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On the Euler Function of Fibonacci Numbers

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Abstract

In this paper, we show that for any positive integer k, the set

$$\left\{\left(\frac{\phi(F_{n+1})}{\phi(F_n)}, \frac{\phi(F_{n+2})}{\phi(F_n)}, \ldots, \frac{\phi(F_{n+k})}{\phi(F_n)}\right) : n \geq 1\right\}$$

is dense in $\mathbb{R}^k_{\geq 0}$, where $\phi(m)$ is the Euler function of the positive integer m and F_n is the nth Fibonacci number.

1 Introduction

Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence given by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. Recently, there has been some interest in investigating values of classical arithmetic functions with Fibonacci numbers. For a positive integer m, we put $\phi(m)$ for the Euler function of m which counts the number of positive integers less than or equal to m and coprime to m, and $\sigma(m)$ for the sum of divisors of m.

Luca [6] proved that there is no perfect Fibonacci number; that is, there is no positive integer n such that $\sigma(F_n) = 2F_n$. Luca and Nicolae [10] proved that if $\phi(F_n) = F_m$, then $n \in \{1, 2, 3, 4\}$. Luca [9] proved that if $\phi(F_n) \mid F_n - 1$, then F_n is prime, confirming in this way that a well-known conjecture of Lehmer holds for the Fibonacci numbers.

It follows from the results from Luca and Shparlinski [12] that the asymptotic

$$\frac{1}{x}\sum_{n\leq x}\left(\frac{\phi(F_n)}{F_n}\right)^k = \Gamma_k + O_k\left(\frac{(\log x)^k}{x}\right)$$

holds for all positive integers k as $x \to \infty$ with some positive constant Γ_k. In particular, the function $\phi(F_n)/F_n$ has a distribution function. That is, for every real number z, the asymptotic density of the set of positive integers n such that $\phi(F_n)/F_n < z$ exists. Luca [7] proved that if we put $M_n = 2^n - 1$, then the set $\{\phi(M_n)/M_n : n \geq 1\}$ is dense in $[0, 1]$. In the same paper, it is said that the same property holds true if one replaces the sequence of Mersenne numbers $(M_n)_{n\geq 0}$ by the sequence of Fibonacci numbers $(F_n)_{n\geq 0}$.

In this paper, we revisit values of the Euler function with Fibonacci numbers and prove the following result.

Theorem 1. *For every positive integer k, the set*

$$\left\{\left(\frac{\phi(F_{n+1})}{\phi(F_n)}, \frac{\phi(F_{n+2})}{\phi(F_n)}, \ldots, \frac{\phi(F_{n+k})}{\phi(F_n)}\right) : n \geq 1\right\} \tag{1}$$

is dense in $\mathbb{R}^k_{\geq 0}$.

Theorem 1 remains true when the Euler function ϕ is replaced by the sum of divisors function σ. It also remains true if the sequence of Fibonacci numbers $(F_n)_{n\geq 0}$ is replaced by other Lucas sequences such as the sequence of Mersenne numbers $(M_n)_{n\geq 0}$. If the sequence of Fibonacci numbers $(F_n)_{n\geq 0}$ is replaced by the sequence of all natural numbers, then the statement that for every positive integer k, the set

$$\left\{\left(\frac{\phi(n+1)}{\phi(n)}, \frac{\phi(n+2)}{\phi(n)}, \ldots, \frac{\phi(n+k)}{\phi(n)}\right) : n \geq 1\right\}$$

is dense in $\mathbb{R}^k_{\geq 0}$ is a result of Schinzel of 1955 [13]. More general forms of this result appear in a paper by Erdős and Schinzel of 1960 [3], and in the recent work of Wong [14].

We point out that the similar looking statement that the set

$$\left\{\left(\frac{\gamma(n+1)}{\gamma(n)}, \ldots, \frac{\gamma(n+k)}{\gamma(n)}\right) : n \geq 1\right\}$$

is dense in $\mathbb{R}^k_{\geq 0}$, where $\gamma(m) = \prod_{p|m} p$ is the square-free kernel of m, or the product of all the distinct primes dividing m, was proved by Luca and Shparlinski [11].

We have the following immediate corollary of Theorem 1.

Corollary 1. *For any positive integer k and every permutation $(i_1, \ldots, i_k)$ of $(1, \ldots, k)$ there exist infinitely many positive integers n such that*

$$\phi(F_{n+i_1}) < \phi(F_{n+i_2}) < \cdots < \phi(F_{n+i_k}). \tag{2}$$

P. Erdős, K. Győry and Z. Papp [2] call two functions $f(n)$ and $g(n)$ defined on the set of positive integers to be *independent* if for every $k \geq 1$ and permutations $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ of $(1, \ldots, k)$, there exist infinitely many n such that

$$f(n + i_1) < f(n + i_2) < \cdots < f(n + i_k),$$

while

$$g(n + j_1) < g(n + j_2) < \cdots < g(n + j_k).$$

N. Doyon and F. Luca [1] proved that the Euler function $\phi(n)$ and the Carmichael function $\lambda(n)$ (also known as the universal exponent modulo n) are independent, while Hernane and Luca [5] proved that the two functions $\sigma(\phi(n))$ and $\phi(\sigma(n))$ are independent. In light of (2), it makes sense to ask if we can pair the function $\phi(F_n)$ independently with some other interesting multiplicative functions. Here are some questions which are left for the reader.

Problem 1.

(i) Are the functions $\phi(F_n)$ and $F_{\phi(n)}$ independent?

(ii) Are the functions $\phi(F_n)$ and $\phi(M_n)$ independent?

2 Notation

We put $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. We let $(L_n)_{n\geq 1}$ be the Lucas companion of the Fibonacci sequence given by $L_1 = 1$, $L_2 = 3$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 1$. It is well-known that F_n and L_n can be expressed in terms of α and β as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad L_n = \alpha^n + \beta^n \qquad \text{for } n = 1, 2, \ldots. \tag{3}$$

It is also well-known that F_n and L_n satisfy various relations such as

$$F_{2n} = F_n L_n \qquad \text{and} \qquad L_n^2 - 5F_n^2 = 4(-1)^n, \tag{4}$$

which will be used in the sequel.

Throughout this paper, we use the letters p and q with or without subscripts for prime numbers and ℓ, m and n for positive integers. We write $\omega(n)$ for the number of distinct prime factors of the positive integer n and $p(n)$ for the smallest prime factor of n. We write $z(n)$ for the order of apparition of n in the Fibonacci sequence, which is the smallest positive integer k such that $n \mid F_k$. It is well-known that $z(n)$ exists and that $n \mid F_m$ if and only if $z(n) \mid m$. Furthermore, if p is an odd prime, then $p \mid F_{p-(5|p)}$, where we write $(a|p)$ for the Legendre symbol of a with respect to p.

We write $\log x$ for the natural logarithm, and for an integer $k \geq 1$ we write $\log_k x$ for the recursively defined function given by $\log_1 x = \max\{1, \log x\}$ and $\log_k x = \max\{1, \log(\log_{k-1} x)\}$ for $k \geq 2$. When $k = 1$ we omit the index of the logarithm. Hence, all logarithms which will appear are ≥ 1. We use the Vinogradov symbols $\ll$, $\gg$ and the Landau symbols O and o with their usual meanings. Recall that the statements $A \ll B$, $B \gg A$ and $A = O(B)$ are all equivalent to the fact that the inequality $|A| < cB$ holds with some positive constant c, while $A = o(B)$ means that $A/B \to 0$. The constants implied by these symbols may be absolute or may depend on some other parameters such as k, or a fixed point $\gamma = (\gamma_1, \ldots, \gamma_k) \in \mathbb{R}^k_{>0}$, etc. We use x_0 for a large positive real number. The meaning of "large" might change from a line to the next.

3 Preliminary Results

Lemma 2. *Let $p \equiv 13 \pmod{20}$ be a prime. Then $p \mid F_{(p+1)/2}$.*

Proof. This should be well-known but we supply a quick proof of it for completeness. Since $p \equiv 3 \pmod 5$, it follows that $(5|p) = -1$, therefore $p \mid F_{p+1} = F_{(p+1)/2} L_{(p+1)/2}$. Suppose for a contradiction that $p \nmid F_{(p+1)/2}$. Then $p \mid L_{(p+1)/2}$. Reducing the relation

$$L_{(p+1)/2}^2 - 5F_{(p+1)/2}^2 = 4(-1)^{(p+1)/2} = -4$$

(see (4)) modulo p, we get $5F_{(p+1)/2}^2 \equiv 4 \pmod p$, leading to the conclusion that $(5|p) = 1$, which is a contradiction. ☐

For every positive integer m let $\mathcal{P}_m = \{p : z(p) = m\}$. The following lemma turns out to be useful.

Lemma 3. *We have the estimate*

$$\sum_{p \in \mathcal{P}_m} \frac{1}{p} \ll \frac{\log m}{m}. \tag{5}$$

Proof. We may assume that $m > 5$. Since $p \mid F_{p\pm1}$, we get that $p \equiv \pm 1 \pmod m$ for all $p \in \mathcal{P}_m$. Thus,

$$\alpha^m > F_m \geq \prod_{p \in \mathcal{P}_m} p \geq (m-1)^{\#\mathcal{P}_m},$$

leading to $\#\mathcal{P}_m \ll m/\log m$.

We split the desired sum at m^2. We have

$$\sum_{p \in \mathcal{P}_m} \frac{1}{p} \leq \sum_{\substack{p \in \mathcal{P}_m \\ p < m^2}} \frac{1}{p} + \sum_{\substack{p \in \mathcal{P}_m \\ p > m^2}} \frac{1}{p} := S_1 + S_2. \tag{6}$$

Observe that if $p \in \mathcal{P}_m \cap (1, m^2)$, then $p = \pm 1 + m\ell$ for some integer $\ell \in [1, m]$. Thus

$$S_1 \le \sum_{1 \le \ell \le m} \left(\frac{1}{-1 + m\ell} + \frac{1}{1 + m\ell} \right) \ll \frac{1}{m} \sum_{1 \le \ell \le m} \frac{1}{\ell} \ll \frac{\log m}{m}, \tag{7}$$

As for S_2, we have that

$$S_2 \le \frac{\#\mathcal{P}_m}{m^2} \ll \frac{1}{m \log m}. \tag{8}$$

The desired conclusion now follows at once from estimates (6), (7) and (8). ☐

Lemma 4. *We have the estimate*

$$\sum_{d|m} \frac{\log d}{d} \ll (\log_2 m)^2. \tag{9}$$

Proof. This is Lemma 2 in Luca [7]. ☐

Lemma 5. *Let $\mathcal{P}$ be the set of all primes p which fulfill the following conditions:*

(i) $p \equiv 13 \pmod{20}$*;*

(ii) $p((p+1)/2) > \log_2 p$*; (Here, we recall that $p(m)$ stands for the smallest prime factor of m.)*

(iii) $p + 1$ *is square-free.*

Let $\mathcal{P}(t) = \mathcal{P} \cap (1, t)$. Then the estimate

$$\#\mathcal{P}(t) \gg \frac{t}{\log t \log_3 t}$$

holds.

Proof. This follows by typographic changes from the proof of Lemma 1 in Luca [7]. ☐

4 $\phi(F_n)/F_n$ is dense in $[0, 1]$ revisited

As we have mentioned in the Introduction, the fact that $\{\phi(F_n)/F_n : n \ge 1\}$ is dense in $[0, 1]$ alluded to in the title of this section was already proved in Luca [7]. Here, we revisit that proof especially since that proof yields a bit more. We follow the arguments from Luca [7] although we slightly change some of the parameters.

We let x be a large positive real number, put $y = \log_2 x$, $z = \log x$, and $\mathcal{P}(y, z) = \mathcal{P} \cap (y, z)$. We let

$$S = \sum_{p \in \mathcal{P}(y,z)} \frac{1}{p}.$$

We first show that

$$S > 2 \log_4 x \tag{10}$$

holds for $x > x_0$. Indeed, observe that, by Abel's summation formula, we have

$$S = \sum_{\substack{p\in\mathcal{P}\\ y<p<z}} \frac{1}{p} = \int_y^z \frac{d(\#\mathcal{P}(t))}{t} = \frac{\#\mathcal{P}(t)}{t}\Big|_{t=y}^{t=z} + \int_y^z \frac{\#\mathcal{P}(t)}{t^2}.$$

Clearly,

$$\Big|\frac{\#\mathcal{P}(t)}{t}\Big|_{t=y}^{t=z}\Big| \le \frac{\pi(y)}{y} = O\left(\frac{1}{\log_3 x}\right).$$

For large x, by Lemma 5, we have that

$$\#\mathcal{P}(t) \ge \frac{3t}{\log t \log_2 t}.$$

Thus, we get that

$$\begin{aligned} S &\ge 3\int_y^z \frac{dt}{t\log t\log_2 t} + O\left(\frac{1}{\log_3 x}\right) = 3(\log_3 t)|_{t=y}^{t=z} + o(1) \\ &= 3\log_4 x - 3\log_5 x + o(1) > 2\log_4 x \end{aligned}$$

for $x > x_0$.

Next let

$$N = \mathrm{lcm}[(p+1)/2 : p \in \mathcal{P}(y,z)].$$

Observe that N is square-free, so we can write it as $N = q_1 q_2 \cdots q_T$, where $q_1 < q_2 < \cdots < q_T$. Since $q_T < (z+1)/2$, we have, by the Prime Number Theorem, that $N \le \prod_{q<(z+1)/2} q < \exp((0.5+o(1))z)$ as $x \to \infty$. In particular, $N < x$ for $x > x_0$.

We next put $n_0 = 1$ and $n_j = \prod_{i\le j} q_i$ for $j = 1,\ldots,T$. Let

$$s_j = \frac{\phi(F_{n_j})}{F_{n_j}}.$$

Since $n_0 \mid n_1 \mid n_2 \mid \cdots \mid n_T$, we have that

$$s_0 \ge s_1 \ge \cdots \ge s_T.$$

Obviously, $s_0 = 1$. We now look at s_T. Observe that, by Lemma 2 and the fact that $F_a \mid F_b$ whenever $a \mid b$, we get that

$$p \mid F_{(p+1)/2} \mid F_N = F_{n_T} \qquad \text{for all } p \in \mathcal{P}(y,z).$$

Thus,

$$\begin{aligned} s_T &= \frac{\phi(F_{n_T})}{F_{n_T}} = \prod_{p\mid F_{n_T}}\left(1-\frac{1}{p}\right) \le \prod_{p\in\mathcal{P}(y,z)}\left(1-\frac{1}{p}\right) \\ &= \exp\left(-\sum_{p\in\mathcal{P}(y,z)}\frac{1}{p} + O\left(\sum_p \frac{1}{p^2}\right)\right) \\ &= \exp(-S+O(1)) < \exp(-2\log_4 x + O(1)) < \frac{1}{\log_3 x} \end{aligned}$$

for $x > x_0$, where the last inequality follows from (10).

Next we show that

$$\frac{s_{j+1}}{s_j} = 1 + O\left(\frac{(\log_5 x)^2}{\log_4 x}\right) \qquad \text{holds for all } j = 0, 1, \ldots, T-1. \tag{11}$$

First of all notice that every prime factor p of F_{n_j} for some $j \geq 1$ has the property that $z(p) \mid n_j$. In particular, $z(p) > \log_2 y = \log_3 z$. Since also $p \equiv \pm 1 \pmod{z(p)}$, we get that $p \geq (\log_3 z) - 1$.

Thus,

$$\begin{aligned}
\frac{s_{j+1}}{s_j} &= \prod_{\substack{p \mid F_{n_{j+1}} \\ p \nmid F_{n_j}}} \left(1 - \frac{1}{p}\right) = \prod_{\substack{z(p) \mid n_{j+1} \\ z(p) \nmid n_j}} \left(1 - \frac{1}{p}\right) = \prod_{d \mid n_j} \prod_{p \in \mathcal{P}_{dq_{j+1}}} \left(1 - \frac{1}{p}\right) \\
&= \exp\left(-\sum_{d \mid n_j} \sum_{p \in \mathcal{P}_{dq_{j+1}}} \frac{1}{p} + O\left(\sum_{p \geq (\log_3 z) - 1} \frac{1}{p^2}\right)\right) \\
&= \exp\left(O\left(\sum_{d \mid n_j} \frac{\log(dq_{j+1})}{dq_{j+1}} + \frac{1}{\log_3 z}\right)\right),
\end{aligned} \tag{12}$$

where in the last estimate above we used Lemma 3. Now note that

$$\begin{aligned}
\sum_{d \mid n_j} \frac{\log(dq_{j+1})}{dq_{j+1}} &= \frac{\log q_{j+1}}{q_{j+1}} \sum_{d \mid n_j} \frac{1}{d} + \frac{1}{q_{j+1}} \sum_{d \mid n_j} \frac{\log d}{d} \\
&\ll \frac{1}{q_{j+1}} \left(\log q_{j+1} \log_2 n_j + (\log_2 n_j)^2\right),
\end{aligned} \tag{13}$$

where here we used Lemma 4 together with the well-known estimate

$$\sum_{d \mid m} \frac{1}{d} = \frac{\sigma(m)}{m} \ll \log_2 m$$

with $m = n_j$. Since the primes $q_1, \ldots, q_{j+1}$ are arranged increasingly, we have that

$$n_j \leq \prod_{q < q_{j+1}} q < \exp(2q_{j+1})$$

for $x > x_0$, therefore $\log_2 n_j \ll \log q_{j+1}$. This last inequality together with inequality (13) yields

$$\sum_{d \mid n_j} \frac{\log(dq_{j+1})}{dq_{j+1}} \ll \frac{(\log q_{j+1})^2}{q_{j+1}} \ll \frac{(\log_3 y)^2}{\log_2 y} = \frac{(\log_4 z)^2}{\log_3 z}.$$

Inserting the above inequality into inequality (12), we get that

$$\frac{s_{j+1}}{s_j} = \exp\left(O\left(\frac{(\log_4 z)^2}{\log_3 z}\right)\right) = 1 + O\left(\frac{(\log_4 z)^2}{\log_3 z}\right) = 1 + O\left(\frac{(\log_5 x)^2}{\log_4 x}\right),$$

which is precisely inequality (11).

Now it is easy to see that $\{\phi(F_n)/F_n : n \geq 1\}$ is dense in $[0,1]$ in the following way. Let $\gamma \in (0,1)$ and $\varepsilon > 0$ arbitrary. Let $j \in \{0, \dots, T\}$ be maximal such that $s_j > \gamma$. Since $s_T = o(1)$ for large x, it follows that $j \neq T$. Let j be maximal with this property. Then $s_{j+1} \leq \gamma$. However,

$$\gamma \geq s_{j+1} = s_j \left(1 + O\left(\frac{(\log_5 x)^2}{\log_4 x}\right)\right) > s_j - \frac{c(\log_5 x)^2}{\log_4 x},$$

where $c > 0$ is some absolute constant. This shows that

$$\frac{\phi(F_{n_j})}{F_{n_j}} = s_j \in \left(\gamma, \gamma + c\frac{(\log_5 x)^2}{\log_4 x}\right).$$

Choosing x large enough such that

$$\frac{c(\log_5 x)^2}{\log_4 x} < \varepsilon,$$

we get that $\phi(F_{n_j})/F_{n_j} \in (\gamma, \gamma + \varepsilon)$.

This certainly implies that the sequence $(\phi(F_n)/F_n)_{n \geq 1}$ is dense in $(0,1)$.

However, notice that we have proved the following stronger statement.

Lemma 6. *Let $\gamma \in (0,1)$. Then for $x > x_0$, there exists a square-free $n < x$ such that $p(n) > \log_2 x$ and such that furthermore*

$$\frac{\phi(F_n)}{F_n} \in \left(\gamma, \gamma + \frac{(\log_5 x)^3}{\log_4 x}\right).$$

Now let $i \geq 1$ be fixed and write $\alpha_i = \alpha^i$ and $\beta_i = \beta^i$. Look at the sequence

$$F_n^{(i)} = \frac{\alpha_i^n - \beta_i^n}{\alpha_i - \beta_i} \qquad \text{for all } n = 1, 2, \dots.$$

Observe that

$$F_n^{(i)} = \frac{\alpha^{in} - \beta^{in}}{\alpha^i - \beta^i} = \frac{F_{ni}}{F_i}.$$

The sequence $(F_n^{(i)})_{n \geq 1}$ shares the same divisibility properties of the sequence of Fibonacci numbers. We record two of them.

Lemma 7. *Let $i \geq 1$ be fixed.*

(i) If $a \mid b$, then $F_a^{(i)} \mid F_b^{(i)}$.

(ii) If p is a prime dividing both F_i and $F_a^{(i)}$, then p divides a.

(ii) If $p \equiv 13 \pmod{20}$ and i and $(p+1)/2$ are coprime, then $p \mid F_{(p+1)/2}^{(i)}$.

Proof. The first one is obvious and the second one is well-known and easy to prove. To see the third one, note that since $p \equiv 13 \pmod{20}$, it follows, by Lemma 2, that $p \mid F_{(p+1)/2} \mid F_{i(p+1)/2}$. Since i and $(p+1)/2$ are coprime and $z(p) \mid (p+1)/2$, it follows, in particular, that $z(p)$ does not divide i. Thus, F_i is not a multiple of p, so

$$p \mid \frac{F_{i(p+1)/2}}{F_i} = F^{(i)}_{(p+1)/2}.$$

$\square$

Now the arguments that proved Lemma 6 also prove the following result.

Lemma 8. *Let $i \geq 1$ be fixed and $\gamma \in (0,1)$. There exists $x_0(i)$ depending on i such that for $x > x_0(i)$ there exists a square-free $n < x$ with $p(n) > \log_2 x$ and*

$$\frac{\phi(F_n^{(i)})}{F_n^{(i)}} \in \left(\gamma, \gamma + \frac{(\log_5 x)^3}{\log_4 x}\right).$$

5 The Proof of Theorem 1

We may assume that $k \geq 5$. In particular, $F_K \geq K$ for all $K \geq k$.

We write $f(n) = \phi(n)/n$. First we make some reductions. Observe that

$$\frac{\phi(F_{n+i})}{\phi(F_n)} = \frac{f(F_{n+i})}{f(F_n)} \cdot \frac{F_{n+i}}{F_n},$$

and

$$\frac{F_{n+i}}{F_n} = \alpha^i(1+o(1)) \qquad \text{as } n \to \infty.$$

Thus, in order to prove that the set of vectors shown in the statement of Theorem 1 is dense in $\mathbb{R}^k_{\geq 0}$, it suffices to show that the set of vectors

$$\left\{\left(\frac{f(F_{n+1})}{f(F_n)}, \frac{f(F_{n+2})}{f(F_n)}, \cdots, \frac{f(F_{n+k})}{f(F_n)}\right) : n \geq 1\right\} \tag{14}$$

is dense in $\mathbb{R}^k_{\geq 0}$.

Next let $K = ((k+1)!)^2$ and choose $n = Kn_0$ with some positive integer n_0. Then

$$n + i = Kn_0 + i = i\left(\frac{K}{i}n_0 + 1\right) := in_i \qquad \text{for all } i = 1, \ldots, k.$$

Furthermore, note that the numbers n_i are coprime with all primes $p \leq k+1$ for $i = 1, \ldots, k$. That is, $p(n_i) > k+1$ for $i = 1, \ldots, k$. Assume in fact more, namely that $p(n_i) > F_k$ for all $i = 1, \ldots, k$. Then

$$F_{n+i} = F_{in_i} = F_i\left(\frac{F_{in_i}}{F_i}\right) = F_i F^{(i)}_{n_i}$$

for all $i = 1, \ldots, k$. By (ii) of Lemma 7, the prime factors of $\gcd(F_i, F_{n_i}^{(i)})$ divide n_i. Since $p(n_i) > F_k$ for all $i = 1, \ldots, k$, it follows that F_i and $F_{n_i}^{(i)}$ are coprime. Thus, by the multiplicativity of $f(n)$, we get that

$$f(F_{n+i}) = f(F_i)f(F_{n_i}^{(i)}) \qquad \text{for all } i = 1, \ldots, k.$$

Similarly, assuming that $p(n_0) > F_K$, we get that $F_n = F_K F_{n_0}^{(K)}$ and

$$f(F_n) = f(F_K)f(F_{n_0}^{(K)}).$$

With these conventions, the set shown at (14) will be dense in $\mathbb{R}_{\geq 0}^k$ provided that the set of vectors

$$\left\{ \left(\frac{f(F_{n_1})}{f(F_{n_0}^{(K)})}, \frac{f(F_{n_2}^{(2)})}{f(F_{n_0}^{(K)})}, \ldots, \frac{f(F_{n_k}^{(k)})}{f(F_{n_0}^{(K)})} \right) \ : \ p(n_0 \cdots n_k) > F_K \right\} \tag{15}$$

is dense in $\mathbb{R}_{\geq 0}^k$.

Now let $\gamma = (\gamma_1, \ldots, \gamma_k) \in \mathbf{R}_{>0}^k$. Write $\gamma = (\delta_1/\delta_0, \ldots, \delta_k/\delta_0)$, where $\delta_i \in (0,1)$ for $i = 1, \ldots, k$. It suffices to take $\delta_0 = 1/(2\Gamma)$, where $\Gamma = \max\{1, \gamma_1, \ldots, \gamma_k\}$ and $\delta_i = \gamma_i \delta_0$ for $i = 1, \ldots, k$. In order to prove that the set shown at (15) is dense in $\mathbb{R}_{>0}^k$, it suffices to exhibit an infinite sequence $\mathcal{N}$ of n's such that

$$\big(f(F_{n_0}^{(K)}), f(F_{n_1}), f(F_{n_2}^{(2)}), \ldots, f(F_{n_k}^{(k)})\big) \to (\delta_0, \delta_1, \ldots, \delta_k)$$

as $n \to \infty$ in the sequence $\mathcal{N}$.

We shall now explain how to do this. We next fix some arbitrary $\varepsilon > 0$, and recursively choose positive square-free integers $m_0, m_1 \ldots, m_k$ satisfying the following properties:

(i) $p(m_0 \cdots m_k) > F_K$;

(ii)
$$f(F_{m_0}^{(K)}) \in (\delta_0, \delta_0 + \varepsilon), \quad f(F_{m_1}) \in (\delta_1, \delta_1 + \varepsilon), \quad \ldots \quad f(F_{m_k}^{(k)}) \in (\delta_k, \delta_k + \varepsilon);$$

(iii) $\gcd(m_i, m_j) = 1$ for all $0 \leq i < j \leq k$.

To do this, we proceed as follows. We take a large $x > x_0(K)$, such that both inequalities $\log_2 x > F_K$ and $(\log_5 x)^3/(\log_4 x) < \varepsilon$ hold, and choose a square-free positive integer $m_0 < x$ as in Lemma 8 with $i = K$ and $\gamma = \delta_0$. Such m_0 will satisfy $p(m_0) > \log_2 x$ and

$$f(F_{m_0}^{(K)}) \in \left(\delta_0, \delta_0 + \frac{(\log_5 x)^3}{\log_4 x} \right) \subset (\delta_0, \delta_0 + \varepsilon).$$

After having chosen m_0, we fix it and take a new x which is larger and such that $x > x_0(1)$ and $\log_2 x > \max\{F_K, m_0\}$. Applying now Lemma 8 with $i = 1$ and $\gamma = \delta_1$, we get some square-free positive integer $m_1 < x$ with $p(m_1) > \log_2 x > \max\{F_K, m_0\}$ such that

$$f(F_{m_1}^{(1)}) \in (\delta_1, \delta_1 + \varepsilon).$$

Note that m_1 is coprime to m_0 because $p(m_1) > m_0$. Now we take a new x, perhaps larger, such that $x > x_0(2)$ and $\log_2 x > \max\{F_K, m_0, m_1\}$, and apply Lemma 8 with $i = 2$

and $\delta = \delta_2$ to get some square-free positive integer $m_2 < x$ such that $p(m_2) > \log_2 x > \max\{F_K, m_0, m_1\}$ and

$$f(F_{m_2}^{(2)}) \in (\delta_2, \delta_2 + \varepsilon).$$

Observe that m_2 is coprime to both m_0 and m_1 since $p(m_2) > \max\{m_0, m_1\}$. Proceeding in this way k times, we end up with some square-free positive integers $m_0, \ldots, m_k$ satisfying the above three properties.

We now fix these numbers $m_0, \ldots, m_k$. Throughout the remaining of this section, the constants implied by the symbols $\gg$, $\ll$ and O will depend on k, γ and $m_0, \ldots, m_k$ (but not on ε).

We start by searching for n_0 satisfying the following congruences

$$n_0 \equiv 0 \pmod{m_0} \quad \text{and} \quad (K/i)n_0 + 1 \equiv 0 \pmod{m_i} \quad \text{for } i = 1, \ldots, k.$$

Since the largest prime factor of K is $\le k + 1 \le K \le F_K < p(m_i)$ for all $i = 1, \ldots, k$, it follows that K/i is invertible modulo m_i. In particular, the congruence $(K/i)n_0 + 1 \equiv 0 \pmod{m_i}$ is solvable for n_0 and puts n_0 into a certain congruence class $N_{0,i} \pmod{m_i}$. Since $m_0, \ldots, m_k$ are pairwise coprime, the above system is solvable by the Chinese Remainder Theorem and puts n_0 into a certain progression $N_0 \pmod{M}$, where $M = m_0 \cdots m_k$. We assume that N_0 is the smallest positive integer in this progression. Write

$$n_0 = N_0 + M\ell.$$

Then

$$\begin{aligned} n_0 &= m_0 \left(\frac{N_0}{m_0} + \left(\frac{M}{m_0} \right) \ell \right); & (16) \\ (K/i)n_0 + 1 &= m_i \left(\frac{(K/i)N_0 + 1}{m_i} + (K/i) \left(\frac{M}{m_i} \right) \ell \right) \quad \text{for } i = 1, \ldots, k. & (17) \end{aligned}$$

Put $A_0 = N_0/m_0$, $B_0 = M/m_0$ and for $i = 1, \ldots, k$ put $A_i = ((K/i)N_0 + 1)/m_i$ and $B_i = (K/i)M/m_i$. It is easy to check that $\gcd(A_i, B_i) = 1$ for all $0, \ldots, k$.

Indeed, say that $p \mid \gcd(A_0, B_0)$ for some prime p. Then $p \mid m_i$ for some $i \in \{1, \ldots, k\}$, so that $p \mid (K/i)n_0+1$. However, we also have that $p \mid n_0$, and so $p \mid 1 = ((K/i)n_0+1)-(K/i)n_0$, which is impossible.

Assume now that $p \mid \gcd(A_i, B_i)$ for some $i \in \{1, \ldots, k\}$. If $p \mid K/i$, then $p \le k + 1$, but such primes cannot divide A_i. Thus, $p \mid M/m_i$, and, in particular, $p \mid m_j$ for some $j \neq i$. But then $p \mid (K/i)n_0 + 1$ and $p \mid (K/j)n_0 + 1$, therefore $p \mid n_0(K/i - K/j) = n_0K(j - i)/ij$. Clearly, $p \nmid n_0$ since $p \mid (K/i)n_0 + 1$. Thus, $p \mid K(i - j)/ij$, and so, in particular, $p \le k + 1$. However, $p(m_j) > F_K \ge K > k + 1$, and we obtained a contradiction.

We next see that

$$\Delta = \prod_{0 \le i < j \le k} (A_iB_j - A_jB_i) \neq 0.$$

Indeed, if $\Delta = 0$, then $A_iB_j = A_jB_i$ holds for some $0 \le i < j \le k$. Thus, $A_i/B_i = A_j/B_j$, and since both fractions are reduced and with positive denominators, we must have $B_i = B_j$. Since all prime factors of m_i are $> K$, this implies $m_i = m_j$, which is impossible.

Thus, we have that

$$n_0 = m_0(A_0 + B_0\ell) \quad \text{and} \quad (K/i)n_0 + 1 = m_i(A_i + B_i\ell) \quad \text{for } i = 1, \ldots, k,$$

and the linear forms $A_i + B_i\ell$ are primitive and distinct for $i = 0, \ldots, k$. Next we show that for each prime p, there is ℓ such that none of the above $k+1$ forms evaluated in ℓ is zero modulo p. Well, if $p \le k+1$, this follows because $A_i + B_i\ell \equiv A_i \not\equiv 0 \pmod p$ for all $i = 0, \ldots, k$, while if $p > k+1$, then since the degree $k+1$ polynomial

$$\prod_{i=0}^{k} (A_i + B_i x)$$

is not the zero polynomial modulo p, there must exists a congruence class $x \equiv \ell \pmod p$ which is not a zero of the above polynomial. Under these conditions, it is known from the Brun sieve (see, for example, Theorem 2.6 on page 85 in H. Halberstam and H. E. Richert [4]), that there exists a constant $c > 0$ depending on k and the numbers $m_0, \ldots, m_k$ such that if $x > x_0$ (depending on $m_0, \ldots, m_k$), then there are $\gg x/(\log x)^{k+1}$ positive integers $\ell < x$ such that

$$p(A_i + B_i\ell) > x^c \qquad \text{for all } i = 0, \ldots, k.$$

In particular, $\omega(A_i + B_i\ell) = O(1)$ holds for all $i = 0, \ldots, k$, once x sufficiently large. We now take such an ℓ. Observe that

$$F_{n_0}^{(K)} = \frac{F_{Kn_0}}{F_K} = \frac{F_{Km_0}}{F_K} \cdot \frac{F_{Km_0(A_0+B_0\ell)}}{F_{Km_0}}.$$

The first factor on the left is $F_{m_0}^{(K)}$. If there is some prime number dividing both F_{Km_0} and $F_{Km_0(A_0+B_0\ell)}/F_{Km_0}$, then this prime must also divide $A_0 + B_0\ell$. This is false for large x since $p(A_0 + B_0\ell) > x^c$ exceeds F_{Km_0}. Thus,

$$f(F_{n_0}^{(K)}) = f(F_{m_0}^{(K)}) f\left(\frac{F_{Km_0(A_0+B_0\ell)}}{F_{Km_0}}\right).$$

Let us look at the primes $p \mid F_{Km_0(A_0+B_0\ell)}/F_{Km_0}$. Such primes have an index of apparition $z(p)$ dividing $Km_0(A_0 + B_0\ell)$ but not Km_0. In particular, $z(p) \ge x^c$, therefore $p \ge x^c - 1$. Now by an argument already used in the proof of Lemma 6 and based on Lemma 3, we have that

$$\begin{aligned} f\left(\frac{F_{Km_0(A_0+B_0\ell)}}{F_{Km_0}}\right) &= \prod_{\substack{z(p) \mid Km_0(A_0+B_0\ell) \\ z(p) \nmid Km_0}} \left(1 - \frac{1}{p}\right) \\ &= \exp\left(O\left(\sum_{\substack{d \mid A_0+B_0\ell \\ d>1}} \sum_{d_1 \mid Km_0} \frac{\log(dd_1)}{dd_1} + \sum_{p > x^c - 1} \frac{1}{p^2}\right)\right). \end{aligned}$$

Now Km_0 has $O(1)$ divisors. Furthermore, since $p(A_0 + B_0\ell) > x^c$, the number $A_0 + B_0\ell$ also has $O(1)$ divisors > 1, the smallest one being $\ge x^c$. Putting all this together, we get that

$$\sum_{\substack{d \mid A_0+B_0\ell \\ d>1}} \sum_{d_1 \mid Km_0} \frac{\log(dd_1)}{dd_1} \ll \frac{\log x}{x^c} = o(1),$$

while also

$$\sum_{p > x^c - 1} \frac{1}{p^2} \ll \frac{1}{x^c} = o(1),$$

both when $x \to \infty$. Thus, we have just showed that

$$f(F_{n_0}^{(K)}) = f(F_{m_0}^{(K)})(1 + o(1)) = \delta_0 + \varepsilon + o(1)$$

as $x \to \infty$. In the same way, one proves that

$$f(F_{n_i}^{(i)}) = f(F_{m_i}^{(i)})(1 + o(1)) = \delta_i + \varepsilon + o(1) \qquad \text{for all } i = 1, \ldots, k,$$

as $x \to \infty$. Thus, choosing a large x, we get an n such that

$$f(F_{n_0}^{(K)}) \in (\delta_0 - 2\varepsilon, \delta_0 + 2\varepsilon) \quad \text{and} \quad f(F_{n_i}^{(i)}) \in (\delta_i - 2\varepsilon, \delta_i + 2\varepsilon) \quad \text{for all } i = 1, \ldots, k.$$

Since $\varepsilon > 0$ was arbitrary, the result follows.

6 Comments and Remarks

As we have already pointed out in the Introduction, typographical changes, such as working with primes $p \equiv 7 \pmod 8$ and with $(p-1)/2$ instead of $(p+1)/2$ show that the conclusion of Theorem 1 remains valid when the Fibonacci numbers F_n are replaced by the Mersenne numbers M_n (see Luca [7]). Quite likely, it also applies to other Lucas sequences of the first kind not only to the Fibonacci and Mersenne numbers. Furthermore, the conclusion of the theorem remains also valid when the Euler function $\phi(n)$ is replaced by the sum of divisors function $\sigma(n)$, or, more generally, by any multiplicative function $g(n)$ such that there exist two constants $c \neq 0$ and $\lambda > 1$ with the property that on prime powers p^a we have $g(p^a) = 1 + c/p + O(1/p^\lambda)$. An example of such a function is the function $\alpha(n)$ which associates to every positive integer n the average order of the elements in the cyclic group of order n. Such function was studied in Luca [8]. On the other hand, it seems very difficult to prove the analogous Theorem 1 when the function $\phi(n)$ is replaced by one of the functions $\omega(n)$, or $\Omega(n)$, which count the number of prime power divisors of n, or $\tau(n)$, which counts the number of divisors of n. We leave such problems as possible research projects for the interested reader.

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