Journal of Integer Sequences, Vol. 12 (2009),
Article 09.7.7

Recursive Generation of k-ary Trees

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Abstract

In this paper we present a construction of every k-ary tree using a forest of $(k-1)$-ary trees satisfying a particular condition. We use this method recursively for the construction of the set of k-ary trees from the set of $(k-1)$-Dyck paths, thus obtaining a new bijection ϕ between these two sets. Furthermore, we introduce a new order on $[k]^*$ which is used for the full description of this bijection. Finally, we study some new statistics on k-ary trees which are transferred by ϕ to statistics concerning the occurrence of strings in $(k-1)$-Dyck paths.

1 Introduction

The notion of k-ary trees has been studied extensively in the literature. Some authors deal with the generation of k-ary trees using some encoding of them as integer sequences, which are generated in a specific order (see for example [3, 10, 11, 17, 20, 21]). In another direction k-ary trees are related to other k-Catalan structures such as staircase tilings, the tennis ball problem, noncrossing contractions and K-trees (see for example [6, 7, 12, 13, 14]). Finally, there are some papers dealing with the enumeration of k-ary trees according to some parameters (see for example [5, 19, 22]).

A well known procedure for the study of trees contained in a certain set $\mathcal{T}$ is to introduce a decomposition of these trees with respect to the size and then, using this decomposition, to rebuild $\mathcal{T}$ from trees of smaller size.

In this paper, we use a different procedure for the construction of the set $\mathcal{T}_k$ of all k-ary trees. First, we present a decomposition of each k-ary tree to a forest of $(k-1)$-ary trees satisfying certain properties and we show how these trees can be reconstructed from the associated forest. Next, by introducing an operation on forests, we present a recursive construction (in terms of k) of $\mathcal{T}_k$ using unary trees, thus obtaining a new bijection from k-ary trees to $(k-1)$-Dyck paths.

In Section 2 we give some definitions and preliminary results.

In Section 3 we associate every k-ary tree with a forest of $(k-1)$-ary trees such that the path with ascent sequence consisting of the sizes of the trees in this forest is a Dyck path. Conversely, every such forest generates the tree uniquely; consequently an algorithmic construction is given.

In Section 4, the method used in the previous section is applied recursively for every k-ary tree, and terminates with a forest of unary trees such that the path with ascent sequence consisting of the sizes of the trees in this forest is a $(k-1)$-Dyck path. Conversely, this forest generates the tree uniquely, so that a new bijection ϕ between $\mathcal{T}_k$ and the set of all $(k-1)$-Dyck paths is obtained.

In Section 5 we fully describe ϕ, by introducing a new order on the set of maximal paths of the k-ary tree.

Finally, in Section 6 we enumerate the set $\mathcal{T}_k$ according to some parameters related to the notions of the previous sections.

2 Preliminaries

A k-ary tree, $k \geq 1$, is either the empty tree $\square$ or a vertex (or internal node), called the *root* of the tree, with k ordered children which are k-ary trees. We define $\mathcal{T}_{k,n}$ to be the set of all k-ary trees with n vertices, and $\mathcal{T}_k = \bigcup\limits_{n\geq 0} \mathcal{T}_{k,n}$. Thus, every $T \in \mathcal{T}_k$ can be uniquely decomposed as follows:

$$T = \square \quad \text{or} \quad T = T_1T_2\cdots T_k, \quad T_i \in \mathcal{T}_k, \quad i \in [k] \tag{1}$$

(see Figure 1).

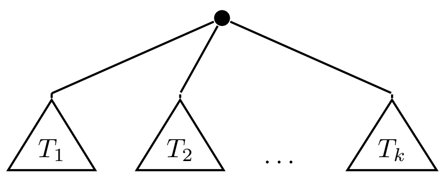

Figure 1: The k-ary tree $T = T_1T_2\cdots T_k$.

An empty child of a vertex is called a *leaf* (or *external node*) of the tree. The *size* of a k-ary tree T is the number of its vertices and it is denoted by $s(T)$; (see for example Figure 2).

Clearly, every $T \in \mathcal{T}_{k,n}$ contains $kn+1$ nodes (k children for each of the n internal nodes plus the root of the tree) and $(k-1)n+1$ leaves.

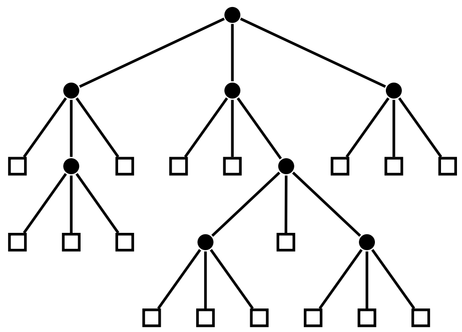

Figure 2: A 3-ary tree of size 8.

It is well known (see for example [8], p.589) that $|\mathcal{T}_{k,n}|$ is equal to the n-th k-Catalan number

$$C_n^{(k)} = \frac{1}{kn+1}\binom{kn+1}{n} = \frac{1}{(k-1)n+1}\binom{kn}{n}.$$

We note that $C_n^{(2)}$ are the ordinary Catalan numbers C_n ([15], A000108).

Furthermore, the generating function $C_k(x) = \sum_{n\geq 0} C_n^{(k)} x^n$ of the k-Catalan sequence satisfies the equation

$$C_k(x) = 1 + x(C_k(x))^k,$$

from which it can be easily shown using the Lagrange inversion formula ([2], Appendix A) that the coefficients of $(C_k(x))^s$, $s \in \mathbb{N}$, are given by the formula

$$[x^n](C_k(x))^s = \frac{s}{kn+s}\binom{kn+s}{n}.$$

Every non-empty $(k-1)$-ary tree can be considered as a k-ary tree which has all its k-th children empty.

A maximal $(k-1)$-ary subtree of a k-ary tree T is any tree obtained by choosing a k-th child in T (or T itself) and by deleting every k-th child in it. Obviously, two maximal $k-1$-ary subtrees of T are disjoint; (see for example Figure 3).

Consequently, if T has n vertices, then it contains $n+1$ maximal $(k-1)$-ary subtrees.

A (totally ordered) *forest* of k-ary trees is an element $\mathcal{F}$ of the cartesian product $\mathcal{T}_k^{\lambda}$, for some $\lambda \in \mathbb{N}^*$. We denote, for simplicity, the forest which consists of a single tree T by T. The *concatenation* of the forests $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\rho$ is a new forest which consists of the trees of each $\mathcal{F}_j$, $j \in [\rho]$, ordered by extending the orders of every $\mathcal{F}_j$, preserving at the same time their natural order. The *length* and the *size* of a forest $\mathcal{F} = (T_1, T_2, \ldots, T_\lambda)$ are defined respectively by

$$|\mathcal{F}| = \lambda \quad \text{and} \quad s(\mathcal{F}) = \sum_{i=1}^{\lambda} s(T_i).$$

A non-empty *i-path*, $i \in \mathbb{N}^*$, is a lattice path starting at the point $(0,0)$ and consisting of steps $u_i = (1,i)$ (rises) and $d = (1,-1)$ (falls). The empty path ε is the path with no steps.

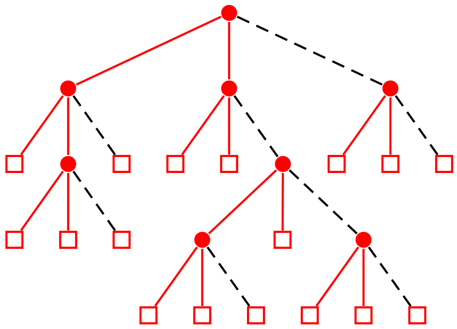

Figure 3: The 3-ary tree of Figure 2 and all its binary maximal subtrees.

For every i-path P we denote by $r(P)$ (respectively $f(P)$) the number of rises (respectively falls) of P. An *i-Dyck path* is an i-path that never falls below the x axis and ends at the x axis. If P is an i-Dyck path, then $f(P) = i \cdot r(P)$ and P ends at the point $((i+1)r(P), 0)$. We will refer to a 1-path (respectively a 1-Dyck path) as a path (respectively a Dyck path); in this case we write u instead of u_1. The set of all i-Dyck paths with n rises is denoted by $\mathcal{D}_n^{(i)}$ and $\mathcal{D}^{(i)} = \bigcup_{n\geq 0} \mathcal{D}_n^{(i)}$. In particular, we write $\mathcal{D}$ (respectively $\mathcal{D}_n$) instead of $\mathcal{D}^{(1)}$ (respectively $\mathcal{D}_n^{(1)}$) for the ordinary Dyck paths.

Every non empty i-Dyck path P is written in the following form, called the *first return decomposition* (for $i = 1$, see [2]):

$$P = u_i Q_1 d Q_2 d \cdots Q_i d Q_{i+1} \tag{2}$$

where $Q_j \in \mathcal{D}^{(i)}$, $j \in [i+1]$. Using this decomposition and the Lagrange inversion formula, it can be easily obtained that i-Dyck paths with n rises are counted by $C_n^{(i+1)}$. A simple bijection θ between the k-ary trees and the $(k-1)$-Dyck paths is given as follows:

$$\theta(\square) = \varepsilon \quad \text{and} \quad \theta(T_1 T_2 \cdots T_k) = u_{k-1}\theta(T_1)d\theta(T_2)d\cdots\theta(T_{k-1})d\theta(T_k).$$

Another well known decomposition of non-empty i-Dyck paths which will be used in this paper is based on the length of the first ascent, i.e.,

$$P = u_i^{\mu} d Q_1 d Q_2 d \cdots Q_{\mu i} \tag{3}$$

where $Q_j \in \mathcal{D}^{(i)}$ and $j \in [\mu i]$.

Every i-path P is uniquely determined by its sequence of ascents $(l_m)_{m\in[\mu]}$, $\mu \in \mathbb{N}^*$, according to the formula $P = u_i^{l_1} d u_i^{l_2} d \cdots u_i^{l_{\mu-1}} d u_i^{l_\mu}$, where $u_i^j = u_i u_i \cdots u_i$ (j times). Clearly, if P ends at the x axis (as in the case of i-Dyck paths), then $l_\mu = 0$. The sum of the elements of this sequence equals the number of rises in the path and the number of its elements is one more than the number of falls; (see for example Figure 4).

We note that the path $P = u_i^{l_1} d u_i^{l_2} d \cdots u_i^{l_{\mu-1}} d u_i^{l_\mu}$ is an $(i-1)$-Dyck path if and only if

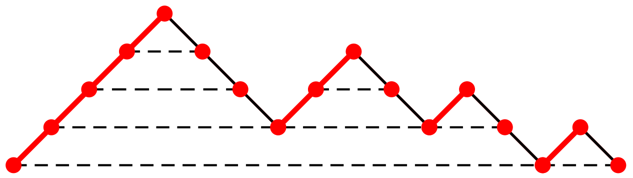

Figure 4: The Dyck path having ascent sequence $(4, 0, 0, 2, 0, 1, 0, 1, 0)$.

the following two conditions hold:

$$(i-1)\sum_{j=1}^{m} l_j \geq m, \text{ for all } m \in [\mu-1] \quad \text{and} \quad (i-1)\sum_{j=1}^{\mu} l_j = \mu - 1.$$

For every forest $\mathcal{F}$ we denote by $P_i(\mathcal{F})$ the i-path with ascent sequence the sequence of sizes of the trees in $\mathcal{F}$. If $i = 1$ we write $P(\mathcal{F})$ instead of $P_1(\mathcal{F})$. It is evident that $r(P_i(\mathcal{F})) = s(\mathcal{F})$ and $f(P_i(\mathcal{F})) = |\mathcal{F}| - 1$.

We note that if $\mathcal{F}$ is the concatenation of the forests $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\rho$, then

$$P_i(\mathcal{F}) = P_i(\mathcal{F}_1)dP_i(\mathcal{F}_2)d\cdots P_i(\mathcal{F}_{\rho-1})dP_i(\mathcal{F}_\rho). \tag{4}$$

3 Generation of k-ary trees from $(k-1)$-ary trees

For every non-empty $T \in \mathcal{T}_k$, $k \geq 2$, we denote by $\mathcal{F}(T)$ the forest consisting of all maximal $(k-1)$-ary subtrees of T, ordered according to the first time visit (in preorder) of the trees of $\mathcal{F}(T)$ in T and by T^* the first component of $\mathcal{F}(T)$; (see for example Figure 5).

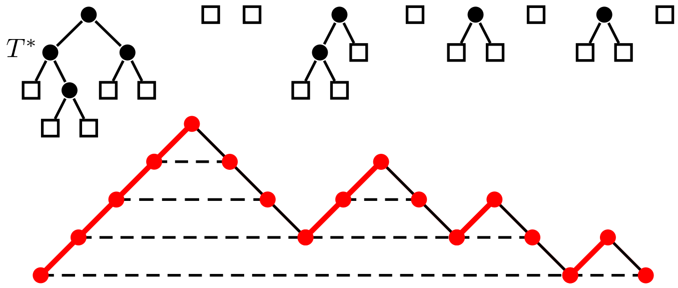

Figure 5: The forest $\mathcal{F}(T)$ and its corresponding path $P(\mathcal{F}(T))$ for the tree T of Figure 2.

Clearly, T^* is rooted at the root of T and can be obtained by deleting every k-th child of T. If T is empty then T^* is empty and $\mathcal{F}(\square) = \square$. It is clear that the last tree of $\mathcal{F}(T)$ is always the empty tree.

Using decomposition (1), we can easily check that $\mathcal{F}(T)$ is the concatenation of T^*, $\widetilde{\mathcal{F}}(T_1)$, $\widetilde{\mathcal{F}}(T_2)$, $\ldots$, $\widetilde{\mathcal{F}}(T_{k-1})$, $\mathcal{F}(T_k)$, where $T^* = T_1^* T_2^* \cdots T_{k-1}^*$ and $\widetilde{\mathcal{F}}(T_j)$, $j \in [k-1]$, is $\mathcal{F}(T_j)$ excluding T_j^*.

We have the following result.

Proposition 1. *For every $T = T_1 T_2 \cdots T_k \in \mathcal{T}_k$ we have*

i) $s(T^*) = 1 + \sum_{j=1}^{k-1} s(T_j^*)$.

ii) $s(\mathcal{F}(T)) = s(T)$.

iii) $|\mathcal{F}(T)| = s(T) + 1$.

iv) $P(\mathcal{F}(T)) = u^{\nu} dP(\widetilde{\mathcal{F}}(T_1)) dP(\widetilde{\mathcal{F}}(T_2)) \cdots dP(\widetilde{\mathcal{F}}(T_{k-1})) dP(\mathcal{F}(T_k))$, *where* $\nu = s(T^*)$.

v) $P(\mathcal{F}(T))$ *is a Dyck path.*

Proof. The proof of (i) is obvious, whereas the proof of (iv) follows immediately from relation (4). The proofs of (ii), (iii) and (v) use induction as follows:

$$\begin{aligned} s(\mathcal{F}(T)) &= s(T^*) + \sum_{j=1}^{k-1} (s(\mathcal{F}(T_j)) - s(T_j^*)) + s(\mathcal{F}(T_k)) \\ &= s(T^*) + \sum_{j=1}^{k} (s(T_j)) - \sum_{j=1}^{k-1} s(T_j^*) = 1 + \sum_{j=1}^{k} s(T_j) = s(T). \end{aligned}$$

$$|\mathcal{F}(T)| = 1 + \sum_{j=1}^{k-1} (|\mathcal{F}(T_j)| - 1) + |\mathcal{F}(T_k)| = 1 + \sum_{j=1}^{k-1} s(T_j) + 1 + s(T_k) = 1 + s(T).$$

Finally, since each $P(\mathcal{F}(T_i)), i \in [k]$ is a Dyck path, it follows that the path

$$uP(\mathcal{F}(T_1))P(\mathcal{F}(T_2)) \cdots P(\mathcal{F}(T_{k-1}))dP(\mathcal{F}(T_k))$$

is also a Dyck path and hence, using the equalities $P(\mathcal{F}(T_j)) = u^{s(T_j^*)} dP(\widetilde{\mathcal{F}}(T_j))$, $j \in [k-1]$, and $(i), (iv)$, we obtain that $P(\mathcal{F}(T))$ is a Dyck path. ◻

In the sequel we will show that k-ary trees can be generated by certain forests of $(k-1)$-ary trees. For this, we will introduce a new decomposition of k-ary trees. For $T \in \mathcal{T}_k$, we denote by Z_i the k-th child in T of the i-th (in postorder) vertex of T^*. Clearly, T can be uniquely recovered by attaching each Z_i as the k-th child to the i-th (in postorder) vertex of T^*. The trees $T^*, Z_1, Z_2, \ldots, Z_\nu$, where $\nu = s(T^*)$, form a decomposition of T called the *first component decomposition*; (see for example Figure 6).

Proposition 2. *For every $T \in \mathcal{T}_k$, the forest $\mathcal{F}(T)$ is the concatenation of the forests $T^*, \mathcal{F}(Z_1), \mathcal{F}(Z_2), \ldots, \mathcal{F}(Z_\nu)$.*

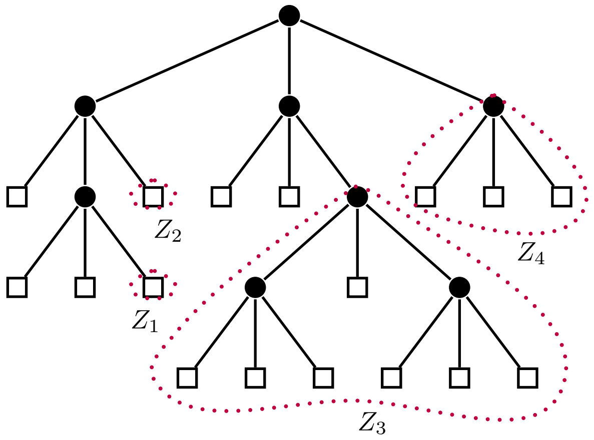

Figure 6: The first component decomposition of the tree of Figure 2.

Proof. It is enough to show that if X, Y are two trees in $\mathcal{F}(Z_i), \mathcal{F}(Z_j)$ respectively then X precedes Y in $\mathcal{F}(T)$ if and only if $i < j$ or $i = j$ and X precedes Y in $\mathcal{F}(Z_i)$.

We will prove this using induction on the size of the tree T.

If $T = T_1 T_2 \cdots T_k$, then it is evident that $Z_\nu = T_k$.

Clearly, each Z_i, $i \in [\nu - 1]$ is a subtree of a unique T_{ξ_i}, $\xi_i \in [k-1]$, such that $\xi_i \leq \xi_j$ whenever $i < j$. Then X, Y belong to $\mathcal{F}(T_{\xi_i})$, $\mathcal{F}(T_{\xi_j})$ respectively and $X \neq T^*_{\xi_i}$, $Y \neq T^*_{\xi_j}$. We consider two cases:

1. If $\xi_i \neq \xi_j$ then X precedes Y in $\mathcal{F}(T)$ if and only if $\xi_i < \xi_j$ or equivalently $i < j$.

2. If $\xi_i = \xi_j$ then X precedes Y in $\mathcal{F}(T)$ if and only if X precedes Y in $\mathcal{F}(T_{\xi_i})$ or equivalently, by the induction hypothesis $i < j$ or $i = j$ and X precedes Y in $\mathcal{F}(Z_i)$.

Hence, in every case X precedes Y in $\mathcal{F}(T)$ if and only if $i < j$ or $i = j$ and X precedes Y in $\mathcal{F}(Z_i)$. $\square$

From Proposition 2 and relation (4) we obtain a new, simpler expression for $P(\mathcal{F}(T))$, using the Dyck paths $P(\mathcal{F}(Z_j))$, $j \in [\nu]$.

$$P(\mathcal{F}(T)) = u^\nu dP(\mathcal{F}(Z_1))dP(\mathcal{F}(Z_2))\ldots dP(\mathcal{F}(Z_\nu)), \quad \text{where } \nu = s(T^*). \tag{5}$$

Proposition 3. *The mapping $T \to \mathcal{F}(T)$ is a size preserving bijection between $\mathcal{T}_k$ and the set of forests $\mathcal{F}$ of $(k-1)$-ary trees with $P(\mathcal{F}) \in \mathcal{D}$.*

Proof. Given a forest $\mathcal{F}$ of $(k-1)$-ary trees such that $P(\mathcal{F}) \in \mathcal{D}$, we will show by induction that there exists a unique tree $T \in \mathcal{T}_k$ such that $\mathcal{F} = \mathcal{F}(T)$.

Using the first ascent decomposition (3) we have $P(\mathcal{F}) = u^\nu dQ_1 dQ_2 \cdots dQ_\nu$, where ν is the size of the first element S of $\mathcal{F}$ and $Q_j \in \mathcal{D}$, for all $j \in [\nu]$. Since

$$\sum_{j=1}^{\nu}(r(Q_j) + 1) = \sum_{j=1}^{\nu} r(Q_j) + \nu = r(P(\mathcal{F})) = s(\mathcal{F}) = |\mathcal{F}| - 1,$$

it follows that there exists a sequence of forests $(\mathcal{F}_j)$, $j \in [\nu]$, such that $\mathcal{F}$ is the concatenation of $S, \mathcal{F}_1, \mathcal{F}_2, \ldots, F_\nu$ and $|\mathcal{F}_j| = r(Q_j) + 1$,

It follows from relation (4) that $P(\mathcal{F}_j) = Q_j$ and hence it is a Dyck path, for every $j \in [\nu]$. Thus, by the induction hypothesis, there exists $Z_j \in \mathcal{T}_k$, $j \in [\nu]$, such that $\mathcal{F}_j = \mathcal{F}(Z_j)$. Then T is the tree constructed by attaching Z_j to the j-th (in postorder) vertex of S as its k-th child; from Proposition 2 it follows immediately that $\mathcal{F} = \mathcal{F}(T)$.

For the proof of the uniqueness, let $\mathcal{F}(X) = \mathcal{F}(T)$; then $T^* = X^*$. If $T^*, Z_1, Z_2, \ldots, Z_\nu$ and $T^*, Y_1, Y_2, \ldots, Y_\nu$ are the first component decompositions of T and X respectively, then since $P(\mathcal{F}(T)) = P(\mathcal{F}(X))$, by relation (5) it follows that $P(\mathcal{F}(Z_i)) = P(\mathcal{F}(Y_i))$, for every $i \in [\nu]$. Furthermore, since $|\mathcal{F}(Z_i)| = f(P(\mathcal{F}(Z_i))) + 1 = f(P(\mathcal{F}(Y_i))) + 1 = |\mathcal{F}(Y_i)|$ for every $i \in [\nu]$, by Proposition 2 we obtain that $\mathcal{F}(Z_i) = \mathcal{F}(Y_i)$, for every $i \in [\nu]$. Thus, by the induction hypothesis, $Z_i = Y_i$ for each $i \in [\nu]$, so that $T = X$. □

We close this section with the following algorithmic construction of the tree $T \in \mathcal{T}_k$ such that $\mathcal{F}(T) = \mathcal{F}$, where $\mathcal{F}$ is a given forest of $(k-1)$-ary trees with $P(\mathcal{F}) \in \mathcal{D}$:

We start with the first tree of $\mathcal{F}$. At each step, we add as the k-th child of the first (in postorder) vertex which does not have a k-th child, the first tree of $\mathcal{F}$ that has not already been used. For example, the tree T of Figure 2 can be constructed from the forest of Figure 5 as shown in Figure 7.

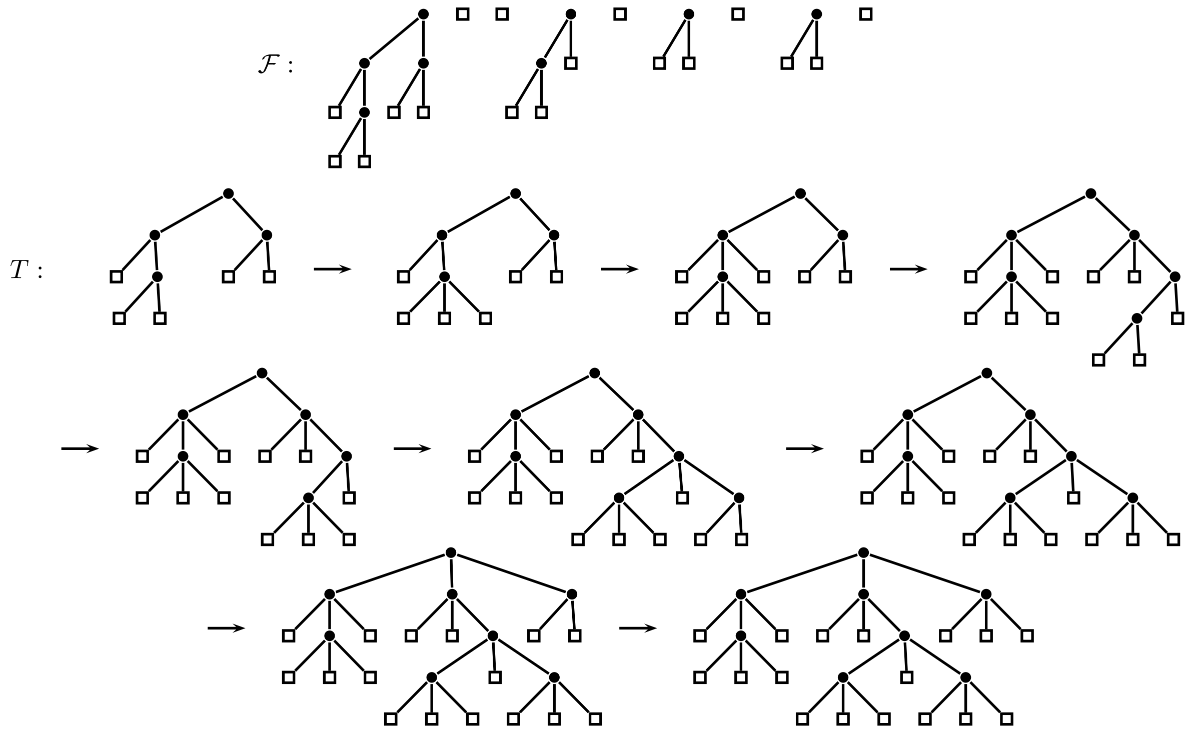

Figure 7: Construction of T from $\mathcal{F}$.

4 Generation of k-ary trees from unary trees

In this section we show how every k-ary tree can be uniquely decomposed into a forest of unary trees which leads to a new bijection between the sets $\mathcal{T}_k$ and $\mathcal{D}^{(k-1)}$. For this, we first introduce a mapping on forests denoted by $(\ )'$.

For every forest $\mathcal{F} = (T_1, T_2, \ldots, T_\lambda)$ of k-ary trees, we define the forest $\mathcal{F}'$ of $(k-1)$-ary trees to be the concatenation of the forests $\mathcal{F}(T_1), \mathcal{F}(T_2), \ldots, \mathcal{F}(T_\lambda)$. Using Proposition 1 $(ii), (iii)$, we deduce the following equalities:

$$s(\mathcal{F}') = s(\mathcal{F}) \qquad \text{and} \qquad |\mathcal{F}'| = s(\mathcal{F}) + |\mathcal{F}|. \tag{6}$$

Furthermore, it can be easily checked that if $\mathcal{F}$ is the concatenation of the forests $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\rho$, then $\mathcal{F}'$ is the concatenation of the forests $\mathcal{F}_1', \mathcal{F}_2', \ldots, \mathcal{F}_\rho'$.

The following two results establish additional properties of $(\ )'$.

Proposition 4. *For any pair of forests $\mathcal{F}, \mathcal{G}$, we have that if $\mathcal{F}' = \mathcal{G}'$ then $\mathcal{F} = \mathcal{G}$.*

Proof. Since $|\mathcal{F}| = |\mathcal{F}'| - s(\mathcal{F}') = |\mathcal{G}'| - s(\mathcal{G}') = |\mathcal{G}|$, we can write

$$\mathcal{F} = (T_1, T_2, \ldots, T_\lambda) \quad \text{and} \quad \mathcal{G} = (X_1, X_2, \ldots, X_\lambda).$$

We assume that $\mathcal{F} \neq \mathcal{G}$ and we choose ρ to be the least element of $[\lambda]$ such that $T_\rho \neq X_\rho$. Since $\mathcal{F}' = \mathcal{G}'$ and $\mathcal{F}(T_\rho) \neq \mathcal{F}(X_\rho)$, it follows that $|\mathcal{F}(T_\rho)| \neq |\mathcal{F}(X_\rho)|$; without loss of generality we assume that $|\mathcal{F}(T_\rho)| < |\mathcal{F}(X_\rho)|$. Then, there exists a forest $\mathcal{H}$ such that $\mathcal{F}(X_\rho)$ is the concatenation of $\mathcal{F}(T_\rho)$ and $\mathcal{H}$. It follows that $P(\mathcal{F}(X_\rho)) = P(\mathcal{F}(T_\rho))dP(\mathcal{H})$ which is not a Dyck path, giving the required contradiction. ∎

Proposition 5. *For every forest $\mathcal{F}$, we have that $P_{i-1}(\mathcal{F}) \in \mathcal{D}^{(i-1)}$ if and only if $P_i(\mathcal{F}') \in \mathcal{D}^{(i)}$.*

Proof. Let $\mathcal{F} = (T_1, T_2, \ldots, T_\lambda)$; then

$$P_{i-1}(\mathcal{F}) = u_{i-1}^{s(T_1)} d u_{i-1}^{s(T_2)} d \cdots u_{i-1}^{s(T_{\lambda-1})} d u_{i-1}^{s(T_\lambda)}$$

and by relation (4)

$$P_i(\mathcal{F}') = P_i(\mathcal{F}(T_1)) d P_i(\mathcal{F}(T_2)) d \cdots P_i(\mathcal{F}(T_{\lambda-1})) d P_i(\mathcal{F}(T_\lambda)).$$

Clearly, since for every $j \in [\lambda]$ the path $P(\mathcal{F}(T_j))$ is a Dyck path, the path $P_i(\mathcal{F}(T_j))$ lies above the x axis and ends at height $(i-1)s(T_j)$. Furthermore, the fall following $P_i(\mathcal{F}(T_j))$ in the path $P_i(\mathcal{F}')$ is at the same height as the fall following the ascent $u_{i-1}^{s(T_j)}$ in the path $P_{i-1}(\mathcal{F})$, for all $j \in [\lambda - 1]$, giving the required result. ∎

We now have the following result.

Proposition 6. *For every forest of $(k-1)$-ary trees $\mathcal{F}$ such that $P_i(\mathcal{F}) \in \mathcal{D}^{(i)}$, there exists a unique forest $\mathcal{G}$ of k-ary trees such that $\mathcal{G}' = \mathcal{F}$ and $|\mathcal{G}| = 1 + (i-1)s(\mathcal{F})$.*

Proof. Clearly, if $\mathcal{F} = \square$, the result holds for $\mathcal{G} = \square$, while , if $i = 1$ the result follows from Proposition 3. Otherwise, since $P_i(\mathcal{F}) \in \mathcal{D}^{(i)}$, the path $P(\mathcal{F})$ starts at the origin with a rise and ends at a point below the x-axis attaining the least possible height; so, there exists a sequence $(Q_j)_{j\in[\lambda]}$ of Dyck paths, such that

$$P(\mathcal{F}) = Q_1 d Q_2 d \cdots Q_{\lambda-1} d Q_\lambda$$

and $Q_1 \neq \varepsilon$. Then, since

$$|\mathcal{F}| = 1 + f(P(\mathcal{F})) = 1 + \lambda - 1 + \sum_{j=1}^{\lambda} f(Q_j) = \sum_{j=1}^{\lambda}(r(Q_j) + 1),$$

there exists a unique sequence $(\mathcal{F}_j)_{j\in[\lambda]}$ of forests of $(k-1)$-ary trees such that $|\mathcal{F}_j| = r(Q_j)+1$, for all $j \in [\lambda]$ and $\mathcal{F}$ is the concatenation of the forests $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\lambda$. Then, by relation (4), it follows that

$$P(\mathcal{F}) = P(\mathcal{F}_1) d P(\mathcal{F}_2) d \cdots P(\mathcal{F}_{\lambda-1}) d P(\mathcal{F}_\lambda).$$

Since for all $j \in [\lambda]$ we have that $f(P(\mathcal{F}_j)) = |\mathcal{F}_j| - 1 = f(Q_j)$, from the above two expressions of $P(\mathcal{F})$ it follows that $P(\mathcal{F}_j) = Q_j$. Thus, $P(\mathcal{F}_j)$ is a Dyck path and by Proposition 3 there exists a unique $T_j \in \mathcal{T}_k$ such that $\mathcal{F}_j = \mathcal{F}(T_j)$. Then, for $\mathcal{G} = (T_1, T_2, \ldots, T_\lambda)$, we obtain $\mathcal{G}' = \mathcal{F}$.

Now, since $P_i(\mathcal{F}) \in \mathcal{D}^{(i)}$ and $Q_j \in \mathcal{D}$ for each $j \in [\lambda]$, we have that

$$ir(P_i(\mathcal{F})) = f(P_i(\mathcal{F})) = f(P(\mathcal{F})) = \lambda - 1 + \sum_{j=1}^{\lambda} f(Q_j) = \lambda - 1 + \sum_{j=1}^{\lambda} r(Q_j) = \lambda - 1 + r(P(\mathcal{F})).$$

Furthermore, since $r(P_i(\mathcal{F})) = r(P(\mathcal{F})) = s(\mathcal{F})$ and $|\mathcal{G}| = \lambda$, it follows that $|\mathcal{G}| = 1 + (i-1)s(\mathcal{F})$.

The uniqueness of $\mathcal{G}$ follows from Proposition 4. $\square$

The next result follows directly from Propositions 5 and 6.

Proposition 7. *The mapping $(\ )'$ from the set of forests of $(k-i+1)$-ary trees with $P_{i-1}(\mathcal{F}) \in \mathcal{D}^{(i-1)}$ to the set of forests of $(k-i)$-ary trees with $P_i(\mathcal{F}) \in \mathcal{D}^{(i)}$ where $i \geq 2$, is a bijection.*

Using the mapping $(\ )'$, for every $T \in \mathcal{T}_k$ and $i \in [k-1]$, we define recursively the forest $\mathcal{F}^i(T)$ by the relations

$$\mathcal{F}^0(T) = T \qquad \text{and} \qquad \mathcal{F}^i(T) = (\mathcal{F}^{i-1}(T))'.$$

For example, for the tree T of Figure 2 for which $\mathcal{F}(T)$ has been already constructed (see Figure 5), we can easily obtain that $\mathcal{F}^2(T)$ is the forest of Figure 8.

Clearly, the forest $\mathcal{F}^i(T)$ consists of $(k-i)$-ary trees. Furthermore, from (6) we obtain inductively the following generalization of equalities $(ii), (iii)$ of Proposition 1:

$$s(\mathcal{F}^i(T)) = s(T) \qquad \text{and} \qquad |\mathcal{F}^i(T)| = is(T) + 1.$$

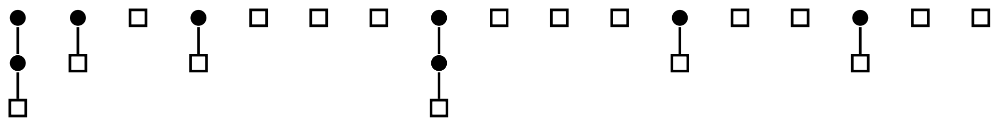

Figure 8: The forest $\mathcal{F}^2(T)$.

In particular, the second equality for $i = k-1$ shows that we have a 1-1 correspondence between the leaves of the tree T and the unary trees of $\mathcal{F}^{k-1}(T)$.

Using Propositions 5 and 6 we can easily show by induction that $P_i(\mathcal{F}^i(T)) \in \mathcal{D}^{(i)}$, for every $i \in [k-1]$. Furthermore, using Proposition 7 we deduce by induction the following result which is a generalization of Proposition 3.

Proposition 8. *For every $i \in [k-1]$, the mapping $T \to \mathcal{F}^i(T)$ is a size preserving bijection between $\mathcal{T}_k$ and the set of forests $\mathcal{F}$ of $(k-i)$-ary trees with $P_i(\mathcal{F}) \in \mathcal{D}^{(i)}$.*

An application of the previous result for $i = k-1$ gives that the mapping $\mathcal{T} \to \mathcal{F}^{k-1}(\mathcal{T})$ is a size preserving bijection between $\mathcal{T}_k$ and the set of forests $\mathcal{F}$ of unary trees with $P_{k-1}(\mathcal{F}) \in \mathcal{D}^{(k-1)}$. Clearly, since any such forest $\mathcal{F}$ can be identified with the associated path $P_{k-1}(\mathcal{F})$, we obtain the following result.

Proposition 9. *The mapping $\phi : \mathcal{T}_k \to \mathcal{D}^{(k-1)}$ with $\phi(T) = P_{k-1}(\mathcal{F}^{k-1}(T))$ is a bijection such that $s(T) = r(\phi(T))$.*

Notice that the classical bijection θ mentioned in Section 2 is different from the bijection ϕ of the previous Proposition. For example, for the tree T of Figure 2 we have

$$\theta(T) = u_2u_2du_2ddddu_2ddu_2u_2ddddu_2dddu_2dd,$$

whereas

$$\phi(T) = u_2u_2du_2ddu_2ddddu_2u_2ddddu_2dddu_2dd.$$

Both bijections use recursion, θ with respect to the size, whereas ϕ with respect to k.

5 Maximal paths of k-ary trees

In this section we show that every k-ary tree can be uniquely expressed by the set of its maximal paths. Furthermore, using this expression, we give an equivalent simple formula for the bijection ϕ.

Let $\mathcal{A}_k$ be the set of all subsets A of $[k]^*$ (the set of all words on the alphabet $[k]$) satisfying the following two conditions:

i) If $x = \rho\alpha \in A$, where $\rho, \alpha \in [k]^*$ and $\alpha \neq \varepsilon$, then, for all $i \in [k]$, the set A contains at least one word of the form $\rho i \gamma_i$, where $\gamma_i \in [k]^*$.

ii) If $\rho \in A$ and $\alpha \in [k]^*$, then $\rho\alpha \in A$ if and only if $\alpha = \varepsilon$.

From the above two conditions, it follows easily that $\{\varepsilon\} \in \mathcal{A}_k$ and $\varepsilon \notin A$ for all $A \in \mathcal{A}_k$ with $A \neq \{\varepsilon\}$.

We define recursively the mapping $\psi : \mathcal{T}_k \to \mathcal{A}_k$ by

$$\psi(\Box) = \{\varepsilon\} \qquad \text{and} \qquad \psi(T_1T_2\cdots T_k) = \{i\alpha : \alpha \in \psi(T_i), i \in [k]\}.$$

For example, for the tree T of Figure 2 we have

$$\psi(T) = \{11, 121, 122, 123, 13, 21, 22, 2311, 2312, 2313, 232, 2331, 2332, 2333, 31, 32, 33\}.$$

It is easy to check that ψ is a bijection, such that $|\psi(T)| = (k-1)s(T)+1$, for all $T \in \mathcal{T}_k$. Furthermore, the elements of $\psi(T)$ code the maximal paths of T. In fact, the maximal path $S = v_1v_2\cdots v_{\ell+1}$ of T or, equivalently, its associated leaf $v_{\ell+1}$, is coded by the word $\alpha = a_1a_2\cdots a_\ell \in \psi(T)$ if and only if v_{i+1} is the a_i-th child of v_i, for all $i \in [\ell]$.

Additionally, since $|\mathcal{F}^{k-1}(T)| = |\psi(T)|$, there exists a 1-1 correspondence between the sequences of $\psi(T)$ and the trees of $\mathcal{F}^{k-1}(T)$ such that every $x \in \psi(T)$ corresponds to a unique unary tree T_x of $\mathcal{F}^{k-1}(T)$, which is the left path of the leaf which is coded by x. For example, the word $x = 2311$ of $\psi(T)$ in the tree T of Figure 2 corresponds to the 8-th element of the forest $\mathcal{F}^2(T)$ of Figure 8.

Using the above expression of k-ary trees, we will give a method for the construction of a $(k-1)$-Dyck path from a set $A \in \mathcal{A}_k$ endowed with a total order. Firstly, for each $x \in [k]^*$, we set $l(x)$ to be the number of trailing 1's of x. Clearly, $l(x) = s(T_x)$, for every $x \in \psi(T)$.

Proposition 10. *Let $\preceq$ be a partial order on $[k]^*$ satisfying the following conditions:*

1. *the restriction of $\preceq$ on A is a total order and* $\min A$ *is the element of A which contains only 1's,*

2. *$i\alpha \preceq i\beta$ if and only if $\alpha \preceq \beta$, for all $\alpha, \beta \in A$ and $i \in [k]$,*

for each $A \in \mathcal{A}_k$. Then we have that

$$u_{k-1}^{l(\alpha_1)}du_{k-1}^{l(\alpha_2)}d\cdots u_{k-1}^{l(\alpha_{\mu-1})}du_{k-1}^{l(\alpha_\mu)} \in \mathcal{D}^{(k-1)},$$

for all $A \in \mathcal{A}_k$, where $A = \{\alpha_1, \alpha_2, \ldots, \alpha_\mu\}$ and $\alpha_1 \preceq \alpha_2 \preceq \cdots \preceq \alpha_\mu$.

Proof. In order to prove that the above path is a $(k-1)$-Dyck path, it suffices to prove that

$$(k-1)\sum_{\substack{x\in A\\ x\preceq y}} l(x) \geq |\{x \in A : x \preceq y\}|, \qquad \text{and} \qquad (k-1)\sum_{x\in A} l(x) = |A|-1,$$

where $y \in A \setminus \{\alpha_\mu\}$. We will use induction with respect to the cardinality of $A \in \mathcal{A}_k$.

For $A = \{\varepsilon\}$, the result is true. For $A \neq \{\varepsilon\}$ we will prove only the inequality, since the equality can be proved analogously.

We define $A_i = \{\alpha \in [k]^* : i\alpha \in A\}$, for each $i \in [k]$. It is easy to check that $A_i \in \mathcal{A}_k$, for every $i \in [k]$. Furthermore, we define $I = \{i \in [k] : i\alpha \preceq y \text{ for some } \alpha \in A_i\}$. Obviously, $1 \in I$. For each $i \in I$, we denote by α_i the maximum element of A_i with $i\alpha_i \preceq y$. Then, we have

$$
\begin{aligned}
(k-1)\sum_{\substack{x\in A\\ x\preceq y}} l(x) &= (k-1)\sum_{i\in I}\sum_{\substack{\alpha\in A_i\\ i\alpha\preceq y}} l(i\alpha) = (k-1)\sum_{i\in I}\sum_{\substack{\alpha\in A_i\\ \alpha\preceq\alpha_i}} l(i\alpha)\\
&= (k-1)\sum_{i\in I\setminus\{1\}}\sum_{\substack{\alpha\in A_i\\ \alpha\preceq\alpha_i}} l(\alpha) + (k-1)(1+\sum_{\substack{\alpha\in A_1\\ \alpha\preceq\alpha_1}} l(\alpha))\\
&= \sum_{i\in I}(k-1)\sum_{\substack{\alpha\in A_i\\ \alpha\preceq\alpha_i}} l(\alpha) + (k-1)\\
&\geq \sum_{i\in I\setminus\{k\}}(|\{\alpha\in A_i:\alpha\preceq\alpha_i\}|-1) + \sum_{i\in I\cap\{k\}}|\{\alpha\in A_i:\alpha\preceq\alpha_i\}| + (k-1)\\
&= \sum_{i\in I}|\{\alpha\in A_i:\alpha\preceq\alpha_i\}| - |I\setminus\{k\}| + k - 1 \geq |\{x\in A: x\preceq y\}|.
\end{aligned}
$$

□

From the previous proposition, it follows that given a partial order "$\preceq$" on $[k]^*$ satisfying conditions 1, 2, we have that

$$
(k-1)\sum_{j=1}^{m} l(\alpha_j) \geq m, \quad \text{for every } m\in[\mu-1] \qquad \text{and} \qquad (k-1)\sum_{j=1}^{\mu} l(\alpha_j) = \mu-1,
$$

for every set $A=\{\alpha_1,\alpha_2,\dots,\alpha_\mu\}\in\mathcal{A}_k$ with $\alpha_t\preceq\alpha_s$ if and only if $t\leq s$. Thus, the mapping χ on $\mathcal{A}_k$ defined by

$$
\chi(\{\varepsilon\})=\varepsilon, \qquad \chi(A)=u_{k-1}^{l(\alpha_1)}du_{k-1}^{l(\alpha_2)}d\cdots u_{k-1}^{l(\alpha_{\mu-1})}d
$$

takes values in $\mathcal{D}^{(k-1)}$.

Clearly, this mapping depends on the choice of the partial order "$\preceq$". If "$\preceq$" is the lexicographic order on $[k]^*$ (which obviously satisfies the conditions of Proposition 10), then the resulting mapping χ may be used in order to give an explicit formula of the bijection θ.

In order to describe the bijection ϕ using the above equivalent expression of k-ary trees, we need an ordering for the elements of each $A\in\mathcal{A}_k$. Thus we define a partial order on $[k]^*$, denoted by "$\preceq$", as follows: If $x=\rho\alpha$ and $y=\rho\beta$, where ρ is the maximal common initial part (possibly empty) of x and y, then

$$
x\preceq y \Leftrightarrow \begin{cases} \alpha=\beta=\varepsilon, \text{ or}\\ \max\alpha<\max\beta, \text{ or}\\ \max\alpha=\max\beta \quad \text{and first element of } \alpha < \text{first element of } \beta. \end{cases}
$$

Clearly, from condition ii) in the definition of $\mathcal{A}_k$, it follows that "$\preceq$" is a total order on each $A\in\mathcal{A}_k$.

For example, the elements of $\psi(T)$ for the tree T of Figure 2 are ordered as follows:

$$
\begin{aligned}
&11\preceq 121\preceq 122\preceq 21\preceq 22\preceq 123\preceq 13\preceq 2311\\
&\quad\preceq 2312\preceq 232\preceq 2313\preceq 2331\preceq 2332\preceq 2333\preceq 31\preceq 32\preceq 33.
\end{aligned}
$$

So, for the set $A = \psi(T)$, we have the following 2-Dyck path:

$$\chi(A) = u_2^2 du_2 ddu_2 ddddu_2^2 ddddu_2 dddu_2 dd.$$

Proposition 11. *For every $k \in \mathbb{N}^*$ and $T \in \mathcal{T}_k$, we have that*

$$\chi(\psi(T)) = \phi(T).$$

Proof. Using the first component decomposition $T^*, Z_1, \ldots, Z_\nu$, where $\nu = s(T^*)$, of a tree $T \in \mathcal{T}_k$, it follows inductively using Proposition 2 that $\mathcal{F}^{k-1}(T)$ is the concatenation of the forests $\mathcal{F}^{k-2}(T^*), \mathcal{F}^{k-1}(Z_1), \ldots, \mathcal{F}^{k-1}(Z_\nu)$. We will show by induction with respect to k and to the size of the tree, that if $x, y \in \psi(T)$ and T_x, T_y are their associated trees in $\mathcal{F}^{k-1}(T)$, then T_x precedes T_y in $\mathcal{F}^{k-1}(T)$ if and only if $x \preceq y$.

We consider the following cases:

1. T_x, T_y are in $\mathcal{F}^{k-2}(T^*)$. Then $x, y \in \psi(T^*)$ and hence, using the induction hypothesis (with respect to k), we deduce that $x \preceq y$.

2. T_x is in $\mathcal{F}^{k-2}(T^*)$ and T_y is in $\mathcal{F}^{k-1}(Z_i)$, for some $i \in [\nu]$. Then $x \prec y$, since $\max x < k = \max y$.

3. T_x, T_y are in $\mathcal{F}^{k-1}(Z_i)$, for some $i \in [\nu]$. Then $x = \rho k \alpha$ and $y = \rho k \beta$, where $\rho \in [k-1]^*$ and $\alpha, \beta \in [k]^*$. It follows that $\alpha, \beta \in \psi(Z_i)$ and hence, using the induction hypothesis (with respect to the size of the tree), we deduce that $\alpha \preceq \beta$ and therefore $x \preceq y$.

4. T_x is in $\mathcal{F}^{k-1}(Z_i)$ and T_y is in $\mathcal{F}^{k-1}(Z_j)$, where $i, j \in [\nu]$ and $i \neq j$. Then $i < j$, so that the parent of Z_i precedes the parent of Z_j (in postorder). Thus, $x = \rho\alpha$ and $y = \rho\beta$, $\rho \in [k-1]^*$ (where ρ is the initial common part of x and y) and $\alpha, \beta \in [k]^*$. It follows that $\max\alpha = \max\beta = k$ and first element of $\alpha <$ first element of β, and hence $x \preceq y$.

This shows that in all cases, $x \preceq y$.

The converse now follows obviously, since $\mathcal{F}^{k-1}(T)$ is totally ordered.

Finally, since $l(x) = s(T_x)$, for every $x \in \psi(T)$, the $(k-1)$-paths $\chi(\psi(T))$ and $\phi(T)$ have the same ascent sequence and therefore they are identical. □

6 Enumerations

In this section, we study some statistics on k-ary trees related to the notions studied in the previous sections.

6.1 Enumeration of $\mathcal{T}_k$ according to the number of non-empty trees of $\mathcal{F}^i(T)$

Given $i, k \in \mathbb{N}^*$ with $i \leq k-1$ and $T \in \mathcal{T}_k$, we denote by $p_{i,k}(T)$ the number of non-empty trees of $\mathcal{F}^i(T)$ and by $F_{i,k}$ the generating function

$$F_{i,k}(x, y) = \sum_{T \in \mathcal{T}_k} x^{s(T)} y^{p_{i,k}(T)}.$$

It follows that

$$p_{i,k}(\square) = 0 \qquad \text{and} \qquad p_{i,k}(T_1T_2\cdots T_k) = 1 + \sum_{j=1}^{k} p_{i,k}(T_j) - \sum_{j=1}^{k-i}[T_j \neq \square],$$

where $[P]$ is the well known Iverson notation defined by $[P] = \begin{cases} 1, & \text{if } P \text{ is true;} \\ 0, & \text{if } P \text{ is false.} \end{cases}$

The above equality can be proved by induction (with respect to k). Indeed, since the forest $\mathcal{F}(T)$ is the concatenation of $T^*, \widetilde{\mathcal{F}}(T_1), \widetilde{\mathcal{F}}(T_2), \ldots, \widetilde{\mathcal{F}}(T_{k-1}), \mathcal{F}(T_k)$, where $T^* = T_1^*T_2^*\cdots T_{k-1}^*$, we can easily check that the forest $\mathcal{F}^i(T)$ is the concatenation of

$$\mathcal{F}^{i-1}(T^*), \widetilde{\mathcal{F}^i}(T_1), \widetilde{\mathcal{F}^i}(T_2), \ldots, \widetilde{\mathcal{F}^i}(T_{k-1}), \mathcal{F}^i(T_k).$$

Furthermore, using the induction hypothesis for the tree $T^* \in \mathcal{T}_{k-1}$, we obtain that

$$\begin{aligned} p_{i,k}(T) &= p_{i-1,k-1}(T^*) + \sum_{j=1}^{k-1}(p_{i,k}(T_j) - p_{i-1,k-1}(T_j^*)) + p_{i,k}(T_k) \\ &= 1 + \sum_{j=1}^{k-1} p_{i-1,k-1}(T_j^*) - \sum_{j=1}^{k-1-(i-1)}[T_j^* \neq \square] + \sum_{j=1}^{k-1} p_{i,k}(T_j) - \sum_{j=1}^{k-1} p_{i-1,k-1}(T_j^*) + p_{i,k}(T_k) \\ &= 1 + \sum_{j=1}^{k} p_{i,k}(T_j) - \sum_{j=1}^{k-i}[T_j \neq \square]. \end{aligned}$$

From the above relation, it follows that the generating function $F_{i,k}(x, y)$ satisfies the following equation:

$$F_{i,k}(x, y) = 1 + xy(F_{i,k}(x, y))^i(1 + \frac{1}{y}(F_{i,k}(x, y) - 1))^{k-i}.$$

Using the Lagrange inversion formula, we obtain the following result.

Proposition 12. *The number of all k-ary trees of size n for which the forest $\mathcal{F}^i(T), i \in [k-1]$, contains exactly j non-empty trees is equal to*

$$[x^ny^j]F_{i,k} = \frac{1}{n}\binom{ni}{j-1}\binom{(k-i)n}{n-j}.$$

The above result is of special interest for the cases $i = k-1$ and $i = 1$. In particular, using the bijection φ, we can easily check that $p_{k-1,k}(T) = N_{ud}(\phi(T))$, where $N_{ud}(\phi(T))$ denotes the number of ud's (peaks) in $\phi(T)$; thus the above two parameters are equidistributed, which implies that the number of $(k-1)$-Dyck paths having n rises and j peaks is equal to $\frac{1}{n}\binom{(k-1)n}{j-1}\binom{n}{j}$.

We note that since the number $p_{k-1,k}(T)-1$, which counts the non-empty trees in $\mathcal{F}^{k-1}(T)$ other than the first one, is equal to $N_{du}(\phi(T))$ we can easily deduce that the number of $(k-1)$-Dyck paths with n rises and j valleys is equal to $\frac{1}{n}\binom{(k-1)n}{j}\binom{n}{j+1}$.

Furthermore, since $|\mathcal{F}^{k-1}(T)| = (k-1)s(T)+1$, we obtain that the number of empty trees in $\mathcal{F}^{k-1}(T)$ other than the last one (which is always empty) is equal to $(k-1)s(T)-p_{k-1,k}(T)$. Since we can easily check that this number is equal to $N_{dd}(\phi(T))$, we deduce that the number of $(k-1)$-Dyck paths having n rises and j doublefalls is equal to $\frac{1}{n}\binom{(k-1)n}{j+1}\binom{n}{(k-1)n-j}$.

On the other hand, using a variation θ' of θ defined by

$$\theta'(\square) = \varepsilon \qquad \text{and} \qquad \theta'(T_1T_2\cdots T_k) = u\theta'(T_k)d\theta'(T_{k-1})\cdots d\theta'(T_2)d\theta'(T_1),$$

we can easily check by induction that $N_{uu}(\theta'(T)) = p_{1,k}(T) - [T \neq \square]$, for every $T \in \mathcal{T}_k$. From this equality, it follows easily that the number of all $(k-1)$-Dyck paths having n rises and j doublerises is equal to $\frac{1}{n}\binom{n}{j}\binom{(k-1)n}{n-j-1}$.

S. Heubach et al. [6] give analogous results on similar generalized Dyck paths.

6.2 Enumeration according to the size of the first element of $\mathcal{F}^i(T)$

For every $T \in \mathcal{T}_k$ we denote by $q_{i,k}(T), i \in [k-1]$, the size of the first element of $\mathcal{F}^i(T)$ and $G_{i,k}(x,y)$ the generating function

$$G_{i,k}(x,y) = \sum_{T\in\mathcal{T}_k} x^{s(T)}y^{q_{i,k}(T)}.$$

It follows that

$$q_{i,k}(\square) = 0 \qquad \text{and} \qquad q_{i,k}(T_1T_2\cdots T_k) = 1 + \sum_{j=1}^{k-i} q_{i,k}(T_j).$$

The above equality can be proved easily by induction (with respect to k), using the equality $q_{i,k}(T) = q_{i-1,k-1}(T^*)$.

From the above relation, it follows that the generating function $G_{i,k}(x,y)$ satisfies the following equation:

$$G_{i,k}(x,y) = 1 + xy(G_{i,k}(x,1))^i(G_{i,k}(x,y))^{k-i} = 1 + xy(C_k(x))^i(G_{i,k}(x,y))^{k-i}.$$

Using the Lagrange inversion formula, we obtain the following result.

Proposition 13. *The number of all k-ary trees T of size n, for which the first element of $\mathcal{F}^i(T), i \in [k-1]$, has size j is equal to*

$$[x^ny^j]G_{i,k}(x,y) = \frac{i}{(n-j)k+ij}\binom{(n-j)k+ij}{n-j}\binom{(k-i)j}{j-1}.$$

For the case $i = k-1$, using the bijection φ, we can easily check that $q_{k-1,k}(T)$ is the length of the first ascent of $\phi(T)$, thus the number of $(k-1)$-Dyck paths having n rises and length of first ascent equal to j is $\frac{(k-1)j}{kn-j}\binom{kn-j}{n-j}$.

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2000 *Mathematics Subject Classification*: Primary 05A15; Secondary 05A19.
Keywords: Generalized Dyck words, k-ary trees, k-Catalan numbers.

(Concerned with sequence A000108.)

Received July 14 2009; revised version received November 3 2009. Published in *Journal of Integer Sequences*, November 4 2009.

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