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Families of Rational Numbers with Predictable Engel Product Expansions

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Abstract

In 1987 Knopfmacher and Knopfmacher published new infinite product expansions for real numbers $0 < A < 1$ and $A > 1$. They called these Engel product expansions. At that time they had difficulty finding rational $0 < A < 1$ for which the Engel product expansion is predictable. Later, in 1993, Arnold Knopfmacher presented many such families of rationals. In this paper we add to Arnold Knopfmacher's list of such families.

1 Introduction

Knopfmacher and Knopfmacher [1] proved that every positive real number $0 < A < 1$ has an expansion of the form

$$A = \prod_{n=1}^{\infty} \left(1 - \frac{1}{a_1 a_2 \cdots a_n}\right), \tag{1}$$

where a_n is a positive integer for $n \geq 1$, $a_1 \geq 2$, $a_{n+1} \geq a_n - 1$ for $n \geq 1$, and $a_n \geq 2$ infinitely often.

These representations were called Engel product expansions. To abbreviate (1), Knopfmacher and Knopfmacher wrote

$$A = ((a_1, a_2, a_3, \ldots)).$$

They remarked that there do not appear to be simple cases where the digits $\{a_i\}$ could be found with explicit formulae. Later, Arnold Knopfmacher [2] gave several families of rationals where, from some point onwards, the digits $\{a_i\}$ in the expansion (1) satisfy the recurrence

$$a_{n+1} = (a_n + 1)\, a_{n-1} - 1. \tag{2}$$

To illustrate, one of Knopfmacher's main results is the following:

Theorem 1. *Let a and m be positive integers with $a \geq 3$ if $m = 1$, and $a \geq 2$ otherwise. Then*

$$\frac{(a-1)\,m - 1}{am - 1} = ((a_1, a_2, a_3, \ldots))\,,$$

where $a_1 = a$, $a_2 = (a-1)m$, $a_3 = am - 1$, $a_4 = a_1 a_2 - 1$, and (2) applies for $n \geq 4$.

Knopfmacher and Knopfmacher [1] showed that the expansion (1) is unique if it has non-decreasing digits. Therefore, in Theorem 1 a unique expansion is guaranteed for every a provided $m \geq 2$.

Knopfmacher [2, page 426] stated that it would be interesting to characterize in some way which rational numbers have expansions where the digits $\{a_i\}$ ultimately satisfy the nonlinear recurrence (2). Based on our investigations it seems difficult to resolve this completely. We have, however, managed to add to Knopfmacher's collection of such rationals. We present our results as families of rationals that require two or more parameters to describe them. Indeed, we have found many one-parameter families, but, to conserve space, we do not present them here.

In Section 2 we state and prove one of our main results. In Sections 3 and 4 we state further results.

2 A Result With Sample Proof

Before proceeding we give the greedy-type algorithm from [1] which was used to derive the expansion (1):

Given any $0 < A < 1$, let $A_1 = A$, $a_1 = 1 + \left\lfloor \frac{1}{1-A_1} \right\rfloor$, where $\lfloor x \rfloor$ denotes the floor of the real number x. Then recursively define, for $n \geq 2$,

$$a_n = 1 + \left\lfloor \frac{1}{(1 - A_n)\, a_1 a_2 \cdots a_{n-1}} \right\rfloor,$$

where

$$A_{n+1} = \frac{a_1 a_2 \cdots a_n}{a_1 a_2 \cdots a_n - 1} A_n.$$

The theorem that follows is similar in spirit to Theorem 1 above. The proof we present proceeds along the same lines as Knopfmacher's beautiful proof of Theorem 1.

Theorem 2. *Let a and m be positive integers with $a \geq 2$ if $m = 1$, and $a \geq 1$ otherwise. Then*

$$\frac{(2a-1)\,m-1}{4am-2} = ((a_1, a_2, a_3, \ldots)),$$

where $a_1 = 2$, $a_2 = a$, $a_3 = (2a-1)m$, $a_4 = 2am-1$, $a_5 = a_1a_2a_3 - 1$, and (2) applies for $n \geq 5$.

Proof. With the use of the algorithm above we write down A_1, a_1, A_2, a_2, A_3, a_3, A_4, and a_4. We present each A_i in a manner that makes it easy to find a_i.

$$A_1 = \frac{(2a-1)m-1}{4am-2} = 1 - \frac{(2a+1)m-1}{2(2am-1)},$$

$$a_1 = 1 + \left\lfloor \frac{1}{1-A_1} \right\rfloor = 1 + \left\lfloor \frac{2(2am-1)}{(2a+1)m-1} \right\rfloor = 1 + \left\lfloor 2 - \frac{2m}{(2a+1)m-1} \right\rfloor = 2,$$

$$A_2 = \frac{a_1}{a_1-1}A_1 = 2\left(1 - \frac{(2a+1)m-1}{2(2am-1)}\right) = 1 - \frac{m}{2am-1},$$

$$a_2 = 1 + \left\lfloor \frac{1}{(1-A_2)\,a_1} \right\rfloor = 1 + \left\lfloor \frac{2am-1}{2m} \right\rfloor = 1 + \left\lfloor a - \frac{1}{2m} \right\rfloor = a,$$

$$A_3 = \frac{a_1a_2}{a_1a_2-1}A_2 = \frac{2a}{2a-1}\left(1 - \frac{m}{2am-1}\right) = 1 - \frac{1}{(2a-1)(2am-1)},$$

$$\begin{aligned} a_3 &= 1 + \left\lfloor \frac{1}{(1-A_3)\,a_1a_2} \right\rfloor = 1 + \left\lfloor \frac{(2a-1)(2am-1)}{2a} \right\rfloor \\ &= 1 + \left\lfloor (2a-1)m - \frac{2a-1}{2a} \right\rfloor = (2a-1)m, \end{aligned}$$

$$\begin{aligned} A_4 &= \frac{a_1a_2a_3}{a_1a_2a_3-1}A_3 = \frac{2a(2a-1)m}{2a(2a-1)m-1}\left(1 - \frac{1}{(2a-1)(2am-1)}\right) \\ &= 1 - \frac{1}{(2am-1)(2a(2a-1)m-1)}, \end{aligned}$$

$$\begin{aligned} a_4 &= 1 + \left\lfloor \frac{1}{(1-A_4)\,a_1a_2a_3} \right\rfloor = 1 + \left\lfloor \frac{(2am-1)(2a(2a-1)m-1)}{2a(2a-1)m} \right\rfloor \\ &= 1 + \left\lfloor 2am - 1 - \frac{2am-1}{2a(2a-1)m} \right\rfloor = 2am-1. \end{aligned}$$

We define a sequence $\{b_n\}$ as follows:

$$(b_1, b_2, b_3, b_4) = (2, a, (2a-1)m, 2am-1)\,, \quad b_{n+2} = b_1 \cdots b_n - 1, \quad n \geq 3.$$

We prove the following assertions for $n \geq 4$ with the use of induction.

(i) $A_n = 1 - \dfrac{1}{b_n b_{n+1}}$ (ii) $a_n = b_n$ (iii) $b_{n+2} = (b_{n+1}+1)\, b_n - 1$.

Firstly, for $n = 4$

(i) $A_4 = 1 - \dfrac{1}{a_4(a_1a_2a_3 - 1)} = 1 - \dfrac{1}{b_4(b_1b_2b_3 - 1)} = \dfrac{1}{b_4b_5}$;

(ii) $a_4 = b_4$ by the definition of b_4;

(iii) $b_6 = (b_1b_2b_3)b_4 - 1 = (b_5+1)b_4 - 1$.

Next we show that the validity of each assertion for $n \geq 4$ implies its validity for $n+1$.

$$\begin{aligned} A_{n+1} &= \frac{a_1 \cdots a_n}{a_1 \cdots a_n - 1} A_n = \left(1 + \frac{1}{b_{n+2}}\right)\left(1 - \frac{1}{b_n b_{n+1}}\right) \\ &= 1 - \frac{b_{n+2} - b_n b_{n+1} + 1}{b_n b_{n+1} b_{n+2}} = 1 - \frac{1}{b_{n+1} b_{n+2}}, \end{aligned}$$

where we have used (iii) in the last step. This proves (i).

$$\begin{aligned} a_{n+1} &= 1 + \left\lfloor \frac{1}{(1 - A_{n+1})\, a_1 \cdots a_n} \right\rfloor = 1 + \left\lfloor \frac{b_{n+1} b_{n+2}}{b_{n+2} + 1} \right\rfloor \\ &= 1 + \left\lfloor b_{n+1} - \frac{b_{n+1}}{b_{n+2} + 1} \right\rfloor = b_{n+1}, \end{aligned}$$

since $b_n \geq 0$ for all n. This proves (ii).

$$b_{n+3} = (b_1 \cdots b_n) b_{n+1} - 1 = (b_{n+2} + 1) b_{n+1} - 1.$$

This establishes (iii), and the proof is complete.

□

Note that, according to the discussion in the paragraph that follows the statement of Theorem 1, a unique expansion is guaranteed in Theorem 2 provided $a \geq 2$.

3 Miscellaneous Results

In this section, and the next, we state our results without proof since all proofs are similar to that shown above. Knopfmacher [2] took 1^a to denote a consecutive occurrences of the digit 1, and we do the same.

Knopfmacher gave the following theorem with $m = 1$.

Theorem 3. *Let m be a positive integer, and let $a \geq 1$ and $b \geq 0$ be integers. Then*

$$\frac{m}{2^b(2^a m + 1)} = \left(\left(2, 1^{a+b-1}, 2^{a-1}m + 1, 2^a m + 1, 2^a m + 1, a_3, a_4, a_5, \ldots\right)\right),$$

where $a_1 = a_2 = 2^a m + 1$, and (2) applies for $n \geq 2$.

Due to the presence of a^2, the following theorem differs in nature from those above.

Theorem 4. *Let a and m be positive integers with $a \geq 2$ and $m \geq 1$. Then*

$$\frac{(a-1)am - 1}{a^2 m} = ((a_1, a_2, a_3, \ldots)),$$

where $a_1 = a$, $a_2 = (a-1)m + 1$, $a_3 = (a-1)m$, $a_4 = a_1 a_2 - 1$, and (2) applies for $n \geq 4$.

In the next three theorems a_1 and a_2 immediately precede a_3, and (2) applies for $n \geq 2$.

Theorem 5. *Let $a \geq 1$ and k be integers. Then the expansion for $\dfrac{m-1}{2^a m}$ is*

$$\left(\left(2, 1^{a-1}, k+1, k, 2k+1, a_3, a_4, a_5, \ldots\right)\right), m = 2k,\ k \geq 1;$$

$$\left(\left(2, 1^{a-1}, k+1, 2k+1, 2k+1, a_3, a_4, a_5, \ldots\right)\right), m = 2k+1,\ k \geq 1;$$

Theorem 6. *Let $a \geq 3$ and k be integers. Then the expansion for $\dfrac{3m-1}{2^a m}$ is*

$$\left(\left(2, 1^{a-3}, 2, 3k+1, 3k, 12k+3, a_3, a_4, a_5, \ldots\right)\right), m = 4k,\ k \geq 1;$$

$$\left(\left(2, 1^{a-3}, 2, 3k+1, 12k+3, 12k+3, a_3, a_4, a_5, \ldots\right)\right), m = 4k+1,\ k \geq 0;$$

$$\left(\left(2, 1^{a-3}, 2, 3k+2, 6k+3, 12k+7, a_3, a_4, a_5, \ldots\right)\right), m = 4k+2,\ k \geq 0;$$

$$\left(\left(2, 1^{a-3}, 2, 3k+3, 4k+3, 12k+11, a_3, a_4, a_5, \ldots\right)\right), m = 4k+3,\ k \geq 0;$$

Theorem 7. *Let $a \geq 5$ and k be integers. Then the expansion for $\dfrac{9m-1}{2^a m}$ is*

$$\left(\left(2, 1^{a-5}, 2, 1, 9k+1, 9k, 36k+3, a_3, a_4, a_5, \ldots\right)\right), m = 4k,\ k \geq 1;$$

$$\left(\left(2, 1^{a-5}, 2, 1, 9k+3, 12k+3, 36k+11, a_3, a_4, a_5, \ldots\right)\right), m = 4k+1,\ k \geq 0;$$

$$\left(\left(2, 1^{a-5}, 2, 1, 9k+5, 18k+9, 36k+19, a_3, a_4, a_5, \ldots\right)\right), m = 4k+2,\ k \geq 0;$$

$$\left(\left(2, 1^{a-5}, 2, 1, 9k+7, 36k+27, 36k+27, a_3, a_4, a_5, \ldots\right)\right), m = 4k+3,\ k \geq 0;$$

4 A Host of Results of a Particular Type

Knopfmacher [2], in Theorem 3(iii), gave an expansion for $\frac{1}{2^a\left(2^n+1\right)}$ when $a \geq 0$ and $n \geq 1$. Inspired by this, we decided to search for triples of positive integers (b, c, d) such that the expansion for $\frac{1}{2^a\left(2^{bn+c}+2d+1\right)}$ is predictable for all integers $a \geq 0$ and $n \geq 0$. We restricted our search to $1 \leq d \leq 50$, first fixing d and then searching for appropriate pairs (b, c). Some d are associated with many pairs (b, c). For example, associated with $d = 48$ are the following eight pairs (b, c): $(48, 24)$, $(48, 27)$, $(240, 121)$, $(336, 172)$, $(336, 173)$, $(528, 270)$, $(528, 303)$, and $(1200, 602)$. On the other hand, for some d we could find no pair (b, c).

In total we found one-hundred and five triples as described in the previous paragraph. For seventy-nine of these triples (b, c, d) we found that the expansion has the form

$$\frac{1}{2^a\left(2^{bn+c}+2d+1\right)} = \left(\left(2, 1^{a+bn+c-1}, a_1, a_2, a_3, \ldots\right)\right), \tag{3}$$

in which

$$\begin{aligned} a_1 &= \left(2^{bn+c-1}+2d+1+e\right)/\left(2d+1\right), \\ a_2 &= \left(2^{bn+c}+2d+1\right)/\left(2d+1+2e\right), \\ a_3 &= 2a_1 - 1, \end{aligned}$$

and (2) applies for $n \geq 3$. Here e is a relatively small positive integer that depends upon the triple (b, c, d), and can be found by observation.

In twenty-four of our triples, $b = 2c$. In these cases it is convenient to write the triples as $(2b, b, d)$. In all such cases we found that

$$\frac{1}{2^a\left(2^{2bn+b}+2d+1\right)} = \left(\left(2, 1^{a+2bn+b-1}, a_1, a_2, a_3, a_4, \ldots\right)\right), \tag{4}$$

in which

$$\begin{aligned} a_1 &= \left(2^{2bn+b-1}+d+1\right)/\left(2d+1\right), \\ a_2 &= 2^{2bn+b}+1, \\ a_3 &= a_2 + 2d, \\ a_4 &= 2a_1a_2 - 1, \end{aligned}$$

and (2) applies for $n \geq 4$.

The reader can now write down the expansion for $\dfrac{1}{2^a\left(2^{2bn+b}+2d+1\right)}$ corresponding to the first of the eight triples mentioned above. For this triple (4) applies. For the remaining seven triples (3) applies, and the values of e are, respectively, 4, 1, 8, 16, 32, 9, and 2. The reader can now use (3) to write down the associated expansions.

Interestingly, we found two triples not of the form $(2b, b, d)$, namely $(16, 4, 8)$ and $(16, 12, 8)$, but where the associated expansions are found with the use of (4). For these two triples we use (4) and replace each occurrence of $2b$ by 16.

Some further examples of triples $(2b, b, d)$ where the associated expansions are found with the use of (4) are $(2, 1, 1)$, $(4, 2, 2)$, $(6, 3, 4)$, $(10, 5, 5)$, $(12, 6, 6)$, and $(18, 9, 9)$. Further examples of triples (b, c, d) where the associated expansions are found with the use of (3) are $(12, 7, 2)$, $(12, 8, 2)$, $(12, 9, 6)$, $(24, 14, 8)$, $(40, 23, 8)$, and $(40, 24, 8)$. Here, for these six triples, the corresponding values of e are, respectively, 1, 2, 4, 2, 4, and 8.

References

[1] A. Knopfmacher and J. Knopfmacher, A new infinite product representation for real numbers, *Monatshefte Math.* **104** (1987), 29–44.

[2] A. Knopfmacher, Rational numbers with predictable Engel product expansions, in G. E. Bergum, A. N. Philippou, and A. F. Horadam, eds., *Applications of Fibonacci Numbers* **5**, Kluwer Academic Publishers, 1993, pp. 421–427.

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