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# Matrix Compositions

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**Abstract**

In this paper we study the class of *$m$-row matrix compositions* ($m$-compositions, for short), i.e., $m$-row matrices with nonnegative integer entries in which every column has at least one non-zero element. We provide several enumerative results, various combinatorial identities, and some combinatorial interpretations. Most of these properties are an extension to matrix compositions of the combinatorial properties of ordinary compositions.

## 1 Introduction

A *composition* (or *ordered partition*) of length $k$ of a natural number $n$ is a $k$-tuple $(x_1, \ldots, x_k)$ of positive integers such that $x_1+\cdots+x_k = n$ . Compositions are very well known combinatorial objects [1, 11, 12, 13, 17, 27], and their study has been improved in several recent papers [7, 8, 14, 18, 21, 25, 26, 29, 30, 37]. Moreover, in certain algebraic contexts [5, 7, 8, 42, 44, 45], compositions are ordered to form a partially ordered set which generalizes Young's lattice for partitions. Finally, compositions have been generalized in various ways: we have the *vector compositions* [1, p.57][2, 3, 4] by P. A. MacMahon [34, 35, 36], the *m-colored compositions* by Drake and Petersen [19], the compositions defined by Lin and Rui [31], and the *packed matrices* by Duchamp, Hivert and Thibon [20].

Another slight generalization of ordinary compositions to the bidimensional case is given by the 2-*compositions*, introduced to encode $L$-convex polyominoes [16], Clearly, 2-compositions are a particular case of $m$-*compositions*. Indeed, more precisely, for any non-negative integer $m$, we define an $m$-*row matrix composition*, or $m$-*composition* for short, as an $m \times k$ matrix with nonnegative integer entries

$$M = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mk} \end{bmatrix}$$

where each column has at least one non-zero element. We say that the *length* of $M$ is the number $k$ of its columns. Moreover, we say that $M$ is an $m$-*composition* of a non-negative integer $n$ when the sum $\sigma(M)$ of all its entries is equal to $n$. For instance, we have the following seven 2-compositions of $n = 2$:

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

The aim of this paper is to give an elementary introduction to the combinatorics of matrix compositions. In particular, by using standard combinatorial techniques, we obtain several enumerative results, such as generating series, recurrences and explicit formulas, for $m$-compositions, $m$-compositions without zero rows, $m$-compositions with palindromic rows and $m$-compositions of Carlitz type (i.e., without equal consecutive columns). Moreover, we give some combinatorial interpretations of matrix compositions in terms colored linear partitions, labelled bargraphs and words of regular languages. Finally, by employing some of the results obtained by these combinatorial interpretations, we also prove a Cassini-like determinantal identity for $m$-compositions.

Other results concerning matrix compositions can be found in paper [23], where the problem of generating efficiently $m$-compositions and $m$-partitions has been treated, and in paper [32], where the probabilistic aspects of $m$-compositions have been studied.

## 2 Enumeration of $m$-compositions

Basic enumeration and combinatorial properties of $m$-compositions can be easily determined by using the technique of generating series.

Since matrix compositions can be expressed in a natural way in terms of multisets, we recall the following definitions. Let $\mathbb{N}$ be the set of all non-negative integer numbers. A *multiset* on a set $X$ is a function $\mu : X \to \mathbb{N}$. The *multiplicity* of an element $x \in X$ is $\mu(x)$. The *order* of $\mu$ is the sum $\mathrm{ord}(\mu)$ of the multiplicities of the elements of $X$, i.e.,

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $m=0$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $m=1$ | 1 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| $m=2$ | 1 | 2 | 7 | 24 | 82 | 280 | 956 | 3264 | 11144 | 38048 | 129904 |
| $m=3$ | 1 | 3 | 15 | 73 | 354 | 1716 | 8318 | 40320 | 195444 | 947380 | 4592256 |
| $m=4$ | 1 | 4 | 26 | 164 | 1031 | 6480 | 40728 | 255984 | 1608914 | 10112368 | 63558392 |
| $m=5$ | 1 | 5 | 40 | 310 | 2395 | 18501 | 142920 | 1104060 | 8528890 | 65885880 | 508970002 |
| $m=6$ | 1 | 6 | 57 | 524 | 4803 | 44022 | 403495 | 3698352 | 33898338 | 310705224 | 2847860436 |

Table 1: The numbers $c_n^{(m)}$. For $m = 3, 4, 5, 6$, we have the sequences A145839, A145840, A145841, A161434 in [43].

$\mathrm{ord}(\mu) = \sum_{x \in X} \mu(x)$. The number of all multisets of order $k$ on a set of size $n$ is the *multiset coefficient* [46]

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} = \frac{n(n+1)\cdots(n+k-1)}{k!}\,.$$

Let $\mathcal{C}_{nk}^{(m)}$ be the set of all $m$-compositions of $n$ of length $k$ and let $\mathcal{C}_n^{(m)}$ be the set of all $m$-compositions of $n$. Then let $c_{nk}^{(m)} = |\mathcal{C}_{nk}^{(m)}|$ and $c_n^{(m)} = |\mathcal{C}_n^{(m)}|$ (see Table 1). For simplicity, sometimes we just write $c_n$ for the coefficients $c_n^{(2)}$.

**Proposition 1.** *The generating series of $m$-compositions according to the sum (marked by $x$) and the length (marked by $y$), is*

$$c^{(m)}(x,y) = \sum_{n \geq 0} c_{nk}^{(m)} x^n = \frac{(1-x)^m}{(1+y)(1-x)^m - y}\,. \tag{1}$$

*In particular, the generating series of $m$-compositions according to the sum is*

$$c^{(m)}(x) = \sum_{n \geq 0} c_n^{(m)} x^n = \frac{(1-x)^m}{2(1-x)^m - 1}\,. \tag{2}$$

*Proof.* An $m$-composition $M$ can always be considered as the concatenation of its columns. Since each column of $M$ is equivalent to a multiset on an $m$-set with non-zero order, the generating series for the columns is

$$h^{(m)}(x) = \sum_{k \geq 1} \left(\!\!\binom{m}{k}\!\!\right) x^k = \frac{1}{(1-x)^m} - 1\,. \tag{3}$$

Hence, $c^{(m)}(x,y) = (1 - h^{(m)}(x)y)^{-1}$, that is (1). Finally, series (2) follows at once by setting $y = 1$ in (1). $\square$

Reading the denominator of the rational generating series (1) and (2), we can immediately obtain a linear recurrence for the numbers $c_{nk}^{(m)}$ and $c_n^{(m)}$, namely we can obtain recurrences (9) and (10) that will be proved in Proposition 4 with a combinatorial argument explaining their form. Here, we obtain two other recurrences just by manipulating series in formal way. Recall that the *incremental ratio* of a formal series $f(x) = \sum_{n \geq 0} f_n x^n$ is the series defined by $Rf(x) = (f(x) - f_0)/x = \sum_{n \geq 0} f_{n+1} x^n$.

**Proposition 2.** *The numbers $c_n^{(m)}$ satisfy the recurrence*

$$c_{n+1}^{(m)} = -\delta_{n,0} + 2c_n^{(m)} + \sum_{k=0}^{n} \binom{m+k-1}{k+1} c_{n-k}^{(m)} . \tag{4}$$

*Proof.* Rewriting series (2) in the following form

$$c^{(m)}(x) = \frac{1}{2 - \frac{1}{(1-x)^m}} = \frac{1-x}{2-2x-\frac{1}{(1-x)^{m-1}}} ,$$

we obtain the identity

$$\left(2 - 2x - \frac{1}{(1-x)^{m-1}}\right) c^{(m)}(x) = 1 - x$$

and hence the equation

$$c^{(m)}(x) = 1 - x + 2xc^{(m)}(x) + \left(\frac{1}{(1-x)^{m-1}} - 1\right) c^{(m)}(x) .$$

Now, taking the incremental ratio of both sides, we have

$$Rc^{(m)}(x) = -1 + 2c^{(m)}(x) + R\left(\frac{1}{(1-x)^{m-1}} - 1\right) c^{(m)}(x)$$

from which we obtain

$$c_{n+1}^{(m)} = -\delta_{n,0} + 2c_n^{(m)} + \sum_{k=1}^{n+1} \left(\!\!\binom{m-1}{k}\!\!\right) c_{n-k+1}^{(m)} = -\delta_{n,0} + 2c_n^{(m)} + \sum_{k=0}^{n} \left(\!\!\binom{m-1}{k+1}\!\!\right) c_{n-k}^{(m)} ,$$

which simplifies in (4). $\square$

Recurrence (4) generalizes the identity $c_{n+2} = 3c_{n+1} + c_n + \cdots + c_0$ , obtained in [16] by simple manipulations of the recurrence $c_{n+2} = 4c_{n+1} - 2c_n$ .

**Proposition 3.** *The numbers $c_{nk}^{(m)}$ satisfy the recurrence*

$$c_{n+1,k+1}^{(m+1)} = c_{n+1,k+1}^{(m)} - c_{n,k+1}^{(m)} + \sum_{i,j=0}^{n,k+1} c_{ij}^{(m)} c_{n-i,k-j+1}^{(m+1)} + \sum_{i,j=0}^{n,k} c_{ij}^{(m)} c_{n-i,k-j}^{(m+1)} . \tag{5}$$

*Similarly, the numbers $c_n^{(m)}$ satisfy the recurrence*

$$c_{n+1}^{(m+1)} = c_{n+1}^{(m)} - c_n^{(m)} + 2\sum_{k=0}^{n} c_k^{(m)} c_{n-k}^{(m+1)} . \tag{6}$$

*Proof.* For simplicity, we just prove identity (6). Identity (5) can be proved in a completely similar way. From (2), we have

$$(1-x)^m = \frac{c^{(m)}(x)}{2c^{(m)}(x)-1}\,.$$

Now, substituting $m$ with $m+1$ and $(1-x)^m$ with the above expression in identity (2), we obtain straightforwardly the relation

$$c^{(m+1)}(x) = \frac{(1-x)c^{(m)}(x)}{1-2xc^{(m)}(x)}\,,$$

and hence the equation

$$c^{(m+1)}(x) = (1-x)c^{(m)}(x) + 2xc^{(m)}(x)c^{(m+1)}(x)\,.$$

Finally, taking the incremental ratio of both sides, we have at once (6). □

# 3 Combinatorial identities

In this section we give a combinatorial interpretation of some formulas concerning $m$-compositions. Most of them can be obtained by employing the classical Principle of Inclusion-Exclusion [41, 46].

**Proposition 4.** *The coefficients $c_{nk}^{(m)}$ satisfy the recurrence*

$$c_{n+m,k+1}^{(m)} = \sum_{i=1}^{n+m-k} \left(\!\binom{m}{i}\!\right) c_{n+m-i,k}^{(m)}\,. \tag{7}$$

*Similarly, the coefficients $c_n^{(m)}$ satisfy the recurrence*

$$c_{n+m}^{(m)} = \sum_{i=1}^{n+m} \left(\!\binom{m}{i}\!\right) c_{n+m-i}^{(m)}\,. \tag{8}$$

*Proof.* Any $m$-composition $M \in \mathcal{C}_{n+m,k+1}^{(m)}$ can always be decomposed into two parts: the first column, equivalent to a multiset on the set $\{1,\ldots,m\}$ of non-zero order $i$ (with $0 \le i \le n+m-k$), and the rest of the matrix, equivalent to an $m$-composition of $n+m-i$ of length $k$. This decomposition implies at once recurrence (7). The same argument also implies (8). □

Recurrences (7) and (8) can be easily obtained, but they have a complex structure since they involve a summation. However, we also have the following linear recurrences.

**Proposition 5.** *The numbers $c_{nk}^{(m)}$ satisfies the recurrence*

$$c_{n+m,k+1}^{(m)} = \sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} c_{n+m-i,k}^{(m)} + \sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} c_{n+m-i,k+1}^{(m)} \,. \tag{9}$$

*Similarly, the numbers $c_n^{(m)}$ satisfies the recurrence*

$$c_{n+m}^{(m)} = 2 \sum_{i=1}^{m} \binom{m}{i} (-1)^{i-1} c_{n+m-i}^{(m)} \,. \tag{10}$$

*Proof.* For simplicity, we only prove recurrence (10) (the same argument, also proves recurrence (9)). Let $A_i$ be the set of all $m$-compositions $M$ of $n+m$ with a positive entry in position $(i,1)$ along the first column. Since the first column of $M$ is non-zero, it follows that $\mathcal{C}_{n+m}^{(m)} = A_1 \cup \cdots \cup A_m$ . Hence, by the Principle of Inclusion-Exclusion, we have

$$c_{n+m}^{(m)} = |A_1 \cup \cdots \cup A_m| = \sum_{\substack{S \subseteq [m] \\ S \neq \emptyset}} (-1)^{|S|-1} \left| \bigcap_{i \in S} A_i \right| \,.$$

The set $\bigcap_{i \in S} A_i$ is formed of all $m$-compositions $M = [x_{ij}]$ of $n+m$ having positive entries in the first column in all positions indexed by $S$ . If, for every $i \in S$ , we replace the entry $x_{i1}$ with $x_{i1} - 1$ , then the first column of $M$ either becomes the zero vector or it remains different from it. In the first case, removing the first column, we have an $m$-compositions of $n+m-|S|$ . In the second case, we have an $m$-composition of $n+m-|S|$ . Hence $\left|\bigcap_{i \in S} A_i\right| = 2\, c_{n+m-|S|}^{(m)}$ . Since this identity depends only on the size of $S$ , we obtain (10). ☐

**Remark 6.** For $m = 2$ , recurrence (10) reduces to the recurrence $c_{n+2} = 4c_{n+1} - 2c_n$ , already obtained in [16]. For $m = 3$ and $4$ , we have the following recurrences

$$c_{n+3}^{(3)} = 6c_{n+2}^{(3)} - 6c_{n+1}^{(3)} + 2c_n^{(3)}, \quad c_{n+4}^{(4)} = 8c_{n+3}^{(4)} - 12c_{n+2}^{(4)} + 8c_{n+1}^{(4)} - 2c_n^{(4)} \,.$$

**Proposition 7.** *The numbers $c_{nk}^{(m)}$ admit the explicit expression*

$$c_{nk}^{(m)} = \sum_{i=0}^{k} \binom{k}{i} \left(\!\!\binom{m(k-i)}{n}\!\!\right) (-1)^i \,. \tag{11}$$

*Similarly, the numbers $c_n^{(m)}$ admit the explicit expression*

$$c_n^{(m)} = \sum_{k=0}^{n} c_{nk}^{(m)} = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{k}{i} \left(\!\!\binom{m(k-i)}{n}\!\!\right) (-1)^i \,. \tag{12}$$

*Proof.* Let $A_i$ be the set of all matrices $M \in \mathcal{M}_{m,k}(\mathbb{N})$ where the $i$-th column is equal to the zero vector and $\sigma(M) = n$ . By the Principle of Inclusion-Exclusion, we have

$$c_{nk}^{(m)} = |A_1' \cap \cdots \cap A_k'| = \sum_{S \subseteq [k]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right| \,.$$

The intersection $\bigcap_{i\in S} A_i$ is the set of all matrices $M \in \mathcal{M}_{mk}(\mathbb{N})$ with a zero vector in all columns indexed by the elements of $S$. So, it is equivalent to the set of all multisets of order $n$ on a set of size $mk - m|S|$, and hence $\left|\bigcap_{i\in S} A_i\right| = \left(\!\binom{m(k-|S|)}{n}\!\right)$. Since this identity depends only on the size of $S$, we have (11). □

The combinatorial argument used in the proof of Proposition (7) can be easily generalized to the set $\mathcal{C}_k^{(m)}(r_1,\ldots,r_m)$ of all $m$-compositions of length $k$ where the $i$-th row has sum equal to $r_i$, for every $i = 1,\ldots,m$. Let $c_k^{(m)}(r_1,\ldots,r_m)$ be the cardinality of such a set.

**Proposition 8.** *The numbers $c_k^{(m)}(r_1,\ldots,r_m)$ admit the explicit expression*

$$c_k^{(m)}(r_1,\ldots,r_m) = \sum_{i=0}^{k} \binom{k}{i} \left(\!\binom{k-i}{r_1}\!\right) \cdots \left(\!\binom{k-i}{r_m}\!\right) (-1)^i \,. \tag{13}$$

*Proof.* Let $A_i$ be the set of all matrices $M \in \mathcal{M}_{m,k}(\mathbb{N})$ having the $i$-th column equal to the zero vector, and row-sums $r_1$, …, $r_m$. Then, by the Principle of Inclusion-Exclusion, we have

$$c_k^{(m)}(r_1,\ldots,r_m) = |A_1' \cap \ldots \cap A_k'| = \sum_{S\subseteq[k]} (-1)^{|S|} \left|\bigcap_{i\in S} A_i\right| .$$

The intersection $\bigcap_{i\in S} A_i$ contains all matrices $M \in \mathcal{M}_{mk}(\mathbb{N})$ with the zero vector in all columns indexed by the elements of $S$. Since the $i$-th row of such a matrix $M$ corresponds to a multiset of order $r_i$ on a set of size $k - |S|$, it follows that

$$\left|\bigcap_{i\in S} A_i\right| = \left(\!\binom{k-|S|}{r_1}\!\right) \cdots \left(\!\binom{k-|S|}{r_m}\!\right) .$$

Since this cardinality depends only on the size of $S$, we have (13). □

Identity (13) already appears in the book [1] where, however, it is proved in a formal way manipulating generating series.

Now, identities (11), (12) and (13) can be rewritten in terms of the *Stirling numbers of the first kind* $\left[{n \atop k}\right]$ ([22], sequences A008275 and A048994 in [43]) the *Stirling numbers of the second kind* $\left\{{n \atop k}\right\}$ ([22], sequences A008277 and A048933 in [43]), and the numbers $t_n$ of *preferential arrangements* [24, 46, 47] (sequence A000670 in [43]).

**Proposition 9.** *The numbers $c_{nk}^{(m)}$, $c_n^{(m)}$ and $c_k^{(m)}(r_1,\ldots,r_m)$ can be expressed as follows:*

$$c_{nk}^{(m)} = \frac{k!}{n!} \sum_{j=k}^{n} \left[{n \atop j}\right] \left\{{j \atop k}\right\} m^j \tag{14}$$

$$c_n^{(m)} = \frac{1}{n!} \sum_{k=0}^{n} \left[{n \atop k}\right] m^k t_k \tag{15}$$

$$c_k^{(m)}(r_1,\ldots,r_m) = \frac{k!}{r_1!\cdots r_m!} \sum_{j_1,\ldots,j_m=0}^{r_1,\ldots,r_m} \sum_{k\geq 0} \left[{r_1 \atop j_1}\right] \cdots \left[{r_m \atop j_m}\right] \left\{{j_1+\cdots+j_m \atop k}\right\} . \tag{16}$$

*Proof.* Using the ordinary expansion

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} {n \brack k} x^k \tag{17}$$

of the *rising factorials*, identity (11) becomes

$$c_{nk}^{(m)} = \frac{1}{n!}\sum_{j=0}^{n} {n \brack j} m^j \sum_{i=0}^{k} \binom{k}{i}(k-i)^j(-1)^i\,.$$

The second sum on the right hand-side is the number of all surjective functions from an $n$-set to a $k$-set [46], and can be expressed as

$$\sum_{i=0}^{k} \binom{k}{i}(k-i)^n(-1)^i = {n \brace k} k!\,. \tag{18}$$

Hence we have identity (14). Now, from (14), we have

$$c_n^{(m)} = \sum_{k=0}^{n} c_{nk}^{(m)} = \sum_{k=0}^{n} \frac{k!}{n!}\sum_{j=k}^{n} {n \brack j}{j \brace k} m^j = \frac{1}{n!}\sum_{j=0}^{n} {n \brack j} m^j \sum_{k=0}^{n} {j \brace k} k!\,.$$

Since

$$t_n = \sum_{k=0}^{n} {n \brace k} k!\,, \tag{19}$$

we obtain at once identity (15). Finally, by using (17) once again, identity (13) can be written as

$$c_k^{(m)}(r_1,\ldots,r_m) = \frac{1}{r_1!\cdots r_m!}\sum_{j_1,\ldots,j_m=0}^{r_1,\ldots,r_m}\sum_{k\geq 0} {r_1 \brack j_1}\cdots{r_m \brack j_m}\sum_{i=0}^{k}\binom{k}{i}(k-i)^{j_1+\cdots+j_m}(-1)^i\,.$$

Hence, from (18), we obtain (16). □

**Remark 10.** From (14) and (15), it follows that both $c_{nk}^{(m)}$ and $c_n^{(m)}$ are polynomial expressions in $m$.

**Remark 11.** Every $m$-composition $M \in \mathcal{C}_{nn}^{(m)}$ is an $m \times n$ $(0,1)$-matrix with exactly a 1 in each column, and hence is equivalent to a function $f : [n] \to [m]$. So, $c_{nn}^{(m)} = m^n$. Now, by using (11), we have

$$\sum_{k=0}^{n} \binom{n}{k}\left(\!\binom{mk}{n}\!\right)(-1)^k = m^n\,.$$

Notice that this identity can also be obtained from (14).

# 4 Binet-like formulas and asymptotics

In this section we will obtain a *Binet-like formula* and an asymptotic expansion for the coefficients $c_n^{(m)}$ .

**Proposition 12.** *The numbers $c_n^{(m)}$ admit the following Binet-like formula*

$$c_n^{(m)} = \frac{1}{2}\left[\delta_{n,0} + \frac{1}{m\sqrt[m]{2}}\sum_{k=0}^{m-1}\frac{\omega_m^k}{x_k^{n+1}}\right] = \frac{1}{2}\,\delta_{n,0} + \frac{1}{2m}\sum_{k=0}^{m-1}\frac{\omega_m^k}{\sqrt[m]{2}-\omega_m^k}\left(\frac{\sqrt[m]{2}}{\sqrt[m]{2}-\omega_m^k}\right)^n \tag{20}$$

*where*

$$x_k = 1 - \frac{1}{\sqrt[m]{2}}\,\omega_m^k \qquad (k = 0, 1, \ldots, m-1) \tag{21}$$

*where $\omega_m = \mathrm{e}^{2\pi\mathrm{i}/m}$ is a primitive root of unity.*

*Proof.* Series (2) can be rewritten as

$$c^{(m)}(x) = \frac{1}{2}\left[1 + \frac{1}{2(1-x)^m - 1}\right].$$

The roots of the polynomial at the denominator are the numbers $x_k$ given in (21). Then we have the expansion in partial fractions

$$\frac{1}{2(1-x)^m - 1} = \frac{A_0}{x - x_0} + \cdots + \frac{A_{m-1}}{x - x_{m-1}}$$

where the coefficients $A_k$ are defined by

$$A_k = \lim_{x\to x_k}\frac{x - x_k}{2(1-x)^m - 1}.$$

By applying De l'Hopital rule, we have

$$A_k = \lim_{x\to x_k}\frac{1}{-2m(1-x)^{m-1}} = \frac{1}{-2m(1-x_k)^{m-1}} = -\frac{\omega_m^k}{m\sqrt[m]{2}}.$$

Hence

$$c^{(m)}(x) = \frac{1}{2}\left[1 - \frac{1}{x_k}\sum_{k=0}^{m-1}\frac{A_k}{1 - x/x_k}\right] = \frac{1}{2}\left[1 + \frac{1}{m\sqrt[m]{2}\,x_k}\sum_{k=0}^{m-1}\frac{\omega_m^k}{1 - x/x_k}\right]$$

from which we obtain (20). □

**Proposition 13.** *For $n\to\infty$ , we have the asymptotic expansion*

$$c_n^{(m)} \sim \frac{1}{2m(\sqrt[m]{2}-1)}\left(\frac{\sqrt[m]{2}}{\sqrt[m]{2}-1}\right)^n.$$

*In particular, we have the limit*

$$\lim_{n\to\infty}\frac{c_n^{(m)}}{c_{n+1}^{(m)}} = 1 - \frac{1}{\sqrt[m]{2}}.$$

*Proof.* The statement follows at once from the fact that the dominant singularity (i.e., the root with minimum modulus) is $x_0 = 1 - 1/\sqrt[m]{2}$. □

# 5 Combinatorial interpretations

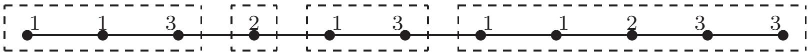
## 5.1 Colored linear partitions

Matrix compositions can be interpreted in terms of linear species [6, 28] as follows. Let $[m] = \{1, \dots, m\}$ be a set of colors, totally ordered in the natural way. We say that a linearly ordered set $[n] = \{1, 2, \dots, n\}$ is *m-colored* when each of its elements is colored with one of the colors in $[m]$ respecting the following condition: for every elements $x$, with color $i$, and $y$ with color $j$, if $x \le y$ then $i \le j$. In other words, an $m$-coloring of $[n]$ is an order-preserving map $\gamma : [n] \to [m]$. We define an *m-colored linear partition* of $[n]$ as a linear partition in which each block is $m$-colored.

The $m$-compositions of length $k$ of $n$ are equivalent to the $m$-colored linear partitions of $[n]$ with $k$ blocks. Indeed, any $M \in \mathcal{C}_{nk}^{(m)}$ corresponds to the $m$-colored linear partition $\pi$ of $[n]$ obtained transforming the $i$-th column $(h_1, \dots, h_m)$ of $M$ into the $i$-th block of $\pi$ of size $h_1 + \cdots + h_m$ with the first $h_1$ elements of color $1$, …, the last $h_m$ elements of color $m$, for every $i = 1, \dots, k$. For instance, the 3-composition

$$M = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \tag{22}$$

corresponds to the following 3-colored partition of the set $\{1, 2, 3, \dots, 11\}$




which can also be represented as $\pi = [[1, 1, 3], [2], [1, 3], [1, 1, 2, 3, 3]]$.

**Proposition 14.** *Let $\mathbf{C}^{(m)}$ be the linear species of $m$-compositions, i.e., the linear species of $m$-colored linear partitions. Let $\mathbf{G}$ be the uniform linear species. Let $\mathbf{Map}_{\neq\emptyset}^{(m)}$ be the linear species of multisets of non-zero order on the set $[m]$. Then*

$$\mathbf{C}^{(m)} = \mathbf{G} \circ \mathbf{Map}_{\neq\emptyset}^{(m)} . \tag{23}$$

*Proof.* To give an $m$-colored linear partition on a linearly ordered set $L$ is equivalent to assign a linear partition $\pi$ on $L$ and then an $m$-coloring (i.e., an order-preserving map in $[m]$ ) on each block of $\pi$. Since an order-preserving map $f : [k] \to [m]$ is equivalent to a multiset of order $k$ on the set $[m]$, we have at once (23). $\square$

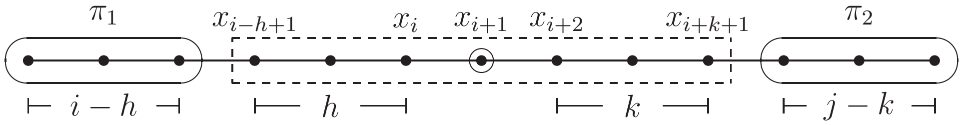
**Remark 15.** From identity (23), we reobtain at once (2). Indeed, since $\mathbf{Card}(\mathbf{G}; x) = 1/(1-x)$ and $\mathbf{Card}(\mathbf{Map}_{\neq\emptyset}^{(m)}; x) = h^{(m)}(x)$, where $h^{(m)}(x)$ is series (3), we have $\mathbf{Card}(\mathbf{C}^{(m)}; x) = \mathbf{Card}(\mathbf{G}; x) \circ \mathbf{Card}(\mathbf{Map}_{\neq\emptyset}^{(m)}; x) = c^{(m)}(x)$.

Using this interpretation, we can obtain the following identities we will employ in Section 6 to prove a Cassini-like identity.

**Proposition 16.** *We have the following identity*

$$c^{(m)}_{i+j+1} = \sum_{h,k\geq 0} \left(\!\!\binom{m}{h+k+1}\!\!\right) c^{(m)}_{i-h} c^{(m)}_{j-k} \,. \tag{24}$$

*Proof.* Let $L = \{x_1, \ldots, x_{i+1}, \ldots, x_{i+j+1}\}$ be a linearly ordered set with size $i+j+1$ and let $\pi \in \mathbf{C}^{(m)}[L]$. The element $x_{i+1}$ belongs to a block $B$ of the form $\{x_{i-h+1}, \ldots, x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k+1}\}$ with $h, k \in \mathbb{N}$, as in the following picture:




Removing the block $B$, $\pi$ splits into an $m$-colored linear partition $\pi_1$ on a linear order of size $i-h$ and into an $m$-colored linear partition $\pi_2$ on a linear order of size $j-k$. ☐

**Proposition 17.** *We have the identity*

$$\left(\!\!\binom{m}{i+j+1}\!\!\right) = \sum_{k=1}^{m} \left(\!\!\binom{k}{i}\!\!\right) \left(\!\!\binom{m-k+1}{j}\!\!\right) = \sum_{k=0}^{m-1} \binom{i+k}{i} \left(\!\!\binom{m-k}{j}\!\!\right). \tag{25}$$

*Proof.* The coefficient $\left(\!\binom{m}{i+j+1}\!\right)$ gives the number of all the order-preserving maps $f : [i+j+1] \to [m]$. Now, suppose that $f(i+1) = k$, with $k \in [m]$. Since $f$ is order-preserving, it follows that $f(x) \in [k]$ for every $x \in [i]$ and $f(x) \in \{k, \ldots, m\}$ for every $x \in \{i+2, \ldots, i+j+1\}$. Hence we have at once identity (25). ☐

## 5.2 Surjective families of order-preserving maps

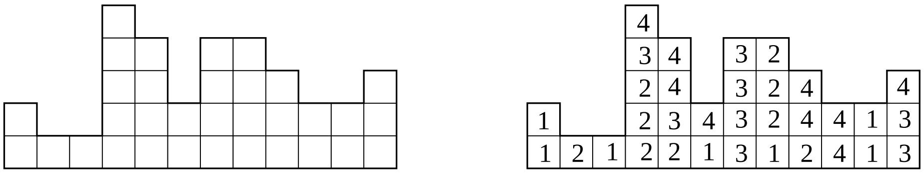
Let $P_1$, ..., $P_m$ and $Q$ be finite linearly ordered sets. We say that $\mathcal{F} = \{f_i : P_i \to Q\}_{i=1}^m$ is a *surjective family* of order-preserving maps when for every element $q \in Q$ there exists at least one index $i$ and one element $p \in P_i$ such that $q = f_i(p)$. The single maps are not necessarily surjective, but every element of the codomain admits at least one preimage along one of the maps of the family.

**Proposition 18.** *Let $P_1$, ..., $P_m$ and $Q$ be finite linearly ordered sets with $|P_1| = r_1$, ..., $|P_m| = r_m$ and $|Q| = k$. Then the number of all surjective families $\mathcal{F} = \{f_i : P_i \to Q\}_{i=1}^m$ is $c_k^{(m)}(r_1, \ldots, r_m)$.*

*Proof.* Let $Q = \{q_1, \ldots, q_k\}$. A surjective family $\mathcal{F} = \{f_i : P_i \to Q\}_{i=1}^m$ is equivalent to the $m$-composition $M$ of length $k$ with row-sum vector $(r_1, \ldots, r_k)$, whose $i$-row is $\mathbf{r}_i = (|f_i^\bullet(q_1)|, \ldots, |f_i^\bullet(q_k)|)$, where $f_i^\bullet(q_j)$ is the set of all preimages of $q_j$ along the map $f_i$. Clearly, the sum of the $i$-row $\mathbf{r}_i$ is $|P_i| = r_i$. Moreover, since $\mathcal{F}$ is a surjective family, any column of $M$ is different from the zero vector. ☐

## 5.3 Labelled bargraphs

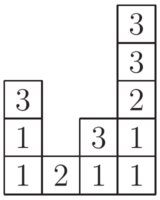
The interpretation of matrix compositions in terms of colored linear partitions can be reformulated in terms of *labelled bargraphs*. A *bargraph* is a column-convex polyomino where all columns are bottom justified (see Figure 1 (a)). A bargraph is completely determined by the height of its columns and gives a graphical representation of an ordinary composition (as already pointed out in [37]). Bargraphs, and more generally polyominoes [9], are well-known combinatorial objects. In particular, the enumeration of bargraphs according to perimeter, area and site-perimeter has been treated in [39, 40], in relation to the study of percolation models, and more recently, from an analytical point of view, in [10].




(a) (b)

Figure 1: (a) a bargraph; (b) a labelled bargraph of degree 4.

Let $M = [a_{ij}]$ be an $m$-composition, equivalent to an $m$-colored linear partition $\pi = [B_1, \ldots, B_k]$ where the block $B_j$ has the form $[1, \ldots, 1, 2, \ldots, 2, \ldots, m, \ldots, m]$, and for every $i = 1, \ldots, m$, $i$ occurs exactly $a_{ij}$ times. Now, draw each block vertically as a stack of cells, and label each cell with the corresponding color. What we obtain is a bargraph in which, along each column, the labels are weakly increasing from the bottom to the top. For instance, the 3-composition (22), equivalent to the 3-colored linear partition $\pi = [[1, 1, 3], [2], [1, 3], [1, 1, 2, 3, 3]]$, is represented by the following labelled bargraph of area 11 with 4 columns:




Similarly, the labelled bargraph in Figure 1(b) represents the following 4-composition of 33:

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 2 & 2 & 0 & 1 \end{bmatrix}.$$

So, we define a *labelled bargraph* as a bargraph in which all cells are labelled with positive integers so that, along each column, the label of a cell is less then or equal to the label of the cell immediately above (if any) (see Figure 1(b)). The *degree* is the maximal label of the bargraph. In this way, an $m$-composition of $n$ of length $k$ is equivalent to a labelled bargraph of area $n$ with $k$ columns and degree at most $m$.

## 5.4 Words of a regular language on finite many letters

Matrix compositions (as concatenation of columns) can be easily encoded as words of a language on infinite letters. However, they can also be encoded as words of a regular language on the finite alphabet $\mathcal{A}_m = \{a_1, \cdots, a_m, b_1, \ldots, b_m\}$ . This encoding extends the encoding described in [8] for the ordinary compositions, Let $\mathcal{C}^{(m)}$ be the set of all $m$-compositions and let $\ell : \mathcal{C}^{(m)} \to \mathcal{A}_m^*$ be the map defined in the following way. First, write an $m$-composition $M$ as the formal sum (juxtaposition) of its columns. Then write each column as juxtaposition of *simple columns*, that is columns containing exactly one non-zero entry. Now, order all simple columns according to the position of the non-zero entry. This convention allows to write each simple column as juxtaposition of *elementary columns*, that is columns containing exactly one non-zero entry, equal to $1$. Finally, substitute each elementary column with a letter according to the following rules

$$\begin{matrix}1\\0\\\vdots\\0\end{matrix} \xmapsto{\ell} a_1\,, \quad \ldots, \quad \begin{matrix}0\\\vdots\\0\\1\end{matrix} \xmapsto{\ell} a_m\,, \qquad +\begin{matrix}1\\0\\\vdots\\0\end{matrix} \xmapsto{\ell} b_1\,, \quad \ldots, \quad +\begin{matrix}0\\\vdots\\0\\1\end{matrix} \xmapsto{\ell} b_m \tag{26}$$

For instance, by applying this procedure to the 3-composition

$$M = \begin{bmatrix}2 & 0 & 1 & 2\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 2\end{bmatrix},$$

we have

$$\begin{aligned}
M \;&\rightsquigarrow\; \begin{matrix}2\\0\\1\end{matrix} + \begin{matrix}0\\1\\0\end{matrix} + \begin{matrix}1\\0\\1\end{matrix} + \begin{matrix}2\\1\\2\end{matrix}\\
&\rightsquigarrow\; \begin{matrix}2 & 0\\0 & 0\\0 & 1\end{matrix} + \begin{matrix}0\\1\\0\end{matrix} + \begin{matrix}1 & 0\\0 & 0\\0 & 1\end{matrix} + \begin{matrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{matrix}\\
&\rightsquigarrow\; \begin{matrix}1 & 1 & 0\\0 & 0 & 0\\0 & 0 & 1\end{matrix} + \begin{matrix}0\\1\\0\end{matrix} + \begin{matrix}1 & 0\\0 & 0\\0 & 1\end{matrix} + \begin{matrix}1 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 1\end{matrix}
\end{aligned}$$

and hence $\ell(M) = a_1a_1a_3b_2b_1a_3b_1a_1a_2a_3a_3$ .

**Proposition 19.** *The language $\mathcal{L}^{(m)} = \ell(\mathcal{C}^{(m)})$ on the alphabet $\mathcal{A}_m$ corresponding to the $m$-compositions is the regular language defined by the unambiguous regular expression*

$$\mathcal{L}^{(m)} = \varepsilon + \mathcal{L}_1^{(m)}\mathcal{L}_2^{(m)} \tag{27}$$

*where $\varepsilon$ , as usual, is the empty word, and*

$$\mathcal{L}_1^{(m)} = \left( a_1^+a_2^* \cdots a_m^* + a_2^+a_3^* \cdots a_m^* + \cdots + a_m^+ \right)$$

$$\mathcal{L}_2^{(m)} = (\, b_1a_1^*a_2^* \cdots a_m^* + b_2a_2^* \cdots a_m^* + \cdots + b_ma_m^* \,)^* \,.$$

*Proof.* Non-empty words in $\mathcal{L}^{(m)}$ are characterized by the following conditions:

1. the first letter is always an $a_i$, with $i = 1, 2, \ldots, m$;
2. the letters $a_i$ and $b_i$ can always be followed by any $b_j$, but they can be followed by an $a_j$ only when $i \le j$.

This characterization implies that the non-empty words in $\mathcal{L}^{(m)}$ have a unique factorization of the form $xy$, where

1. $x$ is a non-empty word of the form $a_1^{i_1} \cdots a_m^{i_m}$, with $i_1, \ldots, i_m \ge 0$;
2. $y$ is a (possibly empty) word $y = y_1 \cdots y_k$, where $y_r = b_j a_j^{q_j} \cdots a_m^{q_m}$ with $q_j, \ldots, q_m \ge 0$, for every $r = 1, \ldots, k$.

This factorization implies at once identity (27). □

**Remark 20.** The encoding just described is the basis for an efficient algorithm for the exhaustive generation of $m$-compositions, and for the definition of a Gray code on the set of $m$-compositions of a given size, as described in [23].

# 6 Cassini-like identities

For $m = 2$, the numbers $c_n$ satisfy the following *Cassini-like identity* [16]:

$$c_n c_{n+2} - c_{n+1}^2 = -2^{n-1} \qquad (\text{for } n \ge 1)\,.$$

This identity can be generalized to arbitrary $m$-compositions, as proved in the following

**Proposition 21.** *For every $m, n \ge 1$, we have the generalized Cassini-like identity:*

$$\begin{vmatrix} c_n^{(m)} & c_{n+1}^{(m)} & \cdots & c_{n+m-1}^{(m)} \\ c_{n+1}^{(m)} & c_{n+2}^{(m)} & \cdots & c_{n+m}^{(m)} \\ \vdots & \vdots & & \vdots \\ c_{n+m-1}^{(m)} & c_{n+m}^{(m)} & \cdots & c_{n+2m-2}^{(m)} \end{vmatrix} = (-1)^{\lfloor m/2 \rfloor}\, 2^{n-1}. \tag{28}$$

*Proof.* Let $C_n^{(m)} = [\ c_{n+i+j}^{(m)}\ ]_{i,j=0}^{m-1}$ be the matrix appearing on the left-hand side of (28). Since the main recurrence (10) is of the form

$$c_{n+m}^{(m)} = \alpha_{m-1} c_{n+m-1}^{(m)} + \cdots + \alpha_1 c_{n+1}^{(m)} + \alpha_0 c_n^{(m)} \qquad \left(\text{where } \alpha_k = (-1)^{m-k-1} 2 \binom{m}{k}\right),$$

we can simplify the last row of the determinant $|C_n^{(m)}|$ simply by subtracting to it a suitable linear combination of the first $m-1$ rows. More precisely, we have

$$|C_n^{(m)}| = \begin{vmatrix} c_n^{(m)} & c_{n+1}^{(m)} & \cdots & c_{n+m-1}^{(m)} \\ c_{n+1}^{(m)} & c_{n+2}^{(m)} & \cdots & c_{n+m}^{(m)} \\ \vdots & \vdots & & \vdots \\ c_{n+m-2}^{(m)} & c_{n+m-1}^{(m)} & \cdots & c_{n+2m-3}^{(m)} \\ \alpha_0 c_{n-1}^{(m)} & \alpha_0 c_n^{(m)} & \cdots & \alpha_0 c_{n+m-2}^{(m)} \end{vmatrix}.$$

Now, we can extract $\alpha_0 = (-1)^{m-1}2$ from the last row and shift cyclically all rows downward, obtaining the identity $|C_n^{(m)}| = 2\,|C_{n-1}^{(m)}|$. From this recurrence, it follows at once that

$$|C_n^{(m)}| = 2^{n-1}\,|C_1^{(m)}| \qquad (\text{for every } \ n \geq 1)\,.$$

It remains to evaluate the determinant of the matrix $C_1^{(m)} = [\ c_{i+j+1}^{(m)}\ ]_{i,j=0}^{m-1}$.

Identity (24) is equivalent to the matrix factorization

$$C_1^{(m)} = L^{(m)} M^{(m)} L_T^{(m)}$$

where

$$L^{(m)} = [\ c_{i-j}^{(m)}\ ]_{i,j=0}^{m-1} \qquad \text{and} \qquad M^{(m)} = \left[\ \left(\!\!\binom{m}{i+j+1}\!\!\right)\ \right]_{i,j=0}^{m-1}.$$

Since $L^{(m)}$ is triangular and its diagonal entries are equal to $c_0^{(m)} = 1$, it follows that

$$|C_1^{(m)}| = |M^{(m)}|.$$

Similarly, identity (25) is equivalent to the matrix factorization

$$M^{(m)} = B^{(m)}\widetilde{B}^{(m)}$$

where

$$B^{(m)} = \left[\ \binom{i+j}{i}\ \right]_{i,j=0}^{m-1} \qquad \text{and} \qquad \widetilde{B}^{(m)} = \left[\ \left(\!\!\binom{m-i}{j}\!\!\right)\ \right]_{i,j=0}^{m-1}.$$

Since $\widetilde{B}^{(m)} = J^{(m)}B^{(m)}$, where $J^{(m)} = [\delta_{i+j,m-1}]_{i,j=0}^{m-1}$, it follows that

$$M^{(m)} = B^{(m)}J^{(m)}B^{(m)}\,.$$

Since $|J^{(m)}| = (-1)^{\lfloor m/2 \rfloor}$ and $|B^{(m)}| = 1$, it follows that $|M^{(m)}| = (-1)^{\lfloor m/2 \rfloor}$. Finally, for every $n \geq 1$, we have

$$|C_n^{(m)}| = 2^{n-1}\,|C_1^{(m)}| = 2^{n-1}\,|M^{(m)}| = (-1)^{\lfloor m/2 \rfloor}2^{n-1}\,,$$

that is we have identity (28). □

# 7 Matrix compositions without zero rows

In this section, we will consider the class of all $m$-compositions where all rows are different from the zero vector. Let $f_n^{(m)}$ be the number of all $m$-compositions of $n$ of this kind.

**Proposition 22.** *The numbers $f_n^{(m)}$ admit the explicit expression*

$$f_n^{(m)} = \sum_{k=0}^{m}\binom{m}{k}(-1)^k c_n^{(m-k)} = \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k} c_n^{(k)}\,. \tag{29}$$

*Proof.* Let $A_i$ be the set of all $m$-compositions $M \in \mathcal{C}_n^{(m)}$ where the $i$-th row is zero. Then, by the Principle of Inclusion-Exclusion, we have

$$f_n^{(m)} = |A'_1 \cap \cdots \cap A'_m| = \sum_{S \subseteq [m]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right| .$$

Since there is an evident bijective correspondence between $\bigcap_{i \in S} A_i$ and the set of all $(m-|S|)$-compositions of $n$, we have at once (29). □

**Remark 23.** Since the set $\mathcal{C}_n^{(m)}$ can be partitioned according to the number of zero rows, we also have the identity

$$c_n^{(m)} = \sum_{k=0}^{m} \binom{m}{k} f_n^{(k)} . \tag{30}$$

Now, by inverting this formula, we reobtain (29). Viceversa, we can obtain (30) by inverting (29).

**Proposition 24.** *The generating series for the numbers $f_n^{(m)}$ is*

$$f^{(m)}(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} c^{(k)}(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{(1-x)^k}{2(1-x)^k - 1} . \tag{31}$$

*Proof.* This is an immediate consequence of identity (29). □

**Proposition 25.** *For $n \geq 1$, the numbers $f_n^{(m)}$ satisfy a homogeneous linear recurrence with constant coefficients of order $\binom{m+1}{2}$.*

*Proof.* Immediate consequence of the fact that the rational series (31) has the form

$$f^{(m)}(x) = \frac{x^m F_m(x)}{(1-2x)(1-4x+2x^2)\cdots(2(1-x)^m - 1)} \tag{32}$$

where $F_m(x)$ is a polynomial with degree (less than or) equal to $\binom{m}{2}$. □

**Remark 26.** The recurrence satisfied by the numbers $f_n^{(m)}$ can be deduced from the denominator of series (32). For instance, for $m = 2$, we have the series

$$f^{(2)}(x) = \frac{3x^2 - 2x^3}{(1-2x)(1-4x+2x^2)} = \frac{3x^2 - 2x^3}{1 - 6x + 10x^2 - 4x^3}$$

and hence the recurrence

$$f_{n+3}^{(2)} = 6f_{n+2}^{(2)} - 10f_{n+1}^{(2)} + 4f_n^{(2)} .$$

Similarly, for $m = 3$, we have the series

$$f^{(3)}(x) = \frac{13x^3 - 24x^4 + 16x^5 - 4x^6}{1 - 12x + 52x^2 - 102x^3 + 96x^4 - 44x^5 + 8x^6}$$

and hence the recurrence

$$f_{n+6}^{(3)} = 12f_{n+5}^{(3)} - 52f_{n+4}^{(3)} + 102f_{n+3}^{(3)} - 96f_{n+2}^{(3)} + 44f_{n+1}^{(3)} - 8f_n^{(3)} .$$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $m=0$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $m=1$ | | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| $m=2$ | | | 3 | 16 | 66 | 248 | 892 | 3136 | 10888 | 37536 | 128880 |
| $m=3$ | | | | 13 | 132 | 924 | 5546 | 30720 | 162396 | 834004 | 4204080 |
| $m=4$ | | | | | 75 | 1232 | 13064 | 114032 | 893490 | 6550112 | 45966744 |
| $m=5$ | | | | | | 541 | 13060 | 195020 | 2327960 | 24418640 | 235804122 |
| $m=6$ | | | | | | | 4683 | 155928 | 3116220 | 48697048 | 657516672 |

Table 2: The numbers $f_n^{(m)}$ .

**Proposition 27.** *The numbers $f_n^{(m)}$ have the following explicit expression*

$$f_n^{(m)} = \sum_{\substack{\rho\in\mathbb{P}^m \\ |\rho|=n}} \sum_{k\geq 0} \sum_{i=0}^{k} \binom{k}{i} \left(\!\!\binom{k-i}{r_1}\!\!\right) \cdots \left(\!\!\binom{k-i}{r_m}\!\!\right) (-1)^i \,, \tag{33}$$

*where $\mathbb{P}$ is the set of all positive integers.*

*Proof.* Since $f_n^{(m)}$ counts all $m$-compositions with non-zero row-sums, we have

$$f_n^{(m)} = \sum_{k\geq 0} \sum_{\substack{(r_1,\ldots,r_m)\in\mathbb{P}^m \\ r_1+\cdots+r_m=n}} c_k^{(m)}(r_1,\ldots,r_m) = \sum_{k\geq 0} \sum_{\substack{\rho\in\mathbb{P}^m \\ |\rho|=n}} c_k^{(m)}(\rho)$$

where $\rho = (r_1,\ldots,r_m)$ and $|\rho| = r_1+\cdots+r_m$ . Hence, by (13), we have at once (33). □

Let $f_{nk}^{(m)}$ be the number of all $m$-compositions, without zero rows, of $n$ of length $k$ .

**Proposition 28.** *The numbers $f_{nk}^{(m)}$ admit the explicit expression*

$$f_{nk}^{(m)} = \sum_{i=0}^{m} \sum_{j=0}^{k} \binom{m}{i} \binom{k}{j} \left(\!\!\binom{(m-i)(k-j)}{n}\!\!\right) (-1)^{i+j} \,. \tag{34}$$

*Proof.* Let $A_{ij}$ be the set of all matrices $M \in \mathcal{M}_{mk}(\mathbb{N})$ with the $i$-th row and the $j$-th column equal to the zero vector. Then, by the Principle of Inclusion-Exclusion, we have

$$f_{nk}^{(m)} = \left| \bigcap_{(i,j)\in[m]\times[k]} A'_{ij} \right| = \sum_{\substack{I\subseteq[m] \\ J\subseteq[k]}} (-1)^{|I|+|J|} \left| \bigcap_{\substack{i\in I \\ j\in J}} A_{ij} \right| .$$

Since the intersection $\bigcap_{i\in I,\, j\in J} A_{ij}$ is in bijective correspondence with the set of all multisets of order $n$ on a set of size $(m-|I|)(k-|J|)$ , identity (34) follows at once. □

Also the numbers $f_{nk}^{(m)}$ and $f_n^{(m)}$ can be expressed in terms of Stirling numbers and of the numbers $t_k$ of preferential arrangements.

**Proposition 29.** *The numbers $f_{nk}^{(m)}$ can be expressed as*

$$f_{nk}^{(m)} = \frac{m!k!}{n!} \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix} \begin{Bmatrix} h \\ m \end{Bmatrix} \begin{Bmatrix} h \\ k \end{Bmatrix} . \tag{35}$$

*Similarly, the numbers $f_n^{(m)}$ can be expressed as*

$$f_n^{(m)} = \frac{m!}{n!} \sum_{k=m}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} t_k . \tag{36}$$

*Proof.* By using (17), identity (34) can be rewritten as

$$f_{nk}^{(m)} = \frac{1}{n!} \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix} \sum_{i=0}^{m} \binom{m}{i} (m-i)^h (-1)^i \sum_{j=0}^{k} \binom{k}{j} (k-j)^h (-1)^j .$$

Now, by (18), we have (35). Finally, using the fact that $f_n^{(m)} = \sum_{k=0}^{n} f_{nk}^{(m)}$ and identities (19) and (35), we have at once identity (36). □

Clearly, $f_n^{(m)} = 0$ whenever $n < m$. In particular, we have

$$f_n^{(n)} = t_n , \qquad f_{n+1}^{(n)} = \frac{n}{2} (t_{n+1} + t_n) ,$$

$$f_{n+2}^{(n)} = \frac{n}{24} [(3n+1)t_{n+2} + 6(n+1)t_{n+1} + (3n+5)t_n] .$$

The identity $f_n^{(n)} = t_n$ can also be proved combinatorially since preferential arrangements can be represented as matrix compositions in a very natural way. Since a *preferential arrangement* is a set partition in which the blocks are linearly ordered [24, 46, 47], with a given preferential arrangement $\pi = (B_1, \ldots, B_k)$ of an $n$-set $X$ we can always associate the matrix $M$ having as columns the characteristic vectors of the blocks of $\pi$. For instance, the preferential arrangement $\pi = (\{2,3\}, \{1,5\}, \{4\})$ of the set $X = \{1,2,3,4,5\}$ corresponds to the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} .$$

So, if $\pi$ is a partition of an $n$-set with $k$ blocks, then the matrix $M$ has $n$ rows each of which contains exactly one non-zero entry equal to $1$, $k$ columns different from the zero vector and $\sigma(M) = n$. This means that $M$ is an $n$-composition of $n$ without zero rows. Since this correspondence is clearly a bijection between the class of preferential arrangements of an $n$-set and the class of $n$-composition of $n$ without zero rows, there follows the identity $f_n^{(n)} = t_n$.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $m=0$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $m=1$ | 1 | 1 | 1 | 3 | 4 | 7 | 14 | 23 | 39 | 71 | 124 |
| $m=2$ | 1 | 2 | 5 | 18 | 53 | 162 | 505 | 1548 | 4756 | 14650 | 45065 |
| $m=3$ | 1 | 3 | 12 | 58 | 255 | 1137 | 5095 | 22749 | 101625 | 454116 | 2028939 |
| $m=4$ | 1 | 4 | 22 | 136 | 793 | 4660 | 27434 | 161308 | 948641 | 5579224 | 32811986 |
| $m=5$ | 1 | 5 | 35 | 265 | 1925 | 14056 | 102720 | 750255 | 5480235 | 40031030 | 292408771 |
| $m=6$ | 1 | 6 | 51 | 458 | 3984 | 34788 | 303902 | 2654064 | 23179743 | 202445610 | 1768099107 |

Table 3: The numbers $z_n^{(m)}$ .

**Remark 30.** Another kind of matrix compositions are *packed matrices* [20], that is matrices with nonnegative integer entries without zero rows or zero columns. Let $b_n$ be the number of all packed matrices $M$ with $\sigma(M) = n$ . These numbers form sequence A120733 in [43], and, by (36), can be expressed as

$$b_n = \sum_{m=0}^{n} f_n^{(m)} = \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} t_k^2 \, .$$

# 8 Matrix compositions of Carlitz type

We say that an $m$-composition is of *Carlitz type* when no two adjacent columns are equal. For $m = 1$ , we have the ordinary *Carlitz compositions* [11] (see also [13, 33, 29] and [18]). Let $z_n^{(m)}$ be the number of all $m$-composition of $n$ of Carlitz type. For $m = 1$ we have sequence A003242 in [43], while for $m \geq 2$ we have new sequences (see Table 3).

**Proposition 31.** *The generating series of the numbers $z_n^{(m)}$ is*

$$z^{(m)}(x) = \sum_{n \geq 0} z_n^{(m)} \, x^n = \frac{1}{1 - \sum_{k \geq 1} \left(\!\!\binom{m}{k}\!\!\right) \frac{x^k}{1+x^k}} \, , \tag{37}$$

*or, equivalently,*

$$z^{(m)}(x) = \frac{1}{1 + \sum_{k \geq 1} (-1)^k \frac{1-(1-x^k)^m}{(1-x^k)^m}} = \frac{1}{1 + \sum_{k \geq 1} (-1)^k h^{(m)}(x^k)} \, . \tag{38}$$

*Proof.* Let $x_\mu$ be an indeterminate marking a column of an $m$-matrix equivalent to a multiset $\mu \in \mathcal{M}_{\neq 0}^{(m)}$ , and let $X$ be the set of all these indeterminates. Let $z^{(m)}(X)$ be the generating series for the set of all $m$-compositions of Carlitz type and let $z_\mu^{(m)}(X)$ be the generating series for the set of all $m$-compositions of Carlitz type whose last column corresponds to the multiset $\mu$. Then we have at once the linear system

$$\begin{cases} z^{(m)}(X) = 1 + \displaystyle\sum_{\mu \in \mathcal{M}^{(m)}_{\neq 0}} z^{(m)}_{\mu}(X) \\ z^{(m)}_{\mu}(X) = \Big(z^{(m)}(X) - z^{(m)}_{\mu}(X)\Big)\, x_{\mu} \qquad \forall\, \mu \in \mathcal{M}^{(m)}_{\neq 0} \end{cases}$$

from which

$$z^{(m)}_{\mu}(X) = \frac{x_{\mu}}{1 + x_{\mu}}\, z^{(m)}(X) \qquad \text{and} \qquad z^{(m)}(X) = \frac{1}{1 - \displaystyle\sum_{\mu \in \mathcal{M}^{(m)}_{\neq 0}} \frac{x_{\mu}}{1 + x_{\mu}}}\,.$$

Now, to obtain (37) it is sufficient to substitute $x_{\mu}$ with $x^{\mathrm{ord}(\mu)}$ in $z^{(m)}(X)$.

Finally, (38) can be obtained with the same argument used in [11] by Carlitz, or simply by rewriting in a suitable way the series at the denominator of (37). ☐

**Proposition 32.** *The numbers $z^{(m)}_n$ can be expressed as*

$$z^{(m)}_n = \sum_{k \geq 0} \sum_{\substack{\alpha, \beta \in \mathbb{P}^k \\ \alpha \cdot \beta = n}} \left(\!\!\binom{m}{\alpha}\!\!\right) (-1)^{|\beta| - k}\,. \tag{39}$$

*where, for every $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_k)$, $\alpha \cdot \beta = a_1 b_1 + \cdots + a_k b_k$, $|\beta| = b_1 + \cdots + b_k$ and $\left(\!\binom{m}{\alpha}\!\right) = \left(\!\!\binom{m}{a_1}\!\!\right) \ldots \left(\!\!\binom{m}{a_k}\!\!\right)$.*

*Proof.* From (37), we have

$$\begin{aligned} z^{(m)}(x) &= \sum_{k \geq 0} \left( \sum_{n \geq 1} \left(\!\!\binom{m}{n}\!\!\right) \frac{x^n}{1 + x^n} \right)^k \\ &= \sum_{k \geq 0} \sum_{a_1 \geq 1} \left(\!\!\binom{m}{a_1}\!\!\right) \frac{x^{a_1}}{1 + x^{a_1}} \cdots \sum_{a_k \geq 1} \left(\!\!\binom{m}{a_k}\!\!\right) \frac{x^{a_k}}{1 + x^{a_k}} \\ &= \sum_{k \geq 0} \sum_{a_1, \ldots, a_k \geq 1} \left(\!\!\binom{m}{a_1}\!\!\right) \cdots \left(\!\!\binom{m}{a_k}\!\!\right) \frac{x^{a_1}}{1 + x^{a_1}} \cdots \frac{x^{a_k}}{1 + x^{a_k}} \\ &= \sum_{k \geq 0} \sum_{\substack{a_1, \ldots, a_k \geq 1 \\ b_1, \ldots, b_k \geq 1}} \left(\!\!\binom{m}{a_1}\!\!\right) \cdots \left(\!\!\binom{m}{a_k}\!\!\right) (-1)^{b_1 + \cdots + b_k - k} x^{a_1 b_1 + \cdots + a_k b_k} \\ &= \sum_{n \geq 0} \left( \sum_{k \geq 0} \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^k \\ \alpha \cdot \beta = n}} \left(\!\!\binom{m}{\alpha}\!\!\right) (-1)^{|\beta| - k} \right) x^n \end{aligned}$$

Hence, we have (39). ☐

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $m=0$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $m=1$ | | 1 | 1 | 3 | 4 | 7 | 14 | 23 | 39 | 71 | 124 |
| $m=2$ | | | 3 | 12 | 45 | 148 | 477 | 1502 | 4678 | 14508 | 44817 |
| $m=3$ | | | | 13 | 108 | 672 | 3622 | 18174 | 87474 | 410379 | 1894116 |
| $m=4$ | | | | | 75 | 1056 | 10028 | 79508 | 570521 | 3850376 | 24966124 |
| $m=5$ | | | | | | 541 | 11520 | 155840 | 1705915 | 16529925 | 148188201 |
| $m=6$ | | | | | | | 4683 | 140256 | 2566554 | 37084794 | 465922722 |

Table 4: The numbers $g_n^{(m)}$ .

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $m=0$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $m=1$ | 1 | 1 | 2 | 2 | 4 | 4 | 8 | 8 | 16 | 16 | 32 |
| $m=2$ | 1 | 2 | 5 | 8 | 18 | 28 | 62 | 96 | 212 | 328 | 724 |
| $m=3$ | 1 | 3 | 9 | 19 | 48 | 96 | 236 | 468 | 1146 | 2270 | 5556 |
| $m=4$ | 1 | 4 | 14 | 36 | 101 | 240 | 648 | 1520 | 4082 | 9560 | 25660 |
| $m=5$ | 1 | 5 | 20 | 60 | 185 | 501 | 1470 | 3910 | 11390 | 30230 | 88002 |
| $m=6$ | 1 | 6 | 27 | 92 | 309 | 930 | 2939 | 8640 | 27048 | 79280 | 247968 |

Table 5: The numbers $p_n^{(m)}$ .

Now, let $g_n^{(m)}$ be the number of all $m$-compositions of Carlitz type of $n$ without zero rows. With arguments completely similar to the ones used in the case of ordinary $m$-compositions, we have

$$z_n^{(m)} = \sum_{k=0}^{m} \binom{m}{k} g_n^{(k)} \qquad \text{and} \qquad g_n^{(m)} = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} z_n^{(k)} . \tag{40}$$

Every $n$-composition of $n$ without zero rows is necessarily of Carlitz type. Indeed, it corresponds to a preferential arrangement and this implies at once that there are no two equal columns. Then $g_n^{(n)} = f_n^{(n)} = t_n$ .

# 9 Matrix compositions with palindromic rows

An ordinary composition is *palindromic* when its elements are the same in the given or in the reverse order [14, 15, 27, 37]. Here, we say that an $m$-composition is palindromic when all its rows are palindromic. For instance,

$$M = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 1 & 1 & 3 \end{bmatrix}$$

is a palindromic 4-composition of length 5 of 24 . Let $p_n^{(m)}$ be the number of all palindromic $m$-compositions of $n$ (see Table 5).

| $m/n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $m=0$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $m=1$ | | 1 | 2 | 2 | 4 | 4 | 8 | 8 | 16 | 16 | 32 |
| $m=2$ | | | 1 | 4 | 10 | 20 | 46 | 80 | 180 | 296 | 660 |
| $m=3$ | | | | 1 | 6 | 24 | 74 | 204 | 558 | 1334 | 3480 |
| $m=4$ | | | | | 1 | 8 | 44 | 192 | 706 | 2384 | 7652 |
| $m=5$ | | | | | | 1 | 10 | 70 | 400 | 1930 | 8182 |
| $m=6$ | | | | | | | 1 | 12 | 102 | 724 | 4404 |

Table 6: The numbers $q_n^{(m)}$ .

**Proposition 33.** *The generating series for the palindromic $m$-compositions is*

$$p^{(m)}(x) = \sum_{n\geq 0} p_n^{(m)}\, x^n = \frac{(1+x)^m}{2(1-x^2)^m - 1}\,. \tag{41}$$

*In particular, the numbers $p_n^{(m)}$ can be expressed as*

$$p_n^{(m)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\!\!\binom{m}{n-2k}\!\!\right) c_k^{(m)}\,. \tag{42}$$

*Proof.* A palindromic $m$-composition of even length has the form $[M|M_s]$ and a palindromic $m$-composition of odd length has the form $[M|\mathbf{v}|M_s]$ , where $M$ is an arbitrary $m$-composition, $M_s$ is the specular $m$-composition obtained from $M$ by reversing every row and $\mathbf{v}$ is an arbitrary non-zero column vector. Hence

$$p^{(m)}(x) = c^{(m)}(x^2) + \left[\frac{1}{(1-x)^m} - 1\right] c^{(m)}(x^2) = \frac{c^{(m)}(x^2)}{(1-x)^m}\,,$$

that is (41). Finally, identity (42) can be obtained by expanding the series on the right-hand side of the above equations. $\square$

Now, let $q_n^{(m)}$ be the number of all $m$-compositions of $n$ with palindromic non-zero rows (see Table 6). With arguments similar to those used in the case of ordinary $m$-compositions, we have

$$p_n^{(m)} = \sum_{k=0}^{m} \binom{m}{k} q_n^{(k)} \qquad \text{and} \qquad q_n^{(m)} = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} p_n^{(k)}\,. \tag{43}$$

When $n = m$ , the column vector with all entries equal to 1 is the only $n$-composition with palindromic rows. So, $q_n^{(n)} = 1$ .

# 10 Matrices generated by $m$-compositions

Identities (29), (30), (40) and (43) can be reformulated in terms of matrices. In particular, we will consider the following pairs of infinite matrices.

1. The matrix $C = [c_n^{(m)}]_{m,n\geq 0}$ generated by $m$-compositions (see Table 1), and the the upper triangular matrix $F = [f_n^{(m)}]_{m,n\geq 0}$ generated by $m$-compositions without zero rows (see Table 2).

2. The matrix $Z = [z_n^{(m)}]_{m,n\geq 0}$ generated by $m$-compositions of Carlitz type (see Table 3), and the upper triangular matrix $G = [g_n^{(m)}]_{m,n\geq 0}$ generated by $m$-compositions of Carlitz type without zero rows (see Table 4).

3. The matrix $P = [p_n^{(m)}]_{m,n\geq 0}$ generated by $m$-compositions with palindromic rows (see Table 5), and the upper triangular matrix $Q = [q_n^{(m)}]_{m,n\geq 0}$ generated by $m$-compositions with palindromic rows without zero rows (see Table 6).

Finally, we also need the ordinary binomial matrix $B = [\binom{m}{n}]_{m,n\geq 0}$ . Moreover, if $X = [x_{ij}]_{i,j\geq 0}$ if an infinite matrix, then we can always consider the partial matrices $X_k = [x_{ij}]_{i,j=0}^{k}$ , for every $k \in \mathbb{N}$ .

**Proposition 34.** *We have the following LU-factorizations over* $\mathbb{N}$ *:* $C = BF$ *,* $Z = BG$ *,* $P = BQ$ *. Similarly,* $C_k = B_k F_k$ *,* $Z_k = B_k G_k$ *,* $P_k = B_k Q_k$ *, for every* $k \in \mathbb{N}$ *.*

**Proposition 35.** *For every* $k \in \mathbb{N}$ *, we have* $\det C_k = \det Z_k = t_0 t_1 \cdots t_k$ *and* $\det P_k = 1$ *.*

*Proof.* Since $B_k$ , $F_k$ , $G_k$ and $Q_k$ are triangular matrices and $f_n^{(n)} = g_n^{(n)} = t_n$ and $q_n^{(n)} = 1$ for every $n \in \mathbb{N}$ , the factorizations in Proposition 34 and Binet's theorem imply at once the stated identities. □

# 11 Final remarks

In this final section, we present some open problems on matrix compositions and some possible lines of research on this topic.

***L*-convex polyominoes.** As remarked in the introduction, 2-compositions have been introduced in [16] to provide a simple encoding of $L$-convex polyominoes. This result led us to consider $m$-compositions with the hope of encoding some larger class of polyominoes. To find such larger classes of polyominoes, however, seems to be much more problematic and is still a completely open problem.

**Labelled bargraphs.** The simple correspondence between $m$-compositions and labelled bargraphs with degree at most $m$ (considered in Subsection 5.3) suggests to study some particular subclasses of matrix compositions arising in a very natural way as subclasses of bargraphs, such as the following ones.

1. The class of bargraphs having all the $m$ labels in each column (Figure 2(a)), corresponding to the set of $m$-compositions containing no 0's.

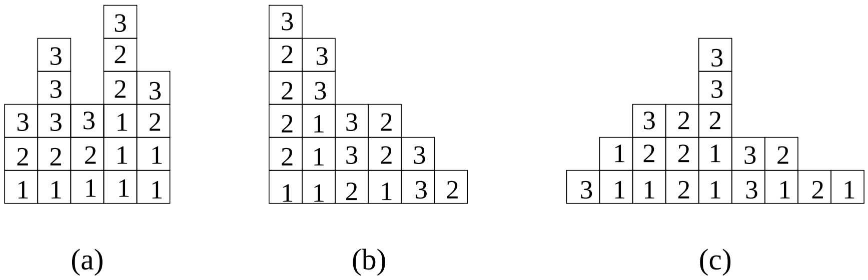



Figure 2: Labelled bargraphs of degree 3: (a) having all the labels in each of its columns; (b) a 3-partition; (c) a labelled stack of degree 3.

2. The class of *labelled Ferrers diagrams*, i.e., labelled bargraphs in which each column has height greater than or equal to the height of the column on its right (see Figure 2 (b)). A labelled Ferrers diagram of degree $m$ corresponds to an $m$-composition such that the sum of the entries of each column is greater than or equal to the sum of the entries of column on its right. We call these objects *$m$-partitions*. This definition is motivated by the fact that the ordinary partitions correspond to Ferrers diagrams, that is labelled Ferrers diagrams of degree $1$. For instance, the bargraph in Figure 2 (b) corresponds to the following 3-partition of $20$:

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 2 & 0 & 2 & 0 \end{bmatrix}.$$

3. The class of *labelled stacks*, that is of labelled bargraphs in which each row is connected. These objects have the shape of stack polyominoes, as can be seen in Figure 2(c). Given a labelled stack, let $c_i$ be the sum of the entries of its $i$-th column. Then a labelled stack of degree $m$ corresponds to an $m$-composition in which the sequence $c_1, \ldots, c_k$ is unimodal.

The problem of generating efficiently the *$m$-partitions* has been studied in [23], while the problem of enumerating labelled Ferrers diagrams and labelled stacks has been solved in [38], in a more general context.

**$m$-colored compositions.** Another interesting problem concerns the generalization to matrix compositions of the poset of ordinary compositions considered by Björner and Stanley [8]. A first step in this direction has been made, independently, by Drake and Petersen in [19], where they introduced the *$m$-colored compositions*. What is relevant here is that the $m$-colored compositions can be considered as a particular kind of matrix compositions. Indeed an $m$-colored composition $\alpha$ is an ordered tuple of "colored" positive integers, that is $\alpha = (a_1\omega^{s_1}, \ldots, a_k\omega^{s_k})$, where the $a_i$'s are positive integers, $\omega$ is a primitive $m$-root of unity and $0 \leq s_i \leq m-1$ for each $i = 1, \ldots, k$. The $i$-th part of $\alpha$ is $a_i\omega^{s_i}$ and has color $\omega^{s_i}$. Moreover, $\alpha$ is an $m$-colored composition of an integer number $n$ when $a_1 + \cdots + a_k = n$. Hence an $m$-colored composition $\alpha$ of a integer $n$ is uniquely represented by the $m$-composition $M(\alpha)$

of $n$ where in column $i$ appears exactly one non-zero entry, equal to $a_i$ , in position $(s_i+1,i)$ , for each $i=1,\ldots,k$ . So, for instance, the 3-colored composition $\alpha=(2\omega,3,1,\omega^2,3\omega)$ of 10 is equivalent to the 3-composition

$$M(\alpha)=\begin{bmatrix}0&3&1&0&0\\2&0&0&0&3\\0&0&0&1&0\end{bmatrix}.$$

This suggests that the poset of $m$-colored compositions can be generalized to a poset of matrix compositions. This generalization will be studied in detail in a further work. Finally, it could be interesting to study the natural generalization of $m$-compositions to $r$-colored $m$-compositions.

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