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A Bijection Between 3-Motzkin Paths and Schröder Paths With No Peak at Odd Height

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Abstract

A new bijection between 3-Motzkin paths and Schröder paths with no peak at odd height is presented, together with numerous consequences involving related combinatorial structures such as 2-Motzkin paths, ordinary Motzkin paths and Dyck paths.

1 Introduction

We define *k-Motzkin paths* as paths from $(0,0)$ to $(n,0)$ not going below x-axis using steps $U=(1,1), D=(1,-1)$ and $L=(1,0)$ where the L steps can be any of k colors. The cases with $k=0,1,2$ and 3 counting these paths yield

$k=0$, Catalan numbers, $1,0,1,0,2,0,5,0,14,0,\ldots$, A097331;

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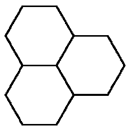
$k=1$, Motzkin numbers, $1,1,2,4,9,21,51,127,\ldots$, A001006;

$k=2$, Catalan numbers(C_{n+1}), $1,2,5,14,42,132,\ldots$, A000108;

$k=3$, Harary-Read Benzene numbers, $1,1,3,10,36,137,543,\ldots$, A002212.

The 'A' numbers are from Sloane's *Encyclopedia* [8].

Harary-Read Benzene numbers count the number of restricted hexagonal polyominoes with n cells and equivalently rooted planar trees where the outdegree of each vertex is $0,1$ or 2. If the outdegree is 1, then it can be a left, right, or middle child. If the outdegree is 2, the vertex must have a left and right child. Denote such trees by *HR-trees*. This last condition prohibits the configuration shown below.



We denote the generating function (GF for short) for the k-Motzkin numbers as $M_{[k]}$. If we change the level step requirement by one kind of level step of length 2, we get the *Schröder paths* and the sequence of numbers counting the Schröder paths starts $1,2,6,22,90,394,\ldots$, A006318.

Returning to the k-Motzkin paths but not allowing any level steps at height 0, we have the k-Riordan paths. This yields

$k=1$, the Riordan numbers, $1,0,1,1,3,6,15,36,91,\ldots$, A005043;

$k=2$, the Fine numbers, $1,0,1,2,6,18,57,\ldots$, A000957;

$k=3$, the 3-Riordan numbers, $1,0,1,3,11,42,167,684,\ldots$, A117641.

For Benzene molecules $A,B,C,D,\ldots$, the 3-Riordan numbers count the number of ways pasting Benzene rings with no attachments occurring on the dark edges. The 3-Riordan numbers also count the number of HR-trees with no middle children on the rightmost branch. Figure 1 is a small example to illustrate these concepts.

The GF for counting k-Riordan paths is denoted $R_{[k]}$. More detail can be found in [9]. For information about $M_{[3]}$, see also [1] and [6].

The basic identities for these GFs are

$$M_{[k]} = 1 + kzM_{[k]} + z^2 M_{[k]}^2 \quad \text{and} \quad R_{[k]} = 1 + z^2 M_{[k]} R_{[k]}.$$

The two generating functions are linked by the identity

$$\begin{aligned} M_{[k]} &= R_{[k]} + R_{[k]} \cdot kz \cdot R_{[k]} + R_{[k]} \cdot kz \cdot R_{[k]} \cdot kz \cdot R_{][k]} + \cdots \\ &= \frac{R_{[k]}}{1-kzR_{[k]}}, \end{aligned}$$

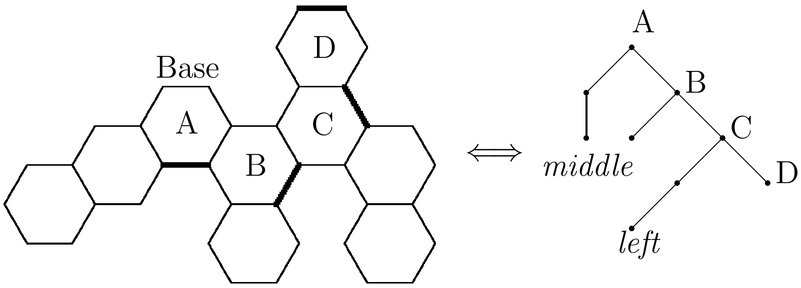


Figure 1: Benzene Molecules and the corresponding HR-tree.

where, for instance, $R_{[k]} \cdot kz \cdot R_{[k]}$ accounts for k-Motzkin paths with one level step at height 0.

This research started while investigating Möbius transformations of generating functions. It was noticed that $\frac{R_{[1]}}{1-zR_{[1]}} = M_{[1]}$ yields the Motzkin numbers, and $\frac{R_{[2]}}{1-2zR_{[2]}}$ yields the Catalan numbers. Next in the series is $M_{[3]} = \frac{R_{[3]}}{1-3zR_{[3]}}$. Consulting Sloane [8] yielded the peculiar fact that these numbers also counted Schröder paths with no peak at odd height. This fact was noticed by Emeric Deutsch who outlined a proof via generating function for us. However such a coincidence seemed worthy of a more combinatorial approach.

In this paper, we present an easy bijection between the Schröder paths of length $2n$ with no peak at odd height and 3-Motzkin paths of length $n-1$. One consequence of this simple bijection is that many new occurrences of paths counted by the Catalan, Motzkin, Fine, and Riordan numbers can be found as Schröder paths with various restrictions. We will examine a few such cases in detail and then list some of the others.

A recent paper of Yan [10] discusses a different bijection between $(2,3)$-Motzkin paths and Schröder paths for the purpose of enumerating the UDD and LD subsequences. For more information about the Fine number sequences, see [2] and [4]. Some useful generating function identities are discussed in [3]. Another paper of definite interest [7] of Mansour, Schork, and Sun discusses Motzkin paths where not only level steps but also up and down steps are colored. In particular if the up steps are bicolored and neither level nor down steps are, then the sequence A025235 which starts $1, 1, 3, 7, 21, 61, 191, \ldots$ is obtained. If both level and up steps are bicolored the result is the sequence A071356 starting $1, 2, 6, 20, 72, 272, 1064, \ldots$.

2 The bijection

We describe the bijection, denoted by ϕ, between Schröder paths of length $2n$ with no peak at odd height and 3-Motzkin paths of length $n-1$. Let ω be a Schröder path of length $2n$ with no peak at odd height. First interchange peaks and level steps. We will call this the *toggle*. We now have a Schröder path with no level steps at even height. Next eliminate the first step (U) and the last step (D), and denote the remaining path by ω'. Thus, ω' is a Schröder path except that it can go down to height -1. We call this step *push down*. At this point we have no level steps at odd height. We define the bijection ϕ as follows. Traverse ω' from left to right two steps at a time where L is regarded as two steps:

- for consecutive UU (resp. DD) steps, set $\phi(UU) = u$ (resp. $\phi(DD) = d$);
- set $\phi(L) = l$;
- set $\phi(UD) = \wedge$;
- set $\phi(DU) = v$.

It is easy to see that we have obtained a 3-Motzkin path and also that the mapping can be easily reversed. More precisely, a Schröder path of length $2n$ without peaks at odd height corresponds to a 3-Motzkin path of length $n-1$. For an example, see Figure 2.

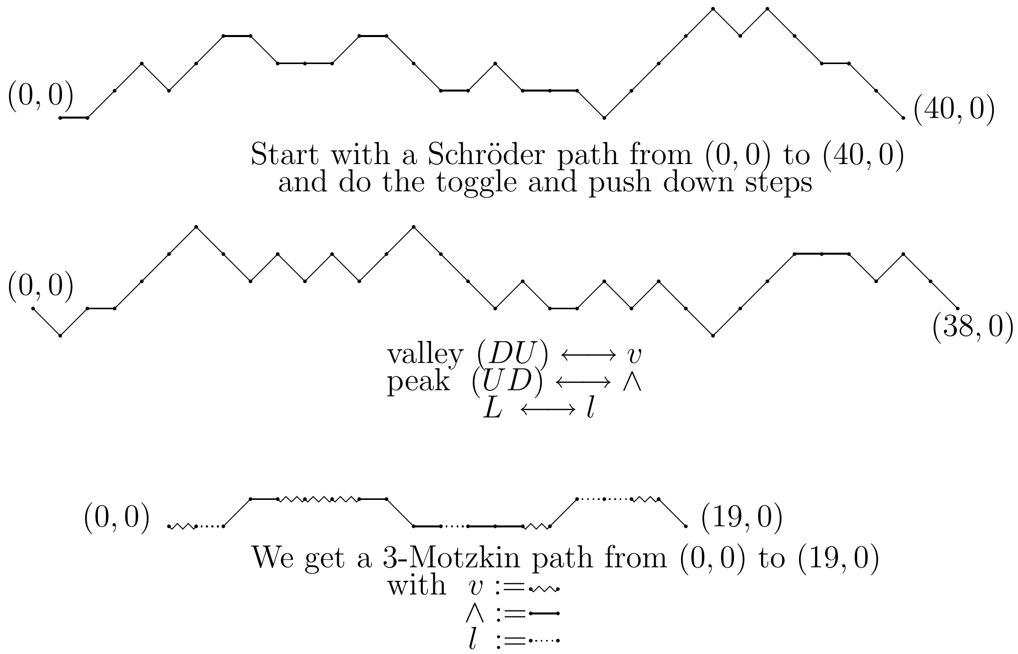


Figure 2: The bijection ϕ

Bijective correspondences. Table 1 lists features carried by the bijection ϕ from Schröder paths to 3-Motzkin paths. For instance, a level step at height 0 becomes a peak at height 1 after the toggle. After the push down we have a peak at height 0. Counting two steps at a time splits the peak into parts of two consecutive valleys. If the original level step is at either end of the Schröder path then half of it would be taken by the push down and the other half would contribute to one valley.

The other cases follow by similar reasoning.

Schröder paths (with no peaks at odd height)	3-Motzkin paths
L steps at height 0	vv consecutive steps at height 0 or a v at either end of the path
L steps at even height $2k$ ($k \geq 1$)	uv, vd, vv, ud consecutive steps where either u or v is at height $k-1$ ($k \geq 1$)
L steps at odd height	$\wedge$ steps
peaks	l steps
UU or UL where the first U is at odd height	u steps
DD or LD where the second D is at odd height	d steps

Table 1: Bijective correspondences.

3 Special Cases

Recall that 2-Motzkin paths are counted by the Catalan numbers, C_{n+1}. If we allow no level steps at height 0, we have the Fine numbers. Similarly we get Motzkin paths by allowing just one kind of level step. If we then forbid level steps at height 0, we obtain the Riordan paths. Thus applying the ϕ^{-1} algorithm will carry all these variations simultaneously.

As one example, we consider the case where the level steps for the 2-Motzkin paths are labeled l and v. Starting to apply ϕ^{-1} gives us the building blocks UU, L, DU and DD with the Catalan condition that the number of DDs never exceeds the number of UUs. We now have L steps possible at even height, peaks at even height. At this stage, the paths could go as low as the line $y = -1$.

We add a single U step at the beginning, a D step at the end (the *push up*) and we now have Schröder paths with level steps and peaks only at odd height. These are counted by the Catalan number, C_n. If we complete the ϕ^{-1} algorithm by interchanging level steps and peaks (the *toggle*), we again have peaks and level steps allowed only at even height. The 5 such (Catalan) Schröder paths from $(0, 0)$ to $(6, 0)$ are in Figure 3.

If we also prohibit L steps at height 0, we get the Fine numbers, $1, 0, 1, 2, 6, 18, 57, \ldots$. The 6 (Fine) Schröder paths from $(0, 0)$ to $(8, 0)$ without toggle and push up steps are shown in Figure 4. After the toggle and the push up, we have the elevated Schröder paths with level steps at height $2k$ ($k \geq 1$) and peaks at height $2k$ ($k \geq 2$).

To obtain one of the settings for the Motzkin numbers, we eliminate l also. This gives us Schröder paths without level steps and peaks only at odd height. If we interchange peaks

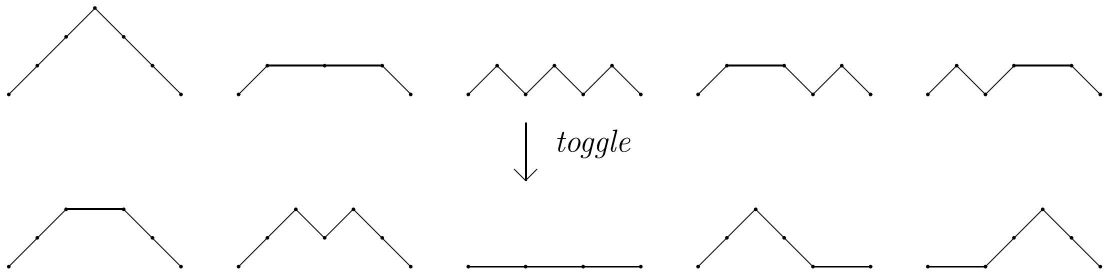


Figure 3: (Catalan) Schröder paths of length 6.

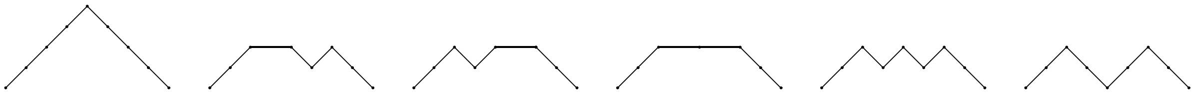


Figure 4: (Fine) Schröder paths of length 8.

and level steps we get Schröder paths with no peaks, and level steps only at even height. Here in Figure 5 are the four (Motzkin) Schröder paths from $(0,0)$ to $(8,0)$.

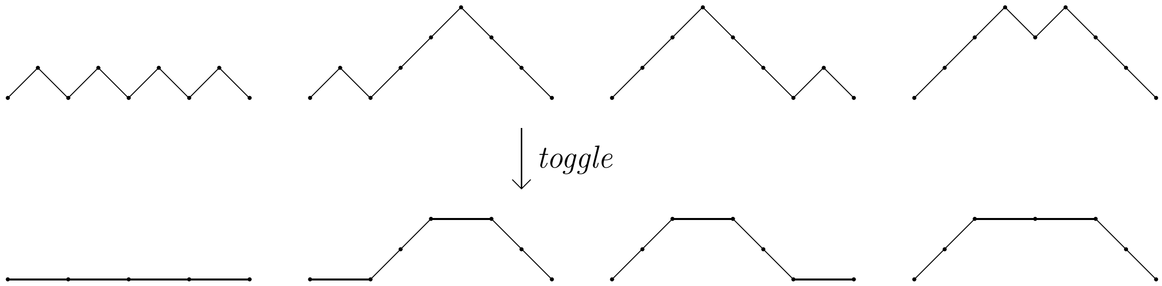


Figure 5: (Motzkin) Schröder paths of length 8.

In these last two settings we can eliminate peaks at height 1 (respectively, level steps at height 0). For paths from $(0,0)$ to $(10,0)$, we get three (Riordan) Schröder paths. We show these in Figure 6 with and without the toggle.

3.1 Remarks

We now list some of the other possibilities. We have $\binom{3}{2}$ ways to choose 2 of 3 kinds of level steps each leading to a Catalan setting and with it a Fine setting. By then eliminating one of the two kinds of level steps, we obtain two settings for the Motzkin numbers and then Riordan numbers. By not using the toggle, we double the numbers of possibilities. When the v step is not one of the level steps, we can eliminate the push down-push up steps. Since we proceed two steps at a time this also yields new settings.

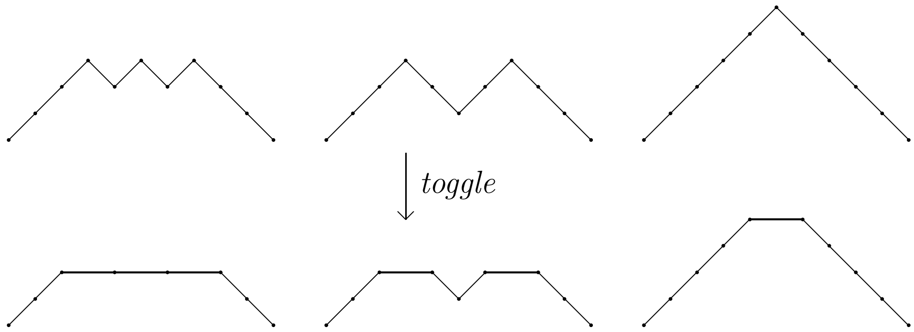


Figure 6: (Riordan) Schröder paths of length 10.

The complete ϕ^{-1} algorithm consists of both the push up and toggle steps. As fascinating as each setting is individually, the list gets quite long quite quickly.

3.2 Catalan numbers

We call a path is elevated if the path is strictly above the x-axis except the initial and end points.

If the level steps for the 2-Motzkin paths are labeled $\wedge$ and l, then by the complete ϕ^{-1} algorithm, we have the elevated Schröder paths with no peaks at odd height, no valley at even height and each L at even height occurring as part of a ULD subsequence.

If the level steps are either $\wedge$ or v, then the complete ϕ^{-1} algorithm will give us the Schröder paths with no peaks.

Eliminating all the level steps from 3-Motzkin paths, we have Dyck paths, which are counted by aerated Catalan numbers, whose first several terms are $1, 0, 1, 0, 2, 0, 5, 0, \ldots$. By the complete ϕ^{-1} algorithm, we have the elevated Schröder paths with no peaks, valleys only at odd height and level steps only at height $2k$ ($k \geq 1$) and occurring as part of a ULD subsequence.

3.3 Motzkin numbers

In this section, we consider Motzkin paths, first with $\wedge$ as the level step, then with l.

If the level steps of Motzkin paths are labeled $\wedge$, then the complete ϕ^{-1} algorithm produces the elevated Schröder paths with neither valley at even height nor peaks and where each L at even height occurs as part of a ULD subsequence.

If we denote the level steps in Motzkin paths by l and apply the complete inverse algorithm, then we obtain the elevated Schröder paths with no peaks at odd height, no valley at even height and L steps only at height $2k$ ($k \geq 1$) occurring as part of ULD subsequences.

3.4 Fine numbers

We next consider 2-Riordan (i.e., Fine) paths with two kinds of level steps, $\wedge$ and l, allowed except at height 0 where there are no level steps. Applying the complete ϕ^{-1} algorithm, we have the elevated Schröder paths with no valley at even height, peaks only at height $2k$ ($k \geq 2$), L steps at height ≥ 2 and each L at even height occurring as part of a ULD subsequence.

If the level steps are either $\wedge$ and v and we apply the complete ϕ^{-1} algorithm, we end up with elevated Schröder paths with no peaks and L steps at height $k \geq 2$.

Eliminating all level steps in 3-Motzkin paths and forbidding peaks at height 1, we will also have the Fine numbers. By applying the complete ϕ^{-1} algorithm, we have the elevated Schröder paths with no peaks, no valleys at even height, L steps only at height $2k$ ($k \geq 2$) and occurring as part of ULD subsequences.

3.5 Riordan numbers

If we eliminate the level steps at height 0 in Motzkin paths, we have the Riordan paths.

If the level steps in Riordan paths are v's, then by the complete ϕ^{-1} algorithm, we have elevated Schröder paths with no peaks and L steps at height $2k$ ($k \geq 1$).

If the level steps are all labeled $\wedge$, then the complete ϕ^{-1} algorithm leads to the elevated Schröder paths with no peaks, no valleys at even height, level steps at height ≥ 2 and level steps at even height occurring as part of ULD subsequences.

If we consider the case where level steps are l's and apply the complete ϕ^{-1} algorithm, then we have the elevated Schröder paths with peaks only at even height ≥ 4, no valley at even height, level steps only at height $2k$ ($k \geq 1$) and occurring as part of ULD subsequences.

Some results, similar to ours, but not in a Schröder context appear in a recent paper of Eu, Liu and Yeh [5].

4 Acknowledgments

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