5)$ and $(n-5, n-6, a_5, \ldots, a_{n-2})$ are in $\mathcal{S}^*(n-4)$. Hence we need to subtract the number of permutations of the form $(a_0, \ldots, a_{n-7})$ and $(a_5, \ldots, a_{n-2})$ that are in $\mathcal{S}^*(n-6)$. At this point, $c^*(n) \geq s^*(n-1) - 2s^*(n-3) + 2s^*(n-4) - 2s^*(n-6)$.

Continue in this manner. If $n \equiv 1 \pmod 3$, then in the last step we add $2s^*(3)$ getting the formula in the statement of the theorem. If $n \equiv 0 \pmod 3$, then on the last step we must add back permutations of length $n = 3$ which begin or end with the value of 2 which have no runs. The only permutations with this property are $(2, 0, 1)$ and $(1, 0, 2)$ and so the formula works with $s^*(2) = 1$. If $n \equiv 2 \pmod 3$, then on the last step we must avoid a permutation of length $n = 4$ which begins with $(3, 2)$ or ends with $(2, 3)$ and has no runs. The only permutations with this property are $(3, 2, 0, 1)$ and $(1, 0, 2, 3)$. So again $s^*(2) = 1$ satisfies the equation.

$\square$

Let $\mathcal{B}^{*n}_{a\ldots b}$ be the set of straight permutations that do not have a run and which start with the sequence a and end with the sequence b. For example, the permutation $(0, 5, 1, 6, 4, 2, 3, 7, 8)$ is in $\mathcal{B}^{*9}_{0\ldots 7,8}$. While we include values for $b^{*n}_{0\ldots 2,1}$ in Table 2, we only use this sequence in the proofs of the following lemmas. We show that $\overline{c^*}(n)$ can be computed using $c^*(n)$ and these $b^{*n}_{a\ldots b}$'s. Since $c^*(n)$ can be computed from $s^*(n)$, sequence A078628 can now be easily computed.

n	$b^{*n}_{0,1\ldots n-2,n-1}$	$b^{*n}_{0\ldots n-2,n-1}$	$b^{*n}_{0\ldots n-1}$	$b^{*n}_{0\ldots 2,1}$
5	0	1	4	1
6	1	3	16	3
7	2	13	86	14
8	11	73	543	75
9	62	470	3934	481

Table 2: Straight permutations which begin and end with certain sequences

Theorem 4. *For* $n \geq 3$, $\overline{c^*}(n) = c^*(n) - 2\left(b^{*n}_{0,1\ldots n-1} + b^{*n}_{0\ldots n-2,n-1} - b^{*n}_{0,1\ldots n-2,n-1}\right)$.

Proof. The permutations in $\mathcal{C}^*(n)$ that are not in $\overline{\mathcal{C}^*}(n)$ are the ones containing the sequence $(n-1, 0, 1)$ and its reverse $(1, 0, n-1)$ or the sequence $(n-2, n-1, 0)$ and its reverse $(0, n-1, n-2)$. Since the number of permutations with one of these sequences is the same as the number with the sequence's reverse, we focus on the sequences $(n-1, 0, 1)$ and $(n-2, n-1, 0)$.

Flattening a permutation containing one of these sequences at 0, we need to subtract from $c^*(n)$, the following: $b^{*n}_{0,1\ldots n-1} + b^{*n}_{0\ldots n-2,n-1}$. Since the permutations in $\mathcal{B}^{*n}_{0,1\ldots n-2,n-1}$ are in both $\mathcal{B}^{*n}_{0,1\ldots n-1}$ and $\mathcal{B}^{*n}_{0\ldots n-2,n-1}$, we need to add $b^{*n}_{0,1\ldots n-2,n-1}$ to get $\overline{c^*}(n) = c^*(n) - 2\Big(b^{*n}_{0,1\ldots n-1} + b^{*n}_{0\ldots n-2,n-1}\Big) + 2b^{*n}_{0,1\ldots n-2,n-1}$. □

The following two lemmas are used only for tidying-up results that we will need later.

Lemma 5. *For* $n \geq 3$, $b^{*n}_{0\ldots 2,1} = b^{*n}_{0\ldots n-2,n-1} + b^{*n-1}_{0,1\ldots n-3,n-2}$.

Proof. Let $\pi = (0, a_1, \ldots, a_{n-3}, 2, 1)$ be in $\mathcal{B}^{*n}_{0\ldots 2,1}$ and so $a_{n-3} \neq 3$. If $a_1 \neq n-1$ or $a_2 \neq n-2$, then multiplying π by -1 gives $-\pi = (0, -a_1, \ldots, -a_{n-3}, -2, -1) = (0, -a_1, \ldots, -a_{n-3}, n-2, n-1)$. Since $a_{n-3} \neq 3$, $-\pi$ is in $\mathcal{B}^{*n}_{0\ldots n-2,n-1}$.

If $a_1 = n-1$ and $a_2 = n-2$, then $-\pi = (0, 1, 2, -a_3, \ldots, -a_{n-3}, n-2, n-1)$. Note that if $-a_3 = 3$, then $(n-1, n-2, n-3)$ would have been a run in π. Removing 0 from $-\pi$ and subtracting 1 from each entry gives a permutation in $\mathcal{B}^{*n-1}_{0,1\ldots n-3,n-2}$. Therefore, $b^{*n}_{0\ldots 2,1} = b^{*n}_{0\ldots n-2,n-1} + b^{*n-1}_{0,1\ldots n-3,n-2}$.

□

Lemma 6. *For* $n \geq 3$, $b^{*n+1}_{0,1\ldots n-1,n} = b^{*n}_{0,1\ldots n-1} - b^{*n}_{0,1\ldots n-2,n-1}$.

Proof. If $\pi = (0, 1, a_2, \ldots, a_{n-2}, n-1)$ is in $\mathcal{B}^{*n}_{0,1\ldots n-1}$, then $a_{n-2} = n-2$ if and only if π is in $\mathcal{B}^{*n}_{0,1\ldots n-2,n-1}$. On the other hand, $a_{n-2} \neq n-2$ if and only if appending n to π gives a permutation in $\mathcal{B}^{*n+1}_{0,1\ldots n-1,n}$. Therefore, $b^{*n}_{0,1\ldots n-1} = b^{*n}_{0,1\ldots n-2,n-1} + b^{*n+1}_{0,1\ldots n-1,n}$. □

The next four results enable us to compute recursively any entry in Table 2 and hence by Corollary 11 to compute $\overline{c^*}(n)$.

Lemma 7. *For* $n \geq 3$, $b^{*n+1}_{0\ldots n} = c^*(n) + b^{*n}_{0\ldots 2,1} - b^{*n}_{0\ldots n-2,n-1}$.

Proof. Let $\pi = (0, a_1, \ldots, a_{n-1}, n)$ be in $\mathcal{B}^{*n+1}_{0\ldots n}$. If $a_{n-2} \neq 2$ or $a_{n-1} \neq 1$, then dropping the n from π and circularizing $(0, a_1, \ldots, a_{n-1})$ gives a permutation in $\mathcal{C}^*(n)$. If $a_{n-2} = 2$ and $a_{n-1} = 1$, then removing n gives a permutation in $\mathcal{B}^{*n}_{0\ldots 2,1}$.

If, however, $\pi = (0, b_1, \ldots, b_{n-3}, n-2, n-1)$ is a circular permutation in $\mathcal{C}^*(n)$, then appending an n to π^f does not give a permutation in $\mathcal{B}^{*n+1}_{0\ldots n}$. The number of such permutations is $b^{*n}_{0\ldots n-2,n-1}$ which must be subtracted from $c^*(n) + b^{*n}_{0\ldots 2,1}$ to obtain $b^{*n+1}_{0\ldots n}$. □

Using Lemma 5 gives us the following corollary to Lemma 7.

Corollary 8. *For* $n \geq 3$, $b^{*n+1}_{0\ldots n} = c^*(n) + b^{*n-1}_{0,1\ldots n-3,n-2}$.

Lemma 9. *For* $n \geq 3$, $b^{*n+1}_{0\ldots n-1,n} = b^{*n}_{0\ldots n-1} - b^{*n}_{0\ldots n-2,n-1}$.

Proof. The proof of this lemma is very similar to the proof of Lemma 6 and is omitted. □

Lemma 10. *For $n \geq 3$, $b^{*n+1}_{0,1\ldots n-1,n} = b^{*n}_{0\ldots n-2,n-1} - b^{*n}_{0,1\ldots n-2,n-1}$.*

Proof. Let $\pi = (0, 1, a_2, \ldots, a_{n-3}, n-1, n)$ be in $\mathcal{B}^{n+1}_{0,1\ldots n-1,n}$. Note that $a_2 \neq 2$. Hence deleting 0 from π and subtracting 1 from each of the remaining entries gives the permutation $(0, a_2-1, \ldots, a_{n-3}-1, n-2, n-1)$ which is in $\mathcal{B}^n_{0\ldots n-2,n-1}$. But a permutation in $\mathcal{B}^n_{0\ldots n-2,n-1}$ could have a 1 in the second position in which case the permutation is in $\mathcal{B}^n_{0,1\ldots n-2,n-1}$. □

Theorem 3 allows us to compute $c^*(n)$ for any n. Next suppose that we have two rows, say $m-1$ and m, of known values in Table 2. By Corollary 8, we can compute row $m+1$ of Column 3 in Table 2 and by Lemma 9 we can compute row $m+1$ of Column 2. Finally, by Lemma 10, we can compute row $m+1$ of Column 1. Therefore we can compute recursively $b^{*n}_{0,1\ldots n-2,n-1}$, $b^{*n}_{0\ldots n-2,n-1}$, and $b^{*n}_{0\ldots n-1}$ for $n > 6$.

Using Theorem 4 and Lemma 6 and then Lemma 10, we can rewrite $\overline{c^*}(n)$ in terms of just $c^*(n)$ and $b^{*n}_{0,1\ldots n-2,n-1}$.

Corollary 11. *For $n \geq 3$, $\overline{c^*}(n) = c^*(n) - 2b^{*n}_{0,1\ldots n-2,n-1} - 4b^{*n+1}_{0,1\ldots n-1,n}$.*

For completeness, we now sketch the proof of the connection between the second and third columns of Table 1. If $\pi = (a_0, \ldots, a_{n-1})$, then for j an integer we define $\pi + j$ to be the permutation $(a_0 + j, \ldots, a_{n-1} + j)$ where all additions are modulo n. Note that π and $\pi + j$ have the same number of modular runs.

Theorem 12. *For $n \geq 3$ and for $x > 0$, $n \cdot c^*(n, x) = \overline{s^*}(n, x)$ where $\overline{\mathcal{S}^*}(n, x)$ is the set of permutations with exactly x modular runs and $\mathcal{C}^*(n, x)$ is the set of circular permutations with exactly x runs.*

Proof. First consider a permutation $\pi = (0, a_1, \ldots, a_{n-1})$ in $\mathcal{C}^*(n, x)$. The basic goal of this proof is to flatten π at 0 to get π^f and create a set of n straight permutations $\Pi = \{\pi^f, \pi^f + 1, \ldots, \pi^f + (n-1)\}$, each of which will have the same number of modular runs as π has runs. But since flattening may remove a run from the original permutation, there are four cases.

Case 1: Either $a_1 \neq n-1$ or $a_2 \neq n-2$ and either $a_{n-2} \neq 2$ or $a_{n-1} \neq 1$. In this case we simply need to flatten the permutation at 0 and create the set Π as above.

Case 2: $a_1 = n-1$ and $a_2 = n-2$ and $a_{n-2} = 2$ and $a_{n-1} = 1$. In this case we flatten the permutation at 0 and create the set Π as above. Note that $(2, 1, 0)$ is a run in the circular permutation and $(0, n-1, n-2)$ is a modular run in the straight permutation π^f. Hence π and π^f have the same number of runs.

Case 3: $a_1 = n-1$ and $a_2 = n-2$ and either $a_{n-2} \neq 2$ or $a_{n-1} \neq 1$. In this case we consider $\gamma = [-\pi']^f$, where $-\pi'$ is $-\pi$ reversed. Hence $\gamma = (0, -a_{n-1}, \ldots, -a_3, 2, 1)$ and has the same number of modular runs as π has runs. We use γ to create the set of n modular permutations that corresponds to π.

Case 4: Either $a_1 \neq n-1$ or $a_2 \neq n-2$ and $a_{n-2} = 2$ and $a_{n-1} = 1$. We again consider $\gamma = [-\pi']^f$, where again $-\pi'$ is $-\pi$ reversed. Hence $\gamma = (0, n-1, n-2, -a_{n-3}, \ldots, -a_3)$ and has the same number of modular runs as π has runs.

It is not difficult to show that the n permutations produced by each of these cases are unique. If $\pi = (a_0, a_1, \ldots, a_{n-1})$ is in $\overline{\mathcal{S}^*}(n, x)$, then $\pi - a_0$ will fit one of the four cases above. Circularize $\pi - a_0$ and it will be the permutation that gives rise to the set of n straight permutations $\{\pi + j : 0 \leq j < n\}$. □

Corollary 13. *For $n \geq 3$, $n \cdot c^*(n) = \overline{s^*}(n)$.*

Proof. By the above theorem, $n \cdot c^*(n) = n \cdot [(n-1)! - \sum_{x>0} c^*(n, x)] = n! - \sum_{x>0} \overline{s^*}(n, x) = \overline{s^*}(n)$, as desired. □

3 Probability

It is not surprising that the probability that a permutation of length n does not have runs is close to 1. In this section we verify this by showing that the probability that our deck of cards from the introduction has no runs is essentially 1.

Since there are two ways to choose a modular run (increasing or decreasing), $n-2$ ways to choose the starting position of the run, n ways to choose the starting value of the run, and $(n-3)!$ ways to place the remaining $n-3$ numbers in the permutation, there are at most $2(n-2)n(n-3)!$ permutations that have a modular run. Hence $\overline{s^*}(n) \geq n! - 2(n-2)n(n-3)!$ and so

$$\frac{\overline{s^*}(n)}{n!} \geq 1 - \frac{2}{n-1}.$$

Hence we have the following:

Theorem 14. *The probability that a permutation of length n does not have a modular run approaches 1 as n approaches infinity.*

Since $s^*(n) > \overline{s^*}(n)$, we also have

Corollary 15. *The probability that a permutation of length n does not have a run approaches 1 as n approaches infinity.*

For circular permutations, position does not matter so there are at most $2n(n-3)!$ circular permutations that contain a modular progression. Hence

$$\frac{\overline{c^*}(n)}{(n-1)!} \geq 1 - \frac{2n}{(n-1)(n-2)}.$$

Again, since $c^*(n) \geq \overline{c^*}(n)$, we have

Theorem 16. *The probability that a circular permutation of length n does not have a modular run approaches 1 as n approaches infinity and the probability that a circular permutation of length n does not have a run approaches 1 as n approaches infinity.*

4 Sequences

This paper deals with sequences A002629, A078628, A095816, A165963, and A165964. The first fifty terms of sequence A078628 are as follows:

$\overline{c^*}(1) = 0$
$\overline{c^*}(2) = 0$
$\overline{c^*}(3) = 0$
$\overline{c^*}(4) = 4$
$\overline{c^*}(5) = 12$
$\overline{c^*}(6) = 76$
$\overline{c^*}(7) = 494$
$\overline{c^*}(8) = 3662$
$\overline{c^*}(9) = 30574$
$\overline{c^*}(10) = 284398$
$\overline{c^*}(11) = 2918924$
$\overline{c^*}(12) = 32791604$
$\overline{c^*}(13) = 400400062$
$\overline{c^*}(14) = 5281683678$
$\overline{c^*}(15) = 74866857910$
$\overline{c^*}(16) = 1135063409918$
$\overline{c^*}(17) = 18330526475060$
$\overline{c^*}(18) = 314169905117860$
$\overline{c^*}(19) = 5695984717957246$
$\overline{c^*}(20) = 108921059813769710$
$\overline{c^*}(21) = 2190998123920252622$
$\overline{c^*}(22) = 46250325111346491694$
$\overline{c^*}(23) = 1022301429750398188716$
$\overline{c^*}(24) = 23613740754886647958180$
$\overline{c^*}(25) = 568950024006846904093598$
$\overline{c^*}(26) = 14274866445575578119743438$
$\overline{c^*}(27) = 372374376152806360290989110$
$\overline{c^*}(28) = 10084828164172773195319256062$
$\overline{c^*}(29) = 283174462307289209810184927092$
$\overline{c^*}(30) = 8233653220849232427790328045876$
$\overline{c^*}(31) = 247614562274689810303882719509278$
$\overline{c^*}(32) = 7693604324191919134660311677872254$
$\overline{c^*}(33) = 246722167225395915065853140640911086$
$\overline{c^*}(34) = 8158170043027413464766703084133765486$
$\overline{c^*}(35) = 277900813774915739274082551738004741004$
$\overline{c^*}(36) = 9743794575197519961922090241025348596308$
$\overline{c^*}(37) = 351363839903830230918432098902818922192894$
$\overline{c^*}(38) = 13021021361761744809803587469208204799416830$
$\overline{c^*}(39) = 495539146920588230240234245219779841743108086$
$\overline{c^*}(40) = 19353425998749866625987965362312555548402003838$

$\overline{c^*}(41) = 775178453748407731849560920571080474371157593332$
$\overline{c^*}(42) = 31822940480660363909632901120497416610409109814276$
$\overline{c^*}(43) = 1338188780079916734821715054077664877231344980041022$
$\overline{c^*}(44) = 57608766415416937793463862738387567270274014040964686$
$\overline{c^*}(45) = 2537585460353397962654355343635847873682378459704820302$
$\overline{c^*}(46) = 114311753947370344920005481482946829653263862575017508398$
$\overline{c^*}(47) = 5263639446015081410706509916195602027644548310012924070828$
$\overline{c^*}(48) = 247629532177056272405843061340997818641369089724829503112772$
$\overline{c^*}(49) = 11897189001344425293916047304884946295425321612277069481144286$
$\overline{c^*}(50) = 583477983942047048226466538472338140101669906331645943031808878$

5 Acknowledgments

The first author was supported by one of Washington and Lee University's Lenfest Summer Grants. Parts of this paper are from Isaac Lambert's honors thesis. We appreciated the suggestions of a referee.

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2010 *Mathematics Subject Classification*: Primary 05A05; Secondary 05A15.
Keywords: permutation, circular permutation, run, arithmetic progression.

(Concerned with sequences A002629, A078628, A095816, A165963, and A165964.)

Received January 15 2010; revised version received July 14 2010; January 19 2011. Published in *Journal of Integer Sequences*, February 8 2011.

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