Journal of Integer Sequences, Vol. 12 (2009),
Article 09.1.7

Polygonal Chain Sequences in the Space of Compact Sets

Steven Schlicker
Grand Valley State University
Allendale, MI 49401
USA
schlicks@gvsu.edu

Lisa Morales
University of California, Riverside
Riverside, CA 92521
USA
lmora008@student.ucr.edu

Daniel Schultheis
University of California, San Diego
La Jolla, CA 92093
USA
dschulth@math.ucsd.edu

Abstract

Configurations in the hyperspace of all non-empty compact subsets of n-dimensional real space provide a potential wealth of examples of familiar and new integer sequences. For example, Fibonacci-type sequences arise naturally in this geometry. In this paper, we introduce integer sequences that are determined by polygonal chain configurations.

1 Introduction

The Hausdorff metric h provides a way of measuring the distance between nonempty closed and bounded subsets of $\mathbb{R}^n$ using the standard Euclidean metric d_E. The metric h imposes a geometry on $\mathcal{H}(\mathbb{R}^n)$, the hyperspace of all non-empty, compact subsets of $\mathbb{R}^n$. The concept of betweenness in this geometry gives rise to interesting examples of integer sequences. In this context we will introduce polygonal chains and new integer sequences related to them. Throughout this article, we let F_k denote the k^{th} Fibonacci number (with $F_0 = 0, F_1 = 1$) and L_k the k^{th} Lucas number (with $L_0 = 2, L_1 = 1$). Recall that the Lucas numbers L_k satisfy

$$L_k = F_{k-1} + F_{k+1} = 2F_{k-1} + F_k.$$

2 The Hausdorff Metric

The Hausdorff metric h was introduced by Felix Hausdorff in the early 20th century as a way to measure the distance between compact sets. The Hausdorff metric imposes a geometry on the space $\mathcal{H}(\mathbb{R}^n)$, which will be the subject of our study. To distinguish between $\mathbb{R}^n$ and $\mathcal{H}(\mathbb{R}^n)$, we will refer to *points* in $\mathbb{R}^n$ and *elements* in $\mathcal{H}(\mathbb{R}^n)$.

Definition 1. Let A and B be elements in $\mathcal{H}(\mathbb{R}^n)$. The *Hausdorff distance*, $h(A, B)$, between A and B is

$$h(A, B) = \max\{d(A, B), d(B, A)\},$$

where $d(A, B) = \max_{a\in A}\{d(a, B)\}$ and $d(a, B) = \min_{b\in B}\{d_E(a, b)\}$.

As an example, let A be the two point set $\{(-1, 0), (1, 0)\}$ and B the circle centered at (0,0) of radius 1 in $\mathbb{R}^2$. Since $A \subset B$, we have $d(A, B) = 0$. However, $d(B, A) = d_E((0, 1), (1, 0)) = \sqrt{2}$. Note that $d(A, B) \neq d(B, A)$, so the function d is not symmetric. That is why we need the maximum of $d(A, B)$ and $d(B, A)$ to obtain a metric. In our example, we have $h(A, B) = d(B, A) = \sqrt{2}$. See Barnsley [1] for a proof that h is a metric on $\mathcal{H}(\mathbb{R}^n)$. The corresponding metric space, $(\mathcal{H}(\mathbb{R}^n), h)$, is then itself a complete metric space [1].

3 Betweenness in $\mathcal{H}(\mathbb{R}^n)$

Betweenness will play a major role in this paper, allowing us to find new integer sequences. In this section we define "betweenness" in $\mathcal{H}(\mathbb{R}^n)$, mimicking the idea of betweenness in $\mathbb{R}^n$ under the Euclidean metric. It is in this context that we will later define polygonal chains and introduce an infinite family of new integer sequences. Before we discuss betweenness, we define the dilation of a set, which will play a critical role in what follows.

Definition 2. Let $A \in \mathcal{H}(\mathbb{R}^n)$ and let $s > 0$ be a real number. The *dilation* of A by s is the set

$$(A)_s = \{x \in \mathbb{R}^n : d_E(x, a) \leq s \text{ for some } a \in A\}.$$

As a simple example, the dilation of a single point set $A = \{a\}$ by s is the closed ball centered at a of radius s. In general, the s-dilation of the set A is the union of all closed s balls centered at points in A. Using dilations, we can alternatively define $h(A, B)$ as the minimum value of s so that the s-dilation of A encloses B and the s-dilation of B encloses A.

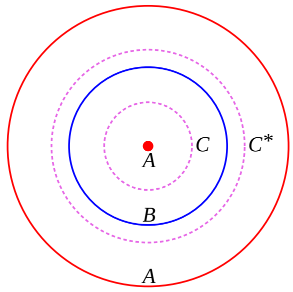

Figure 1: Infinitely many elements at the same location between A and B.

An important and useful result about dilations is the following (Theorem 4 from the article [6]).

Theorem 3. *Let $A \in \mathcal{H}(\mathbb{R}^n)$ and let $s > 0$. Then $(A)_s$ is a compact set with $h(A, (A)_s) = s$. Moreover, if $C \in \mathcal{H}(\mathbb{R}^n)$ and $h(A, C) \leq s$, then $C \subseteq (A)_s$.*

In other words, the element $(A)_s$ is the largest element (in the sense of containment) that is a distance s from A. Now we discuss betweenness. In the standard Euclidean geometry, a point x lies between the points a and b if and only if $d_E(a, b) = d_E(a, x) + d_E(x, b)$. We extend this idea to $\mathcal{H}(\mathbb{R}^n)$.

Definition 4. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $A \neq B$. The element $C \in \mathcal{H}(\mathbb{R}^n)$ *lies between* A and B if $h(A, B) = h(A, C) + h(C, B)$.

Notation: We write ACB to signify that C is between A and B as in Blumenthal [5]. Betweenness in $\mathbb{R}^n$ is quite different than betweenness in $\mathcal{H}(\mathbb{R}^n)$. Consider the following examples.

Example 5. Let A the union of the origin with the circle centered at (0,0) with radius 10 and B the circle centered at (0,0) with radius 5 in $\mathcal{H}(\mathbb{R}^2)$ as shown in Figure 1. Then $h(A, B) = d(A, B) = d(B, A) = 5$. The union $C \cup C^*$ of the two dashed circles is the element $(A)_2 \cap (B)_3$. Theorem 3 shows that any element X satisfying AXB with $h(A, X) = s$ and $h(B, X) = h(A, B) - s$ must be a subset of $(A)_s \cap (B)_{h(A,B)-s}$. It is not difficult to see that if $X_c = \{c\} \cup C^*$ where c is any point on the circle C, then X satisfies AXB with $h(A, X) = 2$. So in this example, there are infinitely many different elements X satisfying AXB and $h(A, X) = 2$.

Example 6. Let $A = \{(-2, 0), (0, 0), (2, 0)\}$ and $B = \{(-1, 0), (1, 0)\}$ in $\mathbb{R}^2$ as shown in Figure 2. Let $0 < s < h(A, B)$ and let $t = h(A, B) - s$. In this case, the set $C = (A)_s \cap (B)_t$ is the four point set $C = \{c_1, c_2, c_3, c_4\}$. Since any element C' that lies between A and B at the distance s from A is a subset of C, there are at most 16 elements C' that could satisfy $AC'B$ with $h(A, C') = s$. However, not every subset of C has this property. For example, if $C' = \{c_1\}$, then $h(A, C') = 3 + t$ and C' does not satisfy $AC'B$. We leave it to the reader to show that the sets C, $\{c_1, c_2, c_4\}$ and $\{c_1, c_3, c_4\}$ are the only sets that lie between A and B at the distance s from A. It is this situation in which we have only finitely many elements

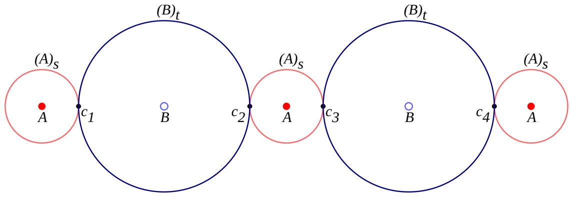

Figure 2: Exactly 3 different elements at the same location between A and B.

between two fixed elements at a specific distance from one of them that will be of interest to us.

The next definition formalizes what is meant when we refer to two elements at the *same location* between sets A and B.

Definition 7. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $A \neq B$ and let $C, C' \in \mathcal{H}(\mathbb{R}^n)$ satisfy ACB and $AC'B$. The elements C, C' are said to be *at the same location* between A and B if $h(A, C) = h(A, C') = s$ for some $0 < s < h(A, B)$.

4 Finding Points Between A and B

Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. If a set $C \in \mathcal{H}(\mathbb{R}^n)$ satisfies ACB with $h(A, C) = s$, then it follows from Theorem 3 that $C \subseteq (A)_s \cap (B)_{h(A,B)-s}$. As we saw in Examples 5 and 6, for some elements A and B there are infinitely many different elements at the same location between A and B and for other A and B there are only finitely many elements at the same location between them. The latter situation is of interest to us. In article [4], the authors show that in order for a pair of elements A and B in $\mathcal{H}(\mathbb{R}^n)$ to have a finite number of elements in $\mathcal{H}(\mathbb{R}^n)$ at a given location between A and B, every point in A must be a common distance $r = h(A, B)$ away from B (that is, $d(a, B) = r$ for all $a \in A$) and every point in B must be that same distance r from A ($d(b, A) = r$ for all $b \in B$). If this condition is not met, there will be infinitely many elements in $\mathcal{H}(\mathbb{R}^n)$ at each location between A and B. If a pair of elements $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfies this condition, we call the pair a configuration.

Definition 8. A *finite configuration* is a pair $[A, B]$ of finite sets $A, B \in \mathcal{H}(\mathbb{R}^n)$ where $d(a, B) = d(b, A) = h(A, B)$ for all $a \in A$ and $b \in B$.

In article [4], the authors also show that if $[A, B]$ is a finite configuration, then the number of elements C satisfying ACB and $h(A, C) = s$ is independent of s for all $0 < s < h(A, B)$. Thus, we denote by $\#([A, B])$ the number of elements between A and B in a finite configuration $[A, B]$ at any distance s.

Note that if $[A, B]$ is a finite configuration, then $C = (A)_s \cap (B)_{h(A,B)-s}$ will be a finite set for any $0 < s < h(A, B)$, so there are only finitely many subsets of C and $\#([A, B])$ must be finite by Theorem 3 (see Figure 2 for an example). There are infinite configurations, but in this paper we will only be concerned with finite ones.

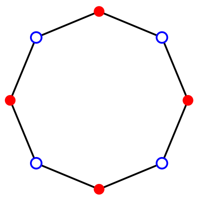

Figure 3: The P_4 configuration.

5 String and Polygonal Configurations

Two basic configurations that will be relevant to our work are string and polygonal configurations. A string configuration is any finite set of points equally spaced on a line segment, with sets A and B consisting of alternate points on the line [9]. Figure 2 is an example. If $|A \cup B| = k$, then we will call the string configuration $[A, B]$ a $k-string$ and denote it as S_k. So Figure 2 shows a 5-string or S_5. In article [9] (Theorem 6.1) the authors show

$$\#(S_k) = F_{k-1}.$$

In Figure 2, the elements $\{c_1, c_2, c_3, c_4\}, \{c_1, c_2, c_4\}$, and $\{c_1, c_3, c_4\}$ are the elements that satisfy ACB at the indicated location in this example. This is as expected, since $\#(S_5) = F_4 = 3$.

Another basic family of configurations is the collection of polygonal configurations. Let A and B be the alternate vertices of a regular $2m$-gon with $m \in \mathbb{N}$. We denote the configuration $[A, B]$ in this case by P_m and call this a polygonal configuration [9]. An example of P_4 is shown in Figure 3, with the set A as the filled points and the set B as the open points. Line segments are drawn between points to indicate the pairs of points $a \in A$ and $b \in B$ satisfying $d_E(a, b) = h(A, B)$. These points will be important in what follows.

Definition 9. Let $[A, B]$ be a finite configuration. The points $a \in A$ and $b \in B$ are *adjacent* if $d_E(a, b) = h(A, B)$. If $a \in A$, the *adjacency set* of a, $[a]_B$, is defined as $[a]_B = \{b \in B : d_E(a, b) = h(A, B)\}$.

Again in the paper [9] (Theorem 6.3) the authors show

$$\#(P_m) = F_{2m} + 2F_{2m-1} = L_{2m}.$$

String and polygonal configurations form the building components of polygonal chains.

6 Polygonal Chains

A polygonal chain $PC^k_{m,l}$ is a configuration obtained by connecting k copies of P_m at antipodal points with string configurations of a fixed length $l + 1$. When $k = 1$, we define $PC^1_{m,l}$ to be P_m. Examples of polygonal chains are shown in Figure 4. In this section we determine recursive formulas to find $\#(PC^k_{m,l})$.

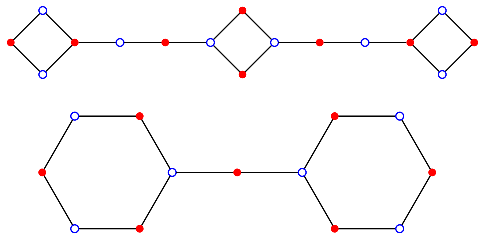

Figure 4: Top: $PC^3_{2,3}$ Bottom: $PC^2_{3,2}$.

6.1 Adjoining Configurations

To compute $\#(PC^k_{m,l})$, we will ultimately reduce the problem to one of determining $\#(X)$ for smaller configurations X created from adjoins of polygonal and string configurations. Before discussing adjoins, we need to understand equivalent configurations. The idea here is that in the computation of $\#([A, B])$ the distance between adjacent points is irrelevant - only adjacencies matter.

Definition 10. The finite configuration $[X, Y]$ is *equivalent* to the finite configuration $[A, B]$ if there are bijections $f : A \to X$ and $g : B \to Y$ such that for all $a \in A$ and $b \in B$, we have a adjacent to b if and only if $f(a)$ is adjacent to $g(b)$. When $[X, Y]$ is equivalent to $[A, B]$ we write $[X, Y] \sim [A, B]$.

It is easy to show that the relation $\sim$ is an equivalence relation on the set of finite configurations. One important result involving equivalent configurations is that if $[X, Y]$ and $[A, B]$ are equivalent configurations, then $\#([X, Y]) = \#([A, B])$ [4] (Theorem 4.1). Now we adjoin a point to a configuration.

Definition 11. Let $[A, B]$ be a finite configuration. The configuration $[A, B](a, y)$ obtained by *adjoining a point* $y \notin A \cup B$ to $[A, B]$ at the point $a \in A$ is the configuration $[A, B']$, where $B' = B \cup \{y\}$, $d_E(y, a) = h(A, B)$, and $d_E(y, x) > h(A, B)$ for all $x \in A \cup B$ with $x \neq a$.

For example, Figure 5 shows the configuration obtained by adjoining a point y to P_4 at point a.

One tool for computing $\#([A, B](a, y))$ for a finite configuration $[A, B]$ is the following [9] (Theorem 6.2).

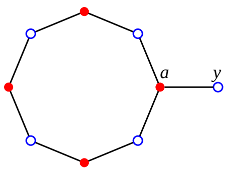

Figure 5: Adjoining a point to P_4.

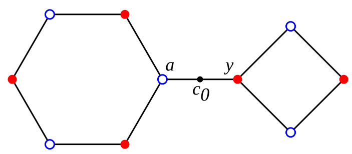

Figure 6: Adjoining P_2 and P_3.

Theorem 12. *Let $[A, B]$ be a finite configuration. Define a new configuration X' by adjoining a point y to $[A, B]$ at the point $a \in A$. Furthermore, assume a is adjacent to k points in B, each of which is adjacent to at least one more point in A. Then*

$$\#(X') = \#([A, B]) + \#([A - \{a\}, B]).$$

As an example, if $[A, B] = P_4$ and $X' = [A, B](a, y)$ as shown in Figure 5, then $\#(X') = \#(P_4) + \#(S_7) = L_8 + F_6 = 55$.

We can also adjoin two configurations.

Definition 13. Let $[A, B]$ and $[X, Y]$ be finite configurations and let $a \in A$ and $y \in Y$. Let $[X', Y']$ be a configuration equivalent to $[X, Y]$ through bijections $f : X \to X'$ and $g : Y \to Y'$ so that

1. $h(X', Y') = h(A, B)$,
2. $d_E(g(y), a) = h(A, B)$,
3. $d(z, A \cup B) > h(A, B)$ for all $z \in X' \cup Y'$, $z \neq g(y)$, and
4. $d(w, X' \cup Y') > h(A, B)$ for all $w \in A \cup B$, $w \neq a$.

The *adjoin* $[A, B][a] \oplus [X, Y][y]$ of the configurations at the points a and y is the configuration $[A \cup X', B \cup Y']$.

As an example, the adjoin of P_2 with P_3 at the points a and y is shown in Figure 6.

Notation: In this paper, we will consider configurations obtained by adjoining string configurations at their endpoints to polygonal configurations. If we adjoin a single string configuration to a polygonal configuration, the point of adjoin does not matter due to symmetry. The notation we use for adjoining a string configuration S_l to a polygonal configuration P_m is $P_m \oplus S_l$. There are cases in which we will adjoin more than one string configuration to

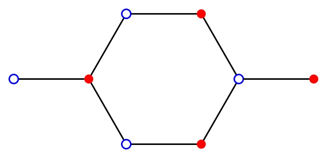

Figure 7: $P_3 \oplus \{S_1[1]; S_1[4]\}$.

a polygonal configuration. In these situations, we will label the vertices of a polygonal configuration counterclockwise in order (the starting point does not matter due to symmetry). The notation we will use to adjoin string configurations of length s and t to a polygonal configuration P_m at the points labeled i and j, respectively is $P_m \oplus \{S_s[i], S_t[j]\}$. An illustration of $P_3 \oplus \{S_1[1]; S_1[4]\}$ is shown in Figure 7.

The next theorem shows us a method for computing $\#(X)$ by breaking X into smaller disconnected configurations. A configuration $[A, B]$ is connected if, given $a \in A$ and $b \in B$, there is a sequence $\{x_k\}_{k=1}^{m} \in A \cup B$ such that $x_1 = a$, $x_m = b$ and x_i is adjacent to x_{i+1} for each i [4]. In other words, there is a string in $A \cup B$ connecting a to b. A configuration is disconnected if it is not connected. The proof of Theorem 15 depends on the next result from the paper [4] (Theorem 6.1).

Theorem 14. *Let $[A, B]$ be a configuration and let $r = h(A, B)$. If $[X, Y]$ is a configuration with $h(A, B) = h(X, Y)$, $d_E(a, y) > h(A, B)$ for all $a \in A$ and $y \in Y$, and $d_E(b, x) > h(A, B)$ for all $b \in B$ and $x \in X$, then $\#([A \cup X, B \cup Y]) = \#([A, B])\#([X, Y])$.*

In other words, if $[U, V]$ is a disconnected configuration constructed from configurations $[A, B]$ and $[X, Y]$, with $U = A \cup X$ and $V = B \cup Y$, then $\#([U, V]) = \#([A, B])\#([X, Y])$. The next theorem describes another important computation technique.

Theorem 15. *Let $[A, B]$ and $[X, Y]$ be finite configurations and let $[U, V] = [A, B][a] \oplus [X, Y][y]$ for some $a \in A$ and $y \in Y$. Then*

$$\#([U, V]) = \#([A, B])\ \#([X, Y]) + \#([A \oplus S_1[a], B])\ \#[X, Y \oplus S_1[y]].$$

Proof. Without loss of generality, we assume $h(A, B) = h(X, Y)$. Let $s, t > 0$ with $s + t = h(A, B)$. Let $C = (A)_s \cap (B)_t$ be the largest element that lies between A and B at a distance s from A, and let $Z = (X)_s \cap (Y)_t$ be the largest element that lies between X and Y at a distance s from X. Then in the configuration $[U, V]$, the largest element between $A \cup X$ and $B \cup Y$ will be $C \cup Z \cup \{c_0\}$, where c_0 is the point which lies between the newly adjacent points a and y. See Figure 6 for an illustration.

We now consider which subsets D of $C \cup Z \cup \{c_0\}$ satisfy UDV. Since D satisfies UDV, we know that $[a']_D \neq \emptyset$, $[b']_D \neq \emptyset$, $[x']_D \neq \emptyset$, and $[y']_D \neq \emptyset$ for all $a' \in A, b' \in B, x' \in X$, and $y' \in Y$. We consider two cases.

Case 1 $c_0 \notin D$. In this case, let $C' = \{d \in D : d \in [a']_D$ for some $a' \in A\}$ and let $Z' = D - C'$. By construction, if $c' \in C'$, then $c' \neq c_0$ and so $c' \in C$. Thus $C' \subseteq C$. If $z' \in Z'$, then $z' \neq c_0$ and $z' \in Z$. Now, if $a' \in A$ and $b' \in B$, we know $[a']_D \neq \emptyset$ and $[b']_D \neq \emptyset$. Therefore, $[a']_{C'} \neq \emptyset$ and $[b']_{C'} \neq \emptyset$. So C' satisfies $AC'B$. Similarly, Z' satisfies $XZ'Y$. By Theorem 14, the number of such sets $D = C' \cup Z'$ is $\#([A, B])\ \#([X, Y])$.

Case 2 $c_0 \in D$. In this case, we have $c_0 \in [a]_D$ and $c_0 \in [y]_D$. The existence of the point c_0 in D allows for the potential removal of other points in D that are adjacent to a or y. In other words, $D = C' \cup \{c_0\} \cup Z'$, where C' satisfies $AC'(B \oplus S_1[a])$ and Z' satisfies $(X \oplus S_1[y])Z'Y$. The number of such sets D is $\#([A \oplus S_1[a], B]) \cdot \#(X, Y \oplus S_1[y]])$.

By combining the results of the two cases here, we arrive at the expected formula $\#([U, V]) = \#([A, B])\ \#([X, Y]) + \#([A \oplus S_1[a], B])\ \#[X, Y \oplus S_1[y]]$. □

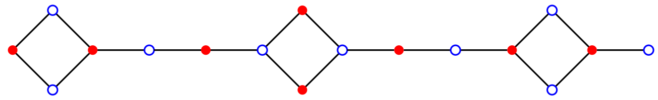

Figure 8: $PC^3_{2,3} \oplus S_1$.

As an example, consider the configuration X obtained by adjoining P_3 and P_2 as shown in Figure 6. Theorem 15 shows

$$\#(X) = \#(P_3)\ \#(P_2) + \#(P_3 \oplus S_1)\ \#(P_2 \oplus S_1).$$

Now Theorem 12 gives us $\#(P_3 \oplus S_1) = \#(P_3) + \#(S_5) = 18 + 3 = 21$ and $\#(P_2 \oplus S_1) = \#(P_2) + \#(S_3) = 7 + 1 = 8$. So

$$\#(X) = (18)(7) + (21)(8) = 294.$$

7 Finding $\#(PC^k_{m,l})$

In this section, we determine a pair of recursive formulas to find $\#(PC^k_{m,l})$. As the first step in our computations, we determine $\#(X \oplus S_t)$ for $t \geq 2$ for any finite configuration X in terms of $\#(X)$ and $\#(X \oplus S_1)$.

Theorem 16. *Let X be a finite configuration and $t \geq 2$. Then*

$$\#(X \oplus S_t) = F_t\#(X \oplus S_1) + F_{t-1}\#(X).$$

Proof. Using Theorem 12, we see that our theorem is certainly true with $t = 2$ and $t = 3$. We proceed by induction and suppose our theorem is true for all s, $1 \leq s \leq t$. Then

$$\begin{aligned}
\#(X \oplus S_{t+1}) &= \#(X \oplus S_t) + \#(X \oplus S_{t-1}) \\
&= [F_t\#(X \oplus S_1) + F_{t-1}\#(X)] + [F_{t-1}\#(X \oplus S_1) + F_{t-2}\#(X)] \\
&= (F_t + F_{t-1})\#(X \oplus S_1) + (F_{t-1} + F_{t-2})\#(X) \\
&= F_{t+1}\#(X \oplus S_1) + F_t\#(X).
\end{aligned}$$

□

By $PC^k_{m,l} \oplus S_t$ we mean the adjoin of S_t with $PC^k_{m,l}$ at a point antipodal to the connected strings. An example of $PC^3_{2,3} \oplus S_1$ is shown in Figure 8. A direct consequence of Theorem 16 is

Corollary 17. *For $m \geq 2$, $k \geq 1$, $l \geq 1$, and $t \geq 2$,*

$$\#(PC^k_{m,l} \oplus S_t) = F_t\#(PC^k_{m,l} \oplus S_1) + F_{t-1}\#(PC^k_{m,l})$$

Now we add in the case when $t = 1$ for polygonal configurations. Note first that

$$L_{2m} + F_{2m-2} = (F_{2m-2} + F_{2m-1}) + F_{2m+1} = F_{2m} + F_{2m+1} = F_{2m+2}.$$

Corollary 18. *For $m \geq 2$ and $t \geq 1$,*

$$\#(P_m \oplus S_t) = F_t F_{2m+2} + F_{t-1} L_{2m}.$$

Proof. When $t = 1$, a direct application of Theorem 12 gives us

$$\#(P_m \oplus S_1) = \#(P_m) + \#(S_{2m-1}) = L_{2m} + F_{2m-2} = F_{2m+2} = F_1 F_{2m+2} + F_0 L_{2m}.$$

For $t \geq 2$, we use Theorem 16 to obtain

$$\#(P_m \oplus S_t) = F_t \#(P_m \oplus S_1) + F_{t-1} \#(P_m) = F_t F_{2m+2} + F_{t-1} L_{2m}.$$

□

We need one more preliminary calculation before we determine $\#(PC^k_{m,l})$.

Lemma 19. *For $m \geq 2$,*

$$\#(P_m \oplus \{S_1[1]; S_1[m+1]\}) = F^2_{m+2}.$$

Proof. Figure 7 shows an example of the configurations in question. By Theorem 12 and Corollary 18,

$$\begin{aligned}\#(P_m \oplus \{S_1[1]; S_1[m+1]\}) &= \#(P_m \oplus S_1) + \#(S_{2m-1} \oplus S_1[m]) \\ &= F_{2m+2} + \#(S_{2m-1}) + \#(S_{m-1})^2 = F_{2m+2} + F_{2m-2} + F^2_{m-2}.\end{aligned}$$

We use the identity

$$F_{2m} = F^2_{m+1} - F^2_{m-1}$$

for $m \geq 1$ (see [7], Corollary 5.4 or [3], Identity 14 (note that $f_m = F_{m+1}$)) to obtain

$$\begin{aligned}F_{2m+2} + F_{2m-2} + F^2_{m-2} &= F_{2(m+1)} + F_{2(m-1)} + F^2_{m-2} \\ &= \left(F^2_{m+2} - F^2_m\right) + \left(F^2_m - F^2_{m-2}\right) + F^2_{m-2} \\ &= F^2_{m+2}.\end{aligned}$$

□

Now we derive recursive formulas for $\#(PC^k_{m,l})$ and $\#(PC^k_{m,l} \oplus S_1)$ in terms of $\#(PC^{k-1}_{m,l})$ and $\#(PC^{k-1}_{m,l} \oplus S_1)$. Remember that $\#(PC^1_{m,l}) = \#(P_m) = L_{2m}$ and $\#(PC^1_{m,l} \oplus S_1) = \#(P_m \oplus S_1) = F_{2m+2}$. To include all cases together, we extend the Fibonacci sequence in the negative direction by defining $F_{-1} = 1$. We also define $X \oplus S_0$ to be X for any finite configuration X.

Theorem 20. *For $m \geq 2$, $k \geq 2$, and $l \geq 1$,*

$$\#(PC^k_{m,l}) = (L_{2m} F_{l-2} + F_{2m+2} F_{l-1})\, \#(PC^{k-1}_{m,l}) + (L_{2m} F_{l-1} + F_{2m+2} F_l)\, \#(PC^{k-1}_{m,l} \oplus S_1)$$

where

$$\begin{aligned}\#(PC^k_{m,l} \oplus S_1) = &\left(F_{2m+2} F_{l-2} + F_{l-1} F^2_{m+2}\right) \#(PC^{k-1}_{m,l}) \\ &+ \left(F_{2m+2} F_{l-1} + F_l F^2_{m+2}\right) \#(PC^{k-1}_{m,l} \oplus S_1).\end{aligned}$$

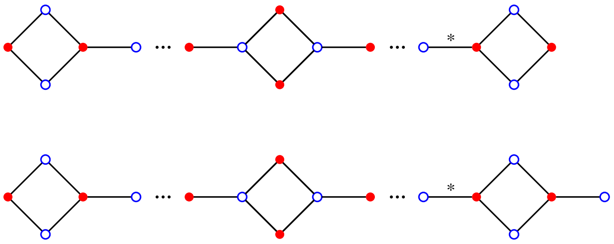

Figure 9: Using Theorem 15 to compute $\#(PC^k_{m,l})$ and $\#(PC^k_{m,l} \oplus S_1)$.

Proof. We use Theorem 15 applied at the adjacent points indicated with an asterisk as illustrated in Figure 9, and Corollary 17:

$$\begin{aligned}
\#(PC^k_{m,l}) &= \#(P_m)\#(PC^{k-1}_{m,l} \oplus S_{l-1}) + \#(P_m \oplus S_1)\#(PC^{k-1}_{m,l} \oplus S_l) \\
&= L_{2m}\left(F_{l-1}\#(PC^{k-1}_{m,l} \oplus S_1) + F_{l-2}\#(PC^{k-1}_{m,l})\right) \\
&\quad + F_{2m+2}\left(F_l\#(PC^{k-1}_{m,l} \oplus S_1) + F_{l-1}\#(PC^{k-1}_{m,l})\right) \\
&= \left(L_{2m}F_{l-1} + F_{2m+2}F_l\right)\#(PC^{k-1}_{m,l} \oplus S_1) + \left(L_{2m}F_{l-2} + F_{2m+2}F_{l-1}\right)\#(PC^{k-1}_{m,l}).
\end{aligned}$$

Finally, we determine the recursive formula for $\#(PC^k_{m,l} \oplus S_1)$ by using Theorem 15, applied at the adjacent points indicated with an asterisk in Figure 9. This gives us

$$\#(PC^k_{m,l} \oplus S_1) = \#(PC^{k-1}_{m,l} \oplus S_{l-1})\#(P_m \oplus S_1) + \#(PC^{k-1}_{m,l} \oplus S_l)\#(P_m \oplus \{S_1[1]; S_1[m+1]\}).$$

Therefore, Lemma 19 and Corollary 17 yield

$$\begin{aligned}
\#(PC^k_{m,l} \oplus S_1) &= F_{2m+2}\#(PC^{k-1}_{m,l} \oplus S_{l-1}) + \#(PC^{k-1}_{m,l} \oplus S_l)F^2_{m+2} \\
&= F_{2m+2}\left(F_{l-1}\#(PC^{k-1}_{m,l} \oplus S_1) + F_{l-2}\#(PC^{k-1}_{m,l})\right) \\
&\quad + \left(F_l\#(PC^{k-1}_{m,l} \oplus S_1) + F_{l-1}\#(PC^{k-1}_{m,l})\right)F^2_{m+2} \\
&= \left(F_{2m+2}F_{l-1} + F_lF^2_{m+2}\right)\#(PC^{k-1}_{m,l} \oplus S_1) \\
&\quad + \left(F_{2m+2}F_{l-2} + F_{l-1}F^2_{m+2}\right)\#(PC^{k-1}_{m,l}).
\end{aligned}$$

□

As examples of the use of Theorem 20,

$$\begin{aligned}
\#(PC^2_{3,1}) &= L_6\#(PC^1_{3,1}) + F_8\#(PC^1_{3,1} \oplus S_1) \\
&= 18\#(P_3) + 21\#(P_3 \oplus S_1) \\
&= 18(18) + 21(21) \\
&= 765,
\end{aligned}$$

$$\begin{aligned}
\#(PC^2_{2,2}) &= F_{2m+2}\#(PC^{k-1}_{m,l}) + [L_{2m} + F_{2m+2}]\#(PC^{k-1}_{m,l} \oplus S_1) \\
&= 8\#(P_2) + [7+8]\#(P_2 \oplus S_1) \\
&= (8)(7) + (15)(8) \\
&= 176,
\end{aligned}$$

Sloane A-number	i	1	2	3	4	5
A152927	$\#(P^i_{2,1})$	7	113	1815	29153	468263
A152928	$\#(P^2_{i,1})$		113	765	5234	35865
A152929	$\#(P^2_{2,i})$	113	176	289	465	754
A152930	$\#(P^i_{2,2})$	7	176	4393	109649	2736832
A152931	$\#(P^3_{i,2})$		4393	80361	1425131	25671393
A152932	$\#(P^3_{3,i})$	32733	80361	215658	559305	1469565

Table 1: Some polygonal chain sequences

and

$$\begin{aligned}\#(PC^2_{2,3}) &= [L_4F_1 + F_6F_2]\,\#(PC^1_{2,3}) + [L_4F_2 + F_6F_l]\,\#(PC^1_{2,3} \oplus S_1)\\ &= [(7)(1) + (8)(1)]\#(P_2) + [(7)(1) + (8)(2)]\#(P_2 \oplus S_1)\\ &= (15)(7) + (23)(8)\\ &= 289.\end{aligned}$$

8 Polygonal Chain Sequences

Theorem 20 gives us recursive formulas for finding $\#(PC^k_{m,l})$. By fixing any two of m, k, or l and letting the other parameter vary, we create infinite families of integer sequences. Examples of the first few terms of several of these sequences are given in Table 1. We suspect that it is possible to define many new types of configurations in this geometry that provide more interesting and previously unknown integer sequences.

9 Asymptotic Behavior

Polygonal chain sequences appear to exhibit asymptotically exponential behavior as the sequences of successive quotients in Table 2 illustrate. These sequences are natural extensions of string and polygonal sequences, both of which involve Fibonacci and Lucas numbers, so we might expect the ratios of polygonal chain sequences to do so as well. To understand this behavior, we investigate these quotients.

9.1 Fixing m and l

In this section we consider the asymptotic behavior of the sequences obtained by letting k vary for fixed values of m and l. For easier notation, let $X_{k,m,l} = \#(PC^k_{m,l})$ and $Y_{k,m,l} = \#(PC^k_{m,l} \oplus S_1)$. Theorem 20 shows that

$$X_{k,m,l} = a_{m,l}X_{k-1,m,l} + b_{m,l}Y_{k-1,m,l} \qquad \text{and} \qquad Y_{k,m,l} = c_{m,l}X_{k-1,m,l} + d_{m,l}Y_{k-1,m,l} \qquad (1)$$

i	6	7	8	9	10
$\frac{\#(P^{i+1}_{2,1})}{\#(P^{i}_{2,1})}$	16.06225775	16.06225775	16.06225775	16.06225775	16.06225775
$\frac{\#(P^{2}_{i+1,1})}{\#(P^{2}_{i,1})}$	6.854063862	6.854096407	6.854101155	6.854101848	6.854101949
$\frac{\#(P^{2}_{2,i+1})}{\#(P^{2}_{2,i})}$	1.618539787	1.617840851	1.618107769	1.618005808	1.618044753
$\frac{\#(P^{i+1}_{2,2})}{\#(P^{i}_{2,2})}$	24.95993579	24.95993579	24.95993579	24.95993579	24.95993579
$\frac{\#(P^{3}_{i+1,2})}{\#(P^{3}_{i,2})}$	17.95473888	17.94023899	17.94580726	17.94368472	17.94449609
$\frac{\#(P^{3}_{3,i+1})}{\#(P^{3}_{3,i})}$	2.619410257	2.617508525	2.618234731	2.617957317	2.618063275

Table 2: Ratios of polygonal chain sequences to 8 decimal places.

for values of $a_{m,l}, b_{m,l}, c_{m,l}$, and $d_{m,l}$ that depend on only m and l. We can determine the asymptotic behavior of $X_{k,m,l}$ by writing this system in matrix form:

$$\begin{bmatrix} X_{k,m,l} \\ Y_{k,m,l} \end{bmatrix} = \begin{bmatrix} a_{m,l} & b_{m,l} \\ c_{m,l} & d_{m,l} \end{bmatrix} \begin{bmatrix} X_{k-1,m,l} \\ Y_{k-1,m,l} \end{bmatrix}.$$

Let $M_{m,l} = \begin{bmatrix} a_{m,l} & b_{m,l} \\ c_{m,l} & d_{m,l} \end{bmatrix}$. Then

$$\begin{bmatrix} X_{k,m,l} \\ Y_{k,m,l} \end{bmatrix} = (M_{m,l})^{k-1} \begin{bmatrix} X_{1,m,l} \\ Y_{1,m,l} \end{bmatrix}.$$

The eigenvalues of $M_{m,l}$ are

$$\lambda_{m,l} = \frac{(d_{m,l} + a_{m,l}) + \sqrt{(d_{m,l} - a_{m,l})^2 + 4b_{m,l}c_{m,l}}}{2}$$

and

$$\overline{\lambda}_{m,l} = \frac{(d_{m,l} + a_{m,l}) - \sqrt{(d_{m,l} - a_{m,l})^2 + 4b_{m,l}c_{m,l}}}{2}.$$

Since $(d_{m,l} - a_{m,l})^2 + 4b_{m,l}c_{m,l} > 0$, we see that $M_{m,l}$ has two distinct real eigenvalues. Thus $M_{m,l}$ is diagonalizable to the matrix $D_{m,l} = \begin{bmatrix} \lambda_{m,l} & 0 \\ 0 & \overline{\lambda}_{m,l} \end{bmatrix}$. If $Q_{m,l}$ is the matrix whose columns are the eigenvectors of $M_{m,l}$, then

$$\begin{bmatrix} X_{k,m,l} \\ Y_{k,m,l} \end{bmatrix} = \left(Q_{m,l} \begin{bmatrix} (\lambda_{m,l})^{k-1} & 0 \\ 0 & \left(\overline{\lambda}_{m,l}\right)^{k-1} \end{bmatrix} Q^{-1}_{m,l} \right) \begin{bmatrix} X_{1,m,l} \\ Y_{1,m,l} \end{bmatrix}.$$

Note that $Q_{m,l} = \begin{bmatrix} 1 & 1 \\ \frac{\lambda_{m,l} - a_{m,l}}{b_{m,l}} & \frac{d_{m,l} - \lambda_{m,l}}{b_{m,l}} \end{bmatrix}$ and $Q^{-1}_{m,l} = \left(\frac{1}{2\lambda_{m,l} - (a_{m,l} + d_{m,l})} \right) \begin{bmatrix} \lambda_{m,l} - d_{m,l} & b_{m,l} \\ \lambda_{m,l} - a_{m,l} & -b_{m,l} \end{bmatrix}$.

Since $X_{1,m,l} = L_{2m}$ and $Y_{1,m,l} = F_{2m+2}$, we have

$$X_{k,m,l} = R_{m,l}\lambda_{m,l}^{k-1} + S_{m,l}\overline{\lambda}_{m,l}^{k-1} \tag{2}$$

where $R_{m,l} = \left(\frac{1}{2\lambda_{m,l}-(a_{m,l}+d_{m,l})}\right)((\lambda_{m,l} - d_{m,l})L_{2m} + b_{m,l}F_{2m+2})$ and $S_{m,l} = \left(\frac{1}{2\lambda_{m,l}-(a_{m,l}+d_{m,l})}\right)((\lambda_{m,l} - a_{m,l})L_{2m} - b_{m,l}F_{2m+2})$. So (2) provides a non-recursive formula for computing $X_{k,m,l}$. Now $\lambda_{m,l} > \overline{\lambda}_{m,l}$, so we have

$$\begin{aligned}
\frac{X_{k,m,l}}{X_{k-1,m,l}} &= \frac{R_{m,l}\lambda_{m,l}^{k-1} + S_{m,l}\overline{\lambda}_{m,l}^{k-1}}{R_{m,l}\lambda_{m,l}^{k-2} + S_{m,l}\overline{\lambda}_{m,l}^{k-2}} \\
&= \frac{R_{m,l}\lambda_{m,l} + S_{m,l}\overline{\lambda}_{m,l}\left(\frac{\overline{\lambda}_{m,l}}{\lambda_{m,l}}\right)^{k-2}}{R_{m,l} + S_{m,l}\left(\frac{\overline{\lambda}_{m,l}}{\lambda_{m,l}}\right)^{k-2}} \\
&\to \frac{R_{m,l}\lambda_{m,l}}{R_{m,l}} \\
&= \lambda_{m,l}
\end{aligned}$$

as $k \to \infty$. Thus,

$$\lim_{k\to\infty} \frac{X_{k,m,l}}{X_{k-1,m,l}} = \lambda_{m,l}.$$

The values of $a_{m,l}, b_{m,l}, c_{m,l}$, and $d_{m,l}$ are given by Theorem 20:

$$a_{m,l} = L_{2m}F_{l-2} + F_{2m+2}F_{l-1},\ b_{m,l} = L_{2m}F_{l-1} + F_{2m+2}F_l,$$
$$c_{m,l} = F_{2m+2}F_{l-2} + F_{l-1}F_{m+2}^2, \text{ and } d_{m,l} = F_{2m+2}F_{l-1} + F_lF_{m+2}^2.$$

Thus the sequence of polygonal chain numbers as k varies for fixed m and l behaves asymptotically as the exponential sequence with ratio $\lambda_{m,l}$. For example, $\lambda_{2,1} = 8 + \sqrt{65} \approx 16.06225775$ and $\lambda_{2,2} = \frac{25+3\sqrt{69}}{2} \approx 24.95993579$ (compare to the ratios in Table 2).

9.2 Limits of Fibonacci and Lucas Sequences

To determine the asymptotic behavior of the sequences $\{X_{k,m,l}\}$ as either m or l vary, we will need some limits of sequences of quotients of Fibonacci and Lucas numbers. Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. The following results concerning Fibonacci and Lucas numbers are well known.

- $\lim_{m\to\infty} \frac{F_{m+1}}{F_m} = \varphi$ ([7] Corollary 8.5, [8] (25))
- $\lim_{m\to\infty} \frac{L_{m+1}}{L_m} = \varphi$ ([7] Corollary 8.7, [8] (25))
- $\varphi^m = F_m\varphi + F_{m-1}$ for $m \geq 1$ ([7] Lemma 5.1, [3] Corollary 33)

- $\lim_{m\to\infty} \frac{F_{m+j}}{F_m} = \varphi^j$ for $j \geq 0$ ([3] Corollary 31, [8] (27))
- For $j \geq 0$, $\lim_{m\to\infty} \frac{L_{m+j}}{L_m} = \varphi^j$ ([8] (27))

We will also need other limits of ratios of Fibonacci and Lucas numbers.

Lemma 21. *For $m \geq 2$ and $j \geq 0$,*

1. *For $j \geq 0$, $\lim_{m\to\infty} \frac{L_m}{F_{m+j}} = \frac{1+\varphi^2}{\varphi^{j+1}}$*
2. $\lim_{m\to\infty} \frac{F_{2m+2}}{F_{m+2}^2} = \frac{\varphi^4-1}{\varphi^4} = \frac{1+\varphi^2}{\varphi^3}$
3. $\lim_{m\to\infty} \frac{F_{2m+2}}{F_{m+3}^2} = \frac{\varphi^4-1}{\varphi^6} = \frac{1+\varphi^2}{\varphi^5}$
4. $\lim_{m\to\infty} \frac{F_{2m+4}}{F_{m+2}^2} = \frac{\varphi^4-1}{\varphi^2} = \frac{1+\varphi^2}{\varphi}$
5. $\lim_{m\to\infty} \frac{L_{2m}}{F_{m+2}^2} = \frac{(\varphi^2+1)^2}{\varphi^6}$

Proof. We prove 2 and 5, the remainder are similar. Recalling that $\varphi^2 - 1 = \varphi$,

$$\begin{aligned}
\lim_{m\to\infty} \frac{F_{2m+2}}{F_{m+2}^2} &= \lim_{m\to\infty} \frac{F_{m+2}^2 - F_m^2}{F_{m+2}^2} \\
&= \lim_{m\to\infty} 1 - \left(\frac{F_m}{F_{m+2}}\right)^2 \\
&= 1 - \left(\frac{1}{\varphi^2}\right)^2 \\
&= \frac{\varphi^4-1}{\varphi^4} \\
&= \frac{(1+\varphi^2)\,\varphi}{\varphi^4} \\
&= \frac{1+\varphi^2}{\varphi^3}.
\end{aligned}$$

Finally, we use Identity 13 from the book [3] ($F_{2m+1} = F_m^2 + F_{m+1}^2$) to see

$$\begin{aligned}
\lim_{m\to\infty} \frac{L_{2m}}{F_{m+2}^2} &= \lim_{m\to\infty} \frac{F_{2m-1} + F_{2m+1}}{F_{m+2}^2} \\
&= \lim_{m\to\infty} \frac{(F_{m-1}^2 + F_m^2) + (F_m^2 + F_{m+1}^2)}{F_{m+2}^2} \\
&= \lim_{m\to\infty} \left(\frac{F_{m-1}}{F_{m+2}}\right)^2 + 2\left(\frac{F_m}{F_{m+2}}\right)^2 + \left(\frac{F_{m+1}}{F_{m+2}}\right)^2 \\
&= \left(\frac{1}{\varphi^3}\right)^2 + 2\left(\frac{1}{\varphi^2}\right)^2 + \left(\frac{1}{\varphi}\right)^2 \\
&= \frac{1 + 2\varphi^2 + \varphi^4}{\varphi^6} \\
&= \frac{(\varphi^2 + 1)^2}{\varphi^6}.
\end{aligned}$$

□

9.3 Fixing k and l

In this section, we fix k and l and let m vary and investigate the asymptotic behavior of the resulting sequence $X_{k,m,l}$. We will need to know limits of ratios of the terms $a_{m,l}, b_{m,l}, c_{m,l}$ and $d_{m,l}$ as m approaches infinity.

Lemma 22. *Let* $G_l = \frac{(1+\varphi^2)F_{l-1}+\varphi^3 F_l}{(1+\varphi^2)F_{l-2}+\varphi^3 F_{l-1}}$. *Then for* $l \geq 1$,

1. $\displaystyle\lim_{m\to\infty} \frac{a_{m+1,l}}{a_{m,l}} = \varphi^2$
2. $\displaystyle\lim_{m\to\infty} \frac{b_{m+1,l}}{a_{m,l}} = \varphi^2 G_l$
3. $\displaystyle\lim_{m\to\infty} \frac{b_{m,l}}{a_{m,l}} = G_l$
4. $\displaystyle\lim_{m\to\infty} \frac{c_{m,l}}{a_{m,l}} = \frac{\varphi^3}{1+\varphi^2}$
5. $\displaystyle\lim_{m\to\infty} \frac{c_{m+1,l}}{a_{m,l}} = \frac{\varphi^5}{1+\varphi^2}$
6. $\displaystyle\lim_{m\to\infty} \frac{d_{m,l}}{a_{m,l}} = \left(\frac{\varphi^3}{1+\varphi^2}\right) G_l.$
7. $\displaystyle\lim_{m\to\infty} \frac{d_{m+1,l}}{a_{m,l}} = \left(\frac{\varphi^5}{1+\varphi^2}\right) G_l.$

Proof. We prove 5, the others are similar:

$$\begin{aligned}
\lim_{m\to\infty}\frac{c_{m+1,l}}{a_{m,l}} &= \lim_{m\to\infty}\frac{F_{2m+4}F_{l-2}+F_{m+3}^2F_{l-1}}{L_{2m}F_{l-2}+F_{2m+2}F_{l-1}} \\
&= \lim_{m\to\infty}\frac{\left(\frac{F_{2m+4}}{F_{2m+2}}\right)F_{l-2}+\left(\frac{F_{m+3}^2}{F_{2m+2}}\right)F_{l-1}}{\left(\frac{L_{2m}}{F_{2m+2}}\right)F_{l-2}+F_{l-1}} \\
&= \frac{(\varphi^2)F_{l-2}+\left(\frac{\varphi^5}{\varphi^2+1}\right)F_{l-1}}{\left(\frac{1+\varphi^2}{\varphi^3}\right)F_{l-2}+F_{l-1}} \\
&= \left(\frac{\varphi^5}{1+\varphi^2}\right)\left(\frac{\left(\frac{1+\varphi^2}{\varphi^3}\right)F_{l-2}+F_{l-1}}{\left(\frac{1+\varphi^2}{\varphi^3}\right)F_{l-2}+F_{l-1}}\right) \\
&= \frac{\varphi^5}{1+\varphi^2}.
\end{aligned}$$

□

Before we can analyze the asymptotic behavior of the polygonal chain sequence, we need to understand how the sequences $\{Y_{k,m,l}\}$ and $\{X_{k,m,l}\}$ are related as m varies.

Lemma 23. *For $k \geq 1$,*

$$\lim_{m\to\infty}\frac{Y_{k,m,l}}{X_{k,m,l}} = \frac{\varphi^3}{1+\varphi^2}.$$

Proof. When $k = 1$ we have $\lim_{m\to\infty}\frac{Y_{1,m,l}}{X_{1,m,l}} = \lim_{m\to\infty}\frac{F_{2m+2}}{L_{2m}} = \frac{1}{\frac{1}{\varphi^3}+\frac{1}{\varphi}} = \frac{\varphi^3}{1+\varphi^2}$. So the claim is true for $k = 1$. We proceed by induction and assume the claim is true for some $k \geq 1$. Then (1) shows

$$\begin{aligned}
\lim_{m\to\infty}\frac{Y_{k+1,m,l}}{X_{k+1,m,l}} &= \lim_{m\to\infty}\frac{c_{m,l}X_{k,m,l}+d_{m,l}Y_{k,m,l}}{a_{m,l}X_{k,m,l}+b_{m,l}Y_{k,m,l}} \\
&= \lim_{m\to\infty}\frac{\left(\frac{c_{m,l}}{a_{m,l}}\right)+\left(\frac{d_{m,l}}{a_{m,l}}\right)\left(\frac{Y_{k,m,l}}{X_{k,m,l}}\right)}{1+\left(\frac{b_{m,l}}{a_{m,l}}\right)\left(\frac{Y_{k,m,l}}{X_{k,m,l}}\right)} \\
&= \frac{\left(\frac{\varphi^3}{1+\varphi^2}\right)+\left(\frac{\varphi^3}{1+\varphi^2}\right)G_l\left(\frac{\varphi^3}{1+\varphi^2}\right)}{1+G_l\left(\frac{\varphi^3}{\varphi^2+1}\right)} \\
&= \frac{\varphi^3}{1+\varphi^2}.
\end{aligned}$$

□

Now we can identify the asymptotic behavior of the polygonal chain sequence when m varies.

Theorem 24. *For $k \geq 1$ and $l \geq 1$,*

$$\lim_{m\to\infty} \frac{X_{k,m+1,l}}{X_{k,m,l}} = \varphi^{2k} \text{ and } \lim_{m\to\infty} \frac{Y_{k,m+1,l}}{X_{k,m,l}} = \frac{\varphi^{2k+3}}{1+\varphi^2}.$$

Proof. The proof will be by induction on k. Now $X_{1,m,l} = L_{2m}$ and $\lim_{m\to\infty} \frac{X_{1,m+1,l}}{X_{1,m,l}} = \lim_{m\to\infty} \frac{L_{2m+2}}{L_{2m}} = \varphi^2$. Also, $Y_{1,m,l} = F_{2m+2}$ and $\lim_{m\to\infty} \frac{Y_{1,m+1,l}}{X_{1,m,l}} = \lim_{m\to\infty} \frac{F_{2m+4}}{L_{2m}} = \frac{1}{\frac{1}{\varphi^5}+\frac{1}{\varphi^3}} = \frac{\varphi^5}{1+\varphi^2}$. Thus, our theorem is true when $k=1$. Assume the theorem is true for some $k \geq 1$. Then

$$\begin{aligned}
\lim_{m\to\infty} \frac{X_{k+1,m+1,l}}{X_{k+1,m,l}} &= \lim_{m\to\infty} \frac{a_{m+1,l}X_{k,m+1,l} + b_{m+1,l}Y_{k,m+1,l}}{a_{m,l}X_{k,m,l} + b_{m,l}Y_{k,m,l}} \\
&= \lim_{m\to\infty} \frac{\left(\frac{a_{m+1,l}}{a_{m,1}}\right)\left(\frac{X_{k,m+1,l}}{X_{k,m,l}}\right) + \left(\frac{b_{m+1,l}}{a_{m,l}}\right)\left(\frac{Y_{k,m+1,l}}{X_{k,m,l}}\right)}{1+\left(\frac{b_{m,l}}{a_{m,1}}\right)\left(\frac{Y_{k,m,l}}{X_{k,m,l}}\right)} \\
&= \frac{(\varphi^2)\left(\varphi^{2k}\right) + (\varphi^2 G_l)\left(\frac{\varphi^{2k+3}}{1+\varphi^2}\right)}{1+(G_l)\left(\frac{\varphi^3}{1+\varphi^2}\right)} \\
&= \varphi^{2k+2}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{m\to\infty} \frac{Y_{k+1,m+1,l}}{X_{k+1,m,l}} &= \lim_{m\to\infty} \frac{c_{m+1,l}X_{k,m+1,l} + d_{m+1,l}Y_{k,m+1,l}}{a_{m,l}X_{k,m,l} + b_{m,l}Y_{k,m,l}} \\
&= \lim_{m\to\infty} \frac{\left(\frac{c_{m+1,l}}{a_{m,l}}\right)\left(\frac{X_{k,m+1,l}}{X_{k,m,l}}\right) + \left(\frac{d_{m+1,l}}{a_{m,l}}\right)\left(\frac{Y_{k,m+1,l}}{X_{k,m,l}}\right)}{1+\left(\frac{b_{m,l}}{a_{m,l}}\right)\left(\frac{Y_{k,m,l}}{X_{k,m,l}}\right)} \\
&= \frac{\left(\frac{\varphi^5}{1+\varphi^2}\right)\left(\varphi^{2k}\right) + \left(\frac{\varphi^5}{1+\varphi^2}\right) G_l \left(\frac{\varphi^{2k+3}}{1+\varphi^2}\right)}{1+(G_l)\left(\frac{\varphi^3}{1+\varphi^2}\right)} \\
&= \frac{\varphi^{2k+5}}{1+\varphi^2}.
\end{aligned}$$

□

So the sequence of polygonal chain numbers as m varies for fixed k and l behaves asymptotically as the exponential sequence with ratio φ^{2k}. For example, $\varphi^4 \approx 6.854101954$ and $\varphi^6 \approx 17.94427186$ (compare to the ratios in Table 2). Note that the limits in these cases are independent of the value of l.

9.4 Fixing k and m

In this section, we fix k and m and let l vary and investigate the asymptotic behavior of the resulting sequence $X_{k,m,l}$. Again, we begin with limits of sequences of ratios of $a_{m,l}$, $b_{m,l}$, $c_{m,l}$, and $d_{m,l}$, this time as l goes to infinity.

Lemma 25. *Let* $H_m = \frac{F_{2m+2}+F_{m+2}^2\varphi}{L_{2m}+F_{2m+2}\varphi}$. *Then for* $m \geq 2$,

1. $\lim_{l\to\infty} \dfrac{a_{m,l+1}}{a_{m,l}} = \varphi$
2. $\lim_{l\to\infty} \dfrac{b_{m,l}}{a_{m,l}} = \varphi$
3. $\lim_{l\to\infty} \dfrac{b_{m,l+1}}{a_{m,l}} = \varphi^2$
4. $\lim_{l\to\infty} \dfrac{c_{m,l}}{a_{m,l}} = H_m$
5. $\lim_{l\to\infty} \dfrac{c_{m,l+1}}{a_{m,l}} = (\varphi)\, H_m$
6. $\lim_{l\to\infty} \dfrac{d_{m,l+1}}{a_{m,l}} = \left(\varphi^2\right) H_m.$

Proof. We verify 5 and leave the others for the reader:

$$\begin{aligned}
\lim_{l\to\infty} \frac{c_{m,l+1}}{a_{m,l}} &= \lim_{l\to\infty} \frac{F_{2m+2}F_{l-1} + F_{m+2}^2 F_l}{L_{2m}F_{l-2} + F_{2m+2}F_{l-1}} \\
&= \lim_{l\to\infty} \frac{F_{2m+2} + F_{m+2}^2\left(\frac{F_l}{F_{l-1}}\right)}{L_{2m}\left(\frac{F_{l-2}}{F_{l-1}}\right) + F_{2m+2}} \\
&= \frac{F_{2m+2} + F_{m+2}^2\varphi}{L_{2m}\left(\frac{1}{\varphi}\right) + F_{2m+2}} \\
&= (\varphi)\left(\frac{F_{2m+2} + F_{m+2}^2\varphi}{L_{2m} + F_{2m+2}\varphi}\right).
\end{aligned}$$

□

Now we need to understand the relationship between the sequence $\{Y_{k,m,l}\}$ and the sequence $\{X_{k,m,l}\}$ as l goes to infinity.

Lemma 26. *For* $k \geq 2$, $m \geq 2$,

$$\lim_{l\to\infty} \frac{Y_{k,m,l}}{X_{k,m,l}} = H_m.$$

Proof. When $k = 2$ we have

$$\begin{aligned}
\lim_{l\to\infty} \frac{Y_{2,m,l}}{X_{2,m,l}} &= \lim_{l\to\infty} \frac{c_{m,l}X_{1,m,l} + d_{m,l}Y_{1,m,l}}{a_{m,l}X_{1,m,l} + b_{m,l}Y_{1,m,l}} \\
&= \lim_{l\to\infty} \frac{\left(\frac{c_{m,l}}{a_{m,l}}\right) X_{1,m,l} + \left(\frac{d_{m,l}}{a_{m,l}}\right) Y_{1,m,l}}{X_{1,m,l} + \left(\frac{b_{m,l}}{a_{m,l}}\right) Y_{1,m,l}} \\
&= \frac{(H_m)\, L_{2m} + (\varphi H_m)\, F_{2m+2}}{L_{2m} + \varphi F_{2m+2}} \\
&= H_m.
\end{aligned}$$

So the lemma is true for $k = 2$. We proceed by induction and assume the lemma is true for some $k \geq 2$. Then (1) shows

$$\begin{aligned}
\lim_{l\to\infty} \frac{Y_{k+1,m,l+1}}{X_{k+1,m,l+1}} &= \lim_{l\to\infty} \frac{c_{m,l}X_{k,m,l} + d_{m,l}Y_{k,m,l}}{a_{m,l}X_{k,m,l} + b_{m,l}Y_{k,m,l}} \\
&= \lim_{l\to\infty} \frac{\left(\frac{c_{m,l}}{a_{m,l}}\right) + \left(\frac{d_{m,l}}{a_{m,l}}\right)\left(\frac{Y_{k,m,l}}{X_{k,m,l}}\right)}{1 + \left(\frac{b_{m,l}}{a_{m,l}}\right)\left(\frac{Y_{k,m,l}}{X_{k,m,l}}\right)} \\
&= \frac{(H_m) + (\varphi H_m)\,(H_m)}{1 + (\varphi)\,(H_m)} \\
&= H_m.
\end{aligned}$$

□

Now we determine the asymptotic behavior of the polygonal chain sequences if we allow l to vary.

Theorem 27. *For $k \geq 2$ and $m \geq 2$,*

$$\lim_{l\to\infty} \frac{X_{k,m,l+1}}{X_{k,m,l}} = \varphi^{k-1} \ \textit{and} \ \lim_{l\to\infty} \frac{Y_{k,m,l+1}}{X_{k,m,l}} = \varphi^{k-1}\,(H_m)\,.$$

Proof. The proof will be by induction on k. Recall that $X_{1,m,l} = L_{2m}$ and $Y_{1,m,l} = F_{2m+2}$ We first verify the $k = 2$ case. Equation (1) shows

$$\begin{aligned}
\lim_{l\to\infty} \frac{X_{2,m,l+1}}{X_{2,m,l}} &= \lim_{l\to\infty} \frac{a_{m,l+1}X_{1,m,l+1} + b_{m,l+1}Y_{1,m,l+1}}{a_{m,l}X_{1,m,l} + b_{m,l}Y_{1,m,l}} \\
&= \lim_{l\to\infty} \frac{\left(\frac{a_{m,l+1}}{a_{m,l}}\right) X_{1,m,l+1} + \left(\frac{b_{m,l+1}}{a_{m,l}}\right) Y_{1,m,l+1}}{X_{1,m,l} + \left(\frac{b_{m,l}}{a_{m,l}}\right) Y_{1,m,l}} \\
&= \frac{\varphi L_{2m} + \varphi^2 F_{2m+2}}{L_{2m} + \varphi F_{2m+2}} \\
&= \varphi
\end{aligned}$$

and

$$\begin{aligned}
\lim_{l\to\infty}\frac{Y_{2,m,l+1}}{X_{2,m,l}} &= \lim_{l\to\infty}\frac{c_{m,l+1}X_{1,m,l+1}+d_{m,l+1}Y_{1,m,l+1}}{a_{m,l}X_{1,m,l}+b_{m,l}Y_{1,m,l}} \\
&= \lim_{l\to\infty}\frac{\left(\frac{c_{m,l+1}}{a_{m,l}}\right)X_{1,m,l+1}+\left(\frac{d_{m,l+1}}{a_{m,l}}\right)Y_{1,m,l+1}}{X_{1,m,l}+\left(\frac{b_{m,l}}{a_{m,l}}\right)Y_{1,m,l}} \\
&= \frac{(\varphi)\left(H_m\right)L_{2m}+\left(\varphi^2 H_m\right)F_{2m+2}}{L_{2m}+\varphi F_{2m+2}} \\
&= \varphi H_m.
\end{aligned}$$

Now the general cases:

$$\begin{aligned}
\lim_{l\to\infty}\frac{X_{k+1,m,l+1}}{X_{k+1,m,l}} &= \lim_{l\to\infty}\frac{a_{m,l+1}X_{k,m,l+1}+b_{m,l+1}Y_{k,m,l+1}}{a_{m,l}X_{k,m,l}+b_{m,l}Y_{k,m,l}} \\
&= \lim_{l\to\infty}\frac{\left(\frac{a_{m,l+1}}{a_{m,l}}\right)\left(\frac{X_{k,m,l+1}}{X_{k,m,l}}\right)+\left(\frac{b_{m,l+1}}{a_{m,l}}\right)\left(\frac{Y_{k,m,l+1}}{X_{k,m,l}}\right)}{1+\left(\frac{b_{m,l}}{a_{m,l}}\right)\left(\frac{Y_{k,m,l}}{X_{k,m,l}}\right)} \\
&= \frac{(\varphi)\left(\varphi^{k-1}\right)+\left(\varphi^2\right)\left(\varphi^{k-1}H_m\right)}{1+(\varphi)\left(H_m\right)} \\
&= \varphi^k
\end{aligned}$$

and

$$\begin{aligned}
\lim_{l\to\infty}\frac{Y_{k+1,m,l+1}}{X_{k+1,m,l}} &= \lim_{l\to\infty}\frac{c_{m,l+1}X_{k,m,l+1}+d_{m,l+1}Y_{k,m,l+1}}{a_{m,l}X_{k,m,l}+b_{m,l}Y_{k,m,l}} \\
&= \lim_{l\to\infty}\frac{\left(\frac{c_{m,l+1}}{a_{m,l}}\right)\left(\frac{X_{k,m,l+1}}{X_{k,m,l}}\right)+\left(\frac{d_{m,l+1}}{a_{m,l}}\right)\left(\frac{Y_{k,m,l+1}}{X_{k,m,l}}\right)}{1+\left(\frac{b_{m,l}}{a_{m,l}}\right)\left(\frac{Y_{k,m,l}}{X_{k,m,l}}\right)} \\
&= \frac{\left(\varphi H_m\right)\left(\varphi^{k-1}\right)+\left(\varphi^2 H_m\right)\left(\varphi^{k-1}H_m\right)}{1+(\varphi)\left(H_m\right)} \\
&= \varphi^k H_m.
\end{aligned}$$

□

So the sequence of polygonal chain numbers as l varies for fixed k and m behaves asymptotically as the exponential sequence with ratio φ^{k-1}. For example, $\varphi \approx 1.618033988$ and $\varphi^2 \approx 2.618033986$ (compare to the ratios in Table 2). Note that the limits in these cases are independent of the value of m.

10 Conclusion

The asymptotic growth rates of polygonal chain sequences are functions of the golden ratio φ. Since polygonal chains are constructed from string and polygonal configurations and $\lim_{l\to\infty} \frac{\#(S_{l+1})}{\#(S_l)} = \varphi$ and $\lim_{m\to\infty} \frac{\#(P_{m+1})}{\#(P_m)} = \varphi^2$, this should not be surprising in hindsight. That the asymptotic ratios of the polygonal chain sequences have such nice closed forms is, however, somewhat surprising given the complex nature of betweenness in this geometry.

Polygonal chain sequences provide a three parameter family of integer sequences previously unidentified in The On-Line Encyclopedia of Integer Sequences [10]. These sequences have interesting geometric interpretations as the number of compact sets at each location between the component sets that make up the polygonal chain. We suspect that there are many other similar families of unknown sequences just waiting to be found.

11 Acknowledgements

This work was supported by National Science Foundation grant DMS-0451254. The authors also thank Grand Valley State University for hosting and supporting a Research Experiences for Undergraduates program.

References

[1] M. Barnsley, *Fractals Everywhere*, Second Edition, Academic Press Professional, 1998.

[2] C. Bay, A. Lembcke, and S. Schlicker, When lines go bad in hyperspace, *Demonstratio Math.*, **38** (2005), 689–701.

[3] A. T. Benjamin and J. J. Quinn, *Proofs That Really Count*, Mathematical Association of America, 2003.

[4] C. Blackburn, K. Lund, S. Schlicker, P. Sigmon, and A. Zupan, A missing prime configuration in the Hausdorff metric geometry, to appear, *J. Geom.*

[5] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford University Press, 1953.

[6] D. Braun, J. Mayberry, A. Powers, and S. Schlicker, A singular introduction to the Hausdorff metric geometry, *Pi Mu Epsilon J.*, **12** No. 3, (2005), 129–138.

[7] T. Koshy, *Fibonacci and Luca Numbers with Applications*, John Wiley & Sons, 2001.

[8] E. Lucas, Théorie des fonctions numériques simplement périodiques, *Amer. J. Math.* **1** (1878), 184–240, 289–321.

[9] K. Lund, P. Sigmon, and S. Schlicker, Fibonacci sequences in the space of compact sets, *Involve*, **1** No. 2 (2008), 197–215.

[10] N. J. A. Sloane, (2007), The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences/

2000 *Mathematics Subject Classification*: Primary 51F99; Secondary 11B83.
Keywords: Hausdorff metric, configuration, metric geometry, polygonal chains, Fibonacci numbers, Lucas numbers.

(Concerned with sequences A000032, A000045, A152927, A152928, A152929, A152930, A152931, and A152932.)

Received August 22 2008; revised version received December 22 2008. Published in *Journal of Integer Sequences*, December 22 2008.

Return to Journal of Integer Sequences home page.