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Two Formulas for Successive Derivatives and Their Applications

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Abstract

We recall two formulas, due to C. Jordan, for the successive derivatives of functions with an exponential or logarithmic inner function. We apply them to get addition formulas for the Stirling numbers of the second kind and for the Stirling numbers of the first kind. Then we show how one can obtain, in a simple way, explicit formulas for the generalized Euler polynomials, generalized Euler numbers, generalized Bernoulli polynomials and the Bell polynomials.

1 Introduction

By $\left\{{n \atop k}\right\}$ we mean the Stirling number of the second kind (the number of ways of partitioning a set of n elements into k nonempty subsets; see Graham et al. [4] and sequence A008277 of Sloane's *On-line Encyclopedia* [10]). As usual, we set $\left\{{n \atop 0}\right\} = 0$ if $n > 0$, $\left\{{0 \atop 0}\right\} = 1$, and $\left\{{n \atop k}\right\} = 0$ for $k > n$ or $k < 0$. Let us recall that the Stirling numbers satisfy the identities

$$\left\{{n \atop k}\right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}(k-j)^n, \tag{1}$$

$$\left\{{n+1 \atop k}\right\} = k\left\{{n \atop k}\right\} + \left\{{n \atop k-1}\right\}, \tag{2}$$

and appear in the Taylor expansion

$$\frac{(e^w-1)^k}{k!} = \sum_{n=k}^{\infty}\left\{{n \atop k}\right\}\frac{w^n}{n!}. \tag{3}$$

Peregrino [9] proved the following addition formula for the Stirling numbers of the second kind

$$\left\{ {u+v \atop k} \right\} = \sum_{j=0}^{v} \sum_{i=0}^{v} \binom{v}{i} k^i (-1)^{v+i+j} \left\{ {v-i \atop j} \right\} \left\{ {u \atop k-j} \right\}. \tag{4}$$

By $\left[{n \atop k} \right]$ we denote the Stirling number of the first kind (number of ways of partitioning a set of n elements into k nonempty cycles, see [4], sequence A008275 in [10]). Similarly $\left[{n \atop 0} \right] = 0$ if $n > 0$, $\left[{0 \atop 0} \right] = 1$, $\left[{n \atop k} \right] = 0$ for $k > n$ or $k < 0$. The Stirling numbers of the first kind fulfil the recurrence formula

$$\left[{n+1 \atop k} \right] = n \left[{n \atop k} \right] + \left[{n \atop k-1} \right]. \tag{5}$$

We use common notation for the falling factorial

$$(x)_k = x(x-1)\cdots(x-k+1)$$

and for the rising factorial (Pochhammer's symbol)

$$x^{(k)} = x(x+1)\cdots(x+k-1).$$

The paper is organized as follows. We recall and prove two formulas, due to C. Jordan [6], in section 2. The formulas involve Stirling numbers of both kinds. Since the original Peregrino's proof of (4), by induction on v, is long we give shorter and more direct proof of his formula and similar formulas in section 3. We prove addition formulas for Stirling numbers of the first kind in section 4. Sections 5,6,7 are devoted to show explicit formulas respectively for generalized Euler polynomials, generalized Bernoulli polynomials and the Bell polynomials. All proofs, in the last three sections, are more simple and direct than the proofs which exist in the literature. In the case of generalized Bernoulli polynomials (and generalized Bernoulli numbers) our results seem to be new.

2 Formulas for successive derivatives

We have the following formulas for successive derivatives of composite functions with the exponential, or the logarithmic, inner function.

Lemma 1. *If $f \in C^\infty(R)$ then the following formulas for the nth order ($n = 1, 2, 3, \ldots$) derivatives hold*

$$\frac{d^n}{dt^n}(f(e^t)) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} f^{(k)}(e^t) e^{kt}, \tag{6}$$

$$\frac{d^n}{dt^n}(f(\log t)) = \frac{1}{t^n} \sum_{k=1}^{n} (-1)^{n-k} \left[{n \atop k} \right] f^{(k)}(\log t). \tag{7}$$

Proof. To prove formula (6), we proceed by induction with respect to n. Denote $g(t) = f(e^t)$. For $n = 1$, (6) is obviously true. Let us suppose that for an integer n formula (6) holds. Then using (2) we have

$$\begin{aligned} g^{(n+1)}(t) &= \frac{d}{dt}\left(\sum_{k=1}^{n} \left\{ {n \atop k} \right\} f^{(k)}(e^t)e^{kt}\right) \\ &= \sum_{k=1}^{n} \left\{ {n \atop k} \right\} (f^{(k+1)}(e^t)e^{(k+1)t} + kf^{(k)}(e^t)e^{kt}) \\ &= f'(e^t)e^t + f^{(n+1)}(e^t)e^{(n+1)t} \\ &+ \sum_{k=2}^{n} \left(k\left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} \right) f^{(k)}(e^t)e^{kt} \\ &= \sum_{k=1}^{n+1} \left\{ {n+1 \atop k} \right\} f^{(k)}(e^t)e^{kt}, \end{aligned}$$

which ends the proof of (6). Analogously, using (5), formula (7) can be shown. □

Formulas (6), (7) are known and can be found, with proofs based on finite differences, in the C. Jordan's book [6, pp. 205–206]. The formulas are given, as exercises without proofs and without direct referencing to [6], also in the L. Comtet's book [2, Ex. 6, p. 157]. It is easy to see that formula (6) holds, with the same proof, for complex variable t and a holomorphic function f. Formula (7) holds for complex t, a branch of logarithm and a holomorphic function f.

For example if $f(x) = x^m$, $g(t) = e^{mt}$ then using (6) we get

$$g^{(n)}(t) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} m(m-1)\cdots(m-k+1)e^{(m-k)t}e^{kt}. \tag{8}$$

From the other side

$$g^{(n)}(t) = m^n e^{mt}, \tag{9}$$

and comparing (8) with (9) we obtain the well-known generating function for the Stirling numbers of the second kind

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\} (m)_k = m^n.$$

3 Addition formulas for Stirling numbers of the second kind

Let us substitute $f(x) = \exp(x)$ in (6). We have

$$(e^{e^t})^{(u+v)} = \sum_{k=1}^{u+v} \left\{ {u+v \atop k} \right\} e^{e^t} \cdot e^{kt}. \tag{10}$$

From the other side

$$(e^{e^t})^{(u)} = \sum_{n=1}^{u} \left\{ {u \atop n} \right\} e^{e^t} \cdot e^{nt}$$

and

$$\begin{aligned}(e^{e^t})^{(u+v)} = [(e^{e^t})^{(u)}]^{(v)} &= \sum_{n=1}^{u} \left\{ {u \atop n} \right\} (e^{e^t} \cdot e^{nt})^{(v)} \\ &= \sum_{n=1}^{u} \left\{ {u \atop n} \right\} \sum_{m=0}^{v} \binom{v}{m} (\sum_{i=0}^{m} \left\{ {m \atop i} \right\} e^{e^t} \cdot e^{it}) \cdot n^{v-m} e^{nt} \\ &= \sum_{n=1}^{u} \left\{ {u \atop n} \right\} \sum_{m=0}^{v} \binom{v}{m} (\sum_{i=0}^{m} \left\{ {m \atop i} \right\} e^{e^t} \cdot e^{(i+n)t} n^{v-m}). \end{aligned} \tag{11}$$

Theorem 2. *The following addition formula for the Stirling numbers of the second kind holds.*

$$\left\{ {u+v \atop k} \right\} = \sum_{n=1}^{k} \left\{ {u \atop n} \right\} \sum_{m=k-n}^{v} \binom{v}{m} \left\{ {m \atop k-n} \right\} n^{v-m}. \tag{12}$$

Proof. Formula (12) follows by comparing the coefficients of $e^{e^t} e^{kt}$ in (10) and (11) for $i + n = k$. □

Denoting in (12) $m = v - i$, $(i = 0, 1, 2, \ldots, v - k + n)$ we have

$$\left\{ {u+v \atop k} \right\} = \sum_{n=1}^{k} \left\{ {u \atop n} \right\} \sum_{i=0}^{v-k+n} \binom{v}{i} \left\{ {v-i \atop k-n} \right\} n^{i},$$

and then letting $n = k - j$, $(j = 0, 1, 2, \ldots, k - 1)$ we obtain

$$\left\{ {u+v \atop k} \right\} = \sum_{j=0}^{k-1} \sum_{i=0}^{v-j} \binom{v}{i} (k-j)^{i} \left\{ {v-i \atop j} \right\} \left\{ {u \atop k-j} \right\}. \tag{13}$$

Formula (4) follows easily from (13). To see this let us observe that the range of the variables i, j in (13) can be changed to the same range as in (4) i.e., $i, j = 0, 1, 2, \ldots, v$. This is allowed because the number $\left\{ {u \atop k-j} \right\}$ is zero if $j \geq k$ and $\left\{ {v-i \atop j} \right\}$ equals zero for $j > v - i$ i.e., if $i + j > v$. We rearrange (13) into (4) using formulas (1), for the symbol $\left\{ {v-i \atop j} \right\}$, respectively in the start and end of the following calculation:

$$
\begin{aligned}
\left\{ {u+v \atop k} \right\} &= \sum_{j=0}^{v} \sum_{i=0}^{v} \binom{v}{i} (k-j)^i \left\{ {v-i \atop j} \right\} \left\{ {u \atop k-j} \right\} \\
&= \sum_{j=0}^{v} \left\{ {u \atop k-j} \right\} \sum_{i=0}^{v} \binom{v}{i} (k-j)^i \frac{1}{j!} \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} l^{v-i} \\
&= \sum_{j=0}^{v} \left\{ {u \atop k-j} \right\} \frac{1}{j!} \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} \sum_{i=0}^{v} \binom{v}{i} (k-j)^i l^{v-i} \\
&= \sum_{j=0}^{v} \left\{ {u \atop k-j} \right\} \frac{1}{j!} \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} (k-j+l)^v \\
&= \sum_{j=0}^{v} \left\{ {u \atop k-j} \right\} \frac{1}{j!} \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} \sum_{i=0}^{v} (-1)^{v-i} \binom{v}{i} k^i (j-l)^{v-i} \\
&= \sum_{j=0}^{v} \left\{ {u \atop k-j} \right\} \sum_{i=0}^{v} \binom{v}{i} k^i (-1)^{v+i+j} \frac{1}{j!} \sum_{l=0}^{j} (-1)^{l} \binom{j}{l} (j-l)^{v-i} \\
&= \sum_{j=0}^{v} \sum_{i=0}^{v} \binom{v}{i} k^i (-1)^{v+i+j} \left\{ {v-i \atop j} \right\} \left\{ {u \atop k-j} \right\}.
\end{aligned}
$$

4 Addition formulas for Stirling numbers of the first kind

Let us substitute $f(x) = \log x$ in (7). We have $(t > 1)$

$$
(\log \log t)^{(u+v)} = \frac{(-1)^{u+v+1}}{t^{u+v}} \sum_{k=1}^{u+v} \left[{u+v \atop k} \right] \frac{(k-1)!}{\log^k t}. \tag{14}
$$

From the other side

$$
(\log \log t)^{(u)} = \frac{(-1)^{u+1}}{t^{u}} \sum_{n=1}^{u} \left[{u \atop n} \right] \frac{(n-1)!}{\log^n t}
$$

and by using the Leibniz formula and formula (7) for $g(t) = (\log t)^{-n}$ we get

$$\begin{aligned}
&(\log\log t)^{(u+v)} = [(\log\log t)^{(u)}]^{(v)} \\
&= (-1)^{u+1} \sum_{n=1}^{u} \begin{bmatrix} u \\ n \end{bmatrix} (n-1)! \left(\frac{1}{\log^n t} \cdot \frac{1}{t^u} \right)^{(v)} \\
&= (-1)^{u+1} \sum_{n=1}^{u} (n-1)! \begin{bmatrix} u \\ n \end{bmatrix} \sum_{m=1}^{v} \binom{v}{m} \left(\frac{1}{\log^n t} \right)^{(m)} \left(\frac{1}{t^u} \right)^{(v-m)} \\
&= (-1)^{u+1} \sum_{n=1}^{u} (n-1)! \begin{bmatrix} u \\ n \end{bmatrix} \sum_{m=1}^{v} \binom{v}{m} \frac{(-u)_{v-m}}{t^{u+v-m}} \\
&\quad \times \frac{1}{t^m} \sum_{i=0}^{m} (-1)^{m-i} \begin{bmatrix} m \\ i \end{bmatrix} \frac{(-n)_i}{(\log t)^{n+i}}.
\end{aligned} \tag{15}$$

Theorem 3. *The following addition formula for the Stirling numbers of the first kind holds.*

$$\begin{bmatrix} u+v \\ k \end{bmatrix} = \sum_{j=0}^{v} \sum_{i=0}^{v} \binom{v}{i} u^{(i)} \begin{bmatrix} v-i \\ j \end{bmatrix} \begin{bmatrix} u \\ k-j \end{bmatrix}. \tag{16}$$

Proof. By comparing the coefficients of $(t^{u+v} \log^k t)^{-1}$ in (14) and (15) for $i+n = k,\ i = k-n$ we get

$$\begin{aligned}
\begin{bmatrix} u+v \\ k \end{bmatrix} (k-1)! &= (-1)^v \sum_{n=1}^{k} (n-1)! \begin{bmatrix} u \\ n \end{bmatrix} (-n)_{k-n} \\
&\quad \times \sum_{m=k-n}^{v} \binom{v}{m} (-u)_{v-m} (-1)^{m-k+n} \begin{bmatrix} m \\ k-n \end{bmatrix}
\end{aligned}$$

and then using the identities

$$\frac{(n-1)!(-n)_{k-n}}{(k-1)!} = (-1)^{k-n}, \quad (-u)_{v-m}(-1)^{v-m} = u^{(v-m)}$$

we obtain the formula

$$\begin{bmatrix} u+v \\ k \end{bmatrix} = \sum_{n=1}^{k} \begin{bmatrix} u \\ n \end{bmatrix} \sum_{m=k-n}^{v} \binom{v}{m} u^{(v-m)} \begin{bmatrix} m \\ k-n \end{bmatrix}. \tag{17}$$

By the same manner as for the Stirling numbers of the second kind, formula (17) can be rearranged to the form (16). □

5 Generalized Euler polynomials

The generalized Euler polynomials $E_n^\mu(z)$ of degree $n = 0, 1, 2, \ldots$, complex order μ and complex argument z (see Nörlund [8]) can be defined by the generating function

$$\sum_{n=0}^{\infty} \frac{E_n^\mu(z)}{n!} w^n = \frac{2^\mu e^{wz}}{(e^w+1)^\mu}, \qquad |w| < \pi.$$

The generalized Euler polynomials play an important role in the calculus of finite differences. We will show, in a very easy way, an explicit formula for the Euler polynomial $E_n^\mu(z)$. By (6) we get

$$\frac{d^m}{dw^m}\frac{1}{(e^w+1)^\mu} = \sum_{k=1}^{m} \left\{ {m \atop k} \right\} (-\mu)_k \frac{e^{kw}}{(e^w+1)^{\mu+k}}$$

and then by the Leibniz formula

$$\frac{d^n}{dw^n}\frac{2^\mu e^{wz}}{(e^w+1)^\mu} = 2^\mu \left(\frac{z^n e^{wz}}{(e^w+1)^\mu} + \sum_{m=1}^{n} \binom{n}{m} \sum_{k=1}^{m} \left\{ {m \atop k} \right\} (-\mu)_k \frac{e^{kw}e^{wz}z^{n-m}}{(e^w+1)^{\mu+k}} \right).$$

Thus

$$E_n^\mu(z) = \left. \frac{d^n}{dw^n}\frac{2^\mu e^{wz}}{(e^w+1)^\mu} \right|_{w=0} = z^n + \sum_{m=1}^{n} \binom{n}{m} z^{n-m} \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \frac{(-\mu)_k}{2^k}. \tag{18}$$

Another approach to formula (18) is presented by Howard [5]. Putting in (18) $z = 0$ we obtain the following explicit formula (see Todorov [12]) for the generalized Euler number E_n^μ

$$E_n^\mu = E_n^\mu(0) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \frac{(-\mu)_k}{2^k}. \tag{19}$$

Luo [7] deals with the Apostol–Euler polynomials, which are a further generalization of the polynomials $\{E_n^\mu(z)\}$. Formula (19) coincides, as a particular case, with formula (32) of this paper.

6 Generalized Bernoulli polynomials

The generalized Bernoulli polynomials $B_n^\mu(z)$ of degree $n = 0, 1, 2, \ldots$, complex order μ and complex argument z (see Nörlund [8]) can be defined by the generating function

$$\sum_{n=0}^{\infty} \frac{B_n^\mu(z)}{n!} w^n = \frac{w^\mu e^{wz}}{(e^w-1)^\mu}, \qquad |w| < 2\pi.$$

We will show that by applying formula (6) one can obtain explicit formulas for the polynomials and then for the generalized Bernoulli numbers $B_n^\mu = B_n^\mu(0)$. Our approach can be seen as consistent with the spirit of the paper of Gould [3].
Using (6) and the Leibniz formula we get successively

$$\frac{d^j}{dw^j}\frac{1}{(e^w-1)^\mu} = \sum_{k=0}^{j} \left\{ {j \atop k} \right\} (-\mu)_k \frac{e^{kw}}{(e^w-1)^{\mu+k}},$$

$$\frac{d^m}{dw^m}\frac{w^\mu}{(e^w-1)^\mu} = \sum_{j=0}^{m} \binom{m}{j} \sum_{k=0}^{j} \left\{ {j \atop k} \right\} \frac{(-\mu)_k e^{kw}}{(e^w-1)^{\mu+k}} (\mu)_{m-j} w^{\mu-m+j} \tag{20}$$

and

$$\frac{d^n}{dw^n}\frac{w^\mu e^{wz}}{(e^w-1)^\mu} = \frac{w^\mu z^n e^{wz}}{(e^w-1)^\mu} + \sum_{m=1}^{n}\binom{n}{m}\left(\frac{d^m}{dw^m}\frac{w^\mu}{(e^w-1)^\mu}\right) z^{n-m}e^{wz}. \tag{21}$$

Rewriting the right hand side of (20) into the form

$$\frac{w^{\mu+m}}{(e^w-1)^{\mu+m}}\left(\frac{1}{w^{2m}}\sum_{j=0}^{m}\binom{m}{j}(\mu)_{m-j}\sum_{k=0}^{j}\left\{ {j \atop k} \right\}(-\mu)_k e^{kw}(e^w-1)^{m-k}w^j\right),$$

using (3) in the expression

$$\begin{aligned} w^j e^{kw}(e^w-1)^{m-k} &= w^j(e^w-1)^m\frac{e^{kw}}{(e^w-1)^k} = w^j(e^w-1)^m\left(1+\frac{1}{e^w-1}\right)^k \\ &= w^j(e^w-1)^m\sum_{l=0}^{k}\binom{k}{l}\frac{1}{(e^w-1)^l} = w^j\sum_{l=0}^{k}\binom{k}{l}(e^w-1)^{m-l} \\ &= w^j\sum_{l=0}^{k}\binom{k}{l}\sum_{i=m-l}^{\infty}(m-l)!\left\{ {i \atop m-l} \right\}\frac{w^i}{i!}, \end{aligned}$$

and grouping terms of power w^{2m} we get

$$\begin{aligned} \lim_{w\to 0}\frac{d^m}{dw^m}\frac{w^\mu}{(e^w-1)^\mu} &= \sum_{j=0}^{m}\binom{m}{j}(\mu)_{m-j}\sum_{k=0}^{j}\left\{ {j \atop k} \right\}(-\mu)_k \\ &\quad\times\sum_{l=0}^{k}\binom{k}{l}\left\{ {2m-j \atop m-l} \right\}\frac{(m-l)!}{(2m-j)!}. \end{aligned}$$

Thus by (21) we obtain the following explicit formula for the generalized Bernoulli polynomials

$$\begin{aligned} B_n^\mu(z) = \frac{d^n}{dw^n}\frac{w^\mu e^{wz}}{(e^w-1)^\mu}\Big|_{w=0} &= z^n + \sum_{m=1}^{n}\binom{n}{m}z^{n-m}\sum_{j=0}^{m}\binom{m}{j}(\mu)_{m-j} \\ &\quad\times\sum_{k=0}^{j}\left\{ {j \atop k} \right\}(-\mu)_k\sum_{l=0}^{k}\binom{k}{l}\left\{ {2m-j \atop m-l} \right\}\frac{(m-l)!}{(2m-j)!}. \end{aligned} \tag{22}$$

Comparing it with the formula given by Srivastava and Todorov [11, Eq. (3), p. 510], we see that formula (22) does not involve any hypergeometric function.

In particular, for $\mu = 1$ we obtain the common Bernoulli polynomials

$$\begin{aligned} B_n(z) &= z^n - \frac{1}{2}z^{n-1} + \sum_{m=2}^{n}\binom{n}{m}z^{n-m} \\ &\quad\times\left(m\sum_{k=1}^{m-1}\left\{ {m-1 \atop k} \right\}(-1)^k k!\sum_{l=0}^{k}\binom{k}{l}\left\{ {m+1 \atop m-l} \right\}\frac{(m-l)!}{(m+1)!}\right. \\ &\quad\left.+\sum_{k=1}^{m}\left\{ {m \atop k} \right\}(-1)^k k!\sum_{l=0}^{k}\binom{k}{l}\left\{ {m \atop m-l} \right\}\frac{(m-l)!}{m!}\right) \end{aligned}$$

and putting $z=0$, the common Bernoulli numbers

$$B_n = n\sum_{k=1}^{n-1} \left\{ {n-1 \atop k} \right\} (-1)^k k! \sum_{l=0}^{k} \binom{k}{l} \left\{ {n+1 \atop n-l} \right\} \frac{(n-l)!}{(n+1)!}$$
$$+\sum_{k=1}^{n} \left\{ {n \atop k} \right\} (-1)^k k! \sum_{l=0}^{k} \binom{k}{l} \left\{ {n \atop n-l} \right\} \frac{(n-l)!}{n!}. \tag{23}$$

Applying, to the expression (23), the two identities

$$\sum_{l=0}^{k} \binom{k}{l} \left\{ {n+1 \atop n-l} \right\} (n-l)! = \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j (n-j)^{n+1},$$

$$\sum_{l=0}^{k} \binom{k}{l} \left\{ {n \atop n-l} \right\} (n-l)! = \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j (n-j)^{n},$$

and then the next two

$$\sum_{k=1}^{n-1} \left\{ {n-1 \atop k} \right\} (-1)^k k! \binom{n-k}{j} = (-1)^{j+n+1} \sum_{k=0}^{j} \binom{n+1}{k} (-1)^k (j+1-k)^{n-1},$$

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\} (-1)^k k! \binom{n-k}{j} = (-1)^{j+n} \sum_{k=0}^{j} \binom{n+1}{k} (-1)^k (j+1-k)^{n}$$
$$= (-1)^{j+n} \left\langle {n \atop j} \right\rangle,$$

where $\left\langle {n \atop j} \right\rangle$ are Eulerian numbers (see [4], and A008292 in [10]), we obtain the following formula for the nth Bernoulli number

$$B_n = \frac{(-1)^n}{(n+1)!} \left[n \sum_{j=0}^{n-1} (n-j)^{n+1} \sum_{k=0}^{j} (-1)^{k+1} \binom{n+1}{k} (j+1-k)^{n-1} \right.$$
$$\left. +(n+1) \sum_{j=0}^{n-1} (n-j)^n \left\langle {n \atop j} \right\rangle \right].$$

7 Bell polynomials

The Bell polynomials $B_n(z)$ can be defined by the generating function (see Bell [1])

$$\sum_{n=0}^{\infty} \frac{B_n(z)}{n!} w^n = e^{(e^w-1)z}. \tag{24}$$

Using (6) we compute the nth derivative of the right hand side of (24)

$$\frac{d^n}{dw^n}e^{(e^w-1)z} = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} e^{(e^w-1)z} e^{kw} z^k. \tag{25}$$

The well known explicit formula for $B_n(z)$

$$B_n(z) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} z^k,$$

follows immediately from (25) by putting $w = 0$.

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