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Equidistribution of Descents, Adjacent Pairs, and Place-Value Pairs on Permutations

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Abstract

An (X,Y)-descent in a permutation is a pair of adjacent elements such that the first element is from X, the second element is from Y, and the first element is greater than the second one. An (X,Y)-adjacency in a permutation is a pair of adjacent elements such that the first one is from X and the second one is from Y. An (X,Y)-place-value pair in a permutation is an element y in position x, such that y is in Y and x is in X. It turns out, that for certain choices of X and Y some of the three statistics above become equidistributed. Moreover, it is easy to derive the distribution formula for (X,Y)-place-value pairs thus providing distribution for other statistics under consideration too. This generalizes some results in the literature. As a result of our considerations, we get combinatorial proofs of several remarkable identities. We also conjecture existence of a bijection between two objects in question preserving a certain statistic.

1 Introduction

Let $\mathcal{S}_n$ denote the set of permutations of $[n]=\{1,\ldots,n\}$ and $\mathbb{N}=\{1,2,\ldots\}$. Also, $\mathbb{E}$ and $\mathbb{O}$ denote the set of even and odd numbers, respectively. For $\sigma=\sigma_1\cdots\sigma_n\in\mathcal{S}_n$ and $X,Y\subseteq\mathbb{N}$ define the following permutation statistics

$$\begin{aligned}
\mathrm{des}_{X,Y}(\sigma) &= |\{i : \sigma_i>\sigma_{i+1},\ \&\ \sigma_i\in X\ \&\ \sigma_{i+1}\in Y\}|,\\
\mathrm{adj}_{X,Y}(\sigma) &= |\{i : \sigma_i\in X\ \&\ \sigma_{i+1}\in Y\}|,\\
\mathrm{val}_{X,Y}(\sigma) &= |\{i : i\in X\ \&\ \sigma_i\in Y\}|,\\
\mathrm{exc}_{X,Y}(\sigma) &= |\{i : \sigma_i>i\ \&\ i\in X\ \&\ \sigma_i\in Y\}|,
\end{aligned}$$

and the following corresponding polynomials

$$\begin{aligned}
D_n^{X,Y}(x) &= \sum_{\sigma\in\mathcal{S}_n} x^{\mathrm{des}_{X,Y}(\sigma)} = \sum_{s=0}^{n-1} D_{n,s}^{X,Y}x^s,\\
A_n^{X,Y}(x) &= \sum_{\sigma\in\mathcal{S}_n} x^{\mathrm{adj}_{X,Y}(\sigma)} = \sum_{s=0}^{n-1} A_{n,s}^{X,Y}x^s,\\
V_n^{X,Y}(x) &= \sum_{\sigma\in\mathcal{S}_n} x^{\mathrm{val}_{X,Y}(\sigma)} = \sum_{s=0}^{n-1} V_{n,s}^{X,Y}x^s,\\
E_n^{X,Y}(x) &= \sum_{\sigma\in\mathcal{S}_n} x^{\mathrm{exc}_{X,Y}(\sigma)} = \sum_{s=0}^{n-1} E_{n,s}^{X,Y}x^s.
\end{aligned}$$

Objects counted by $\mathrm{des}_{X,Y}$ are called (X,Y)-*descents* in [2]. Similarly, we can talk of (X,Y)-*adjacencies*, (X,Y)-*place-value pairs*, and (X,Y)-*excedances*.

Foata's first transformation [1] exchanging excedances and descents (to be used in the paper) can most easily be explained with an example. The permutation $w=61437258$ has three excedances: 6, 4, and 7 in positions 1, 3, and 5, respectively. We write w in cycle form: (162)(34)(57)(8). Next, write each cycle with largest element last, and order the cycles by increasing largest element: (34)(216)(57)(8). Finally, reverse each cycle and erase the parentheses to get the outcome permutation 43612758 with the descents 43, 61, and 75.

Remark 1. Using Foata's first transformation, one obtains that $D_n^{X,Y}(x) = E_n^{Y,X}(x)$. Thus, we do not need to provide any arguments for the polynomial $E_n^{X,Y}(x)$ and its coefficients, instead studying the other three polynomials.

In this paper, we use the following notation for any $X \subseteq \mathbb{N}$ and integer $n \geq 1$:

$$X_n \;=\; [n] \cap X, x_n = |X_n|, X_n^c = [n] - X, \text{ and } x_n^c = |X_n^c|.$$

Collecting some data on the polynomials, we noticed several equidistributions among the statistics, and nice formulas associated with them, for particular choices of sets X and Y. We collect those observations in Table 1, where $a_{n,k}$ denotes the number of permutations in $\mathcal{S}_n$ with k occurrences of the corresponding statistic.

Many of formulas listed in Table 1 are known (see, e.g., [4]). Others are new but quite easy to prove. Our idea to establish the equidistribution results is to prove general recurrence relations for the statistics for arbitrary choice of sets X and Y. Then we will get the equidistributions in Table 1 as a simple corollary to the fact that the recurrences for the statistics in a given block are the same for a particular choice of X and Y. For example, we will show that whenever X and Y are disjoint subsets of $\mathbb{N}$, then $A_{n,s}^{X,Y} = V_{n,s}^{X,Y}$ for all n and s. Indeed, the recursions that we develop will allow us to give a bijective proof of this fact. Other equidistribution results follow from simple bijections. For example, it is easy to see that for any X and Y, $A_{n,s}^{X,Y} = A_{n,s}^{Y,X}$ since if $\sigma_i\sigma_{i+1}$ is an (X,Y)-adjacency in $\sigma = \sigma_1 \cdots \sigma_n$, then $\sigma_{i+1}\sigma_i$ is a (Y,X)-adjacency in the reverse of σ, $\sigma^r = \sigma_n\sigma_{n-1}\cdots\sigma_1$.

Several of our formulas are quite easy to prove for one of our three statistics. For example, it is always easy to compute $V_{n,s}^{X,Y}$.

Theorem 2. *For any $X, Y \subseteq \mathbb{N}$, $n \geq 1$, and $0 \leq s \leq n$,*

$$V_{n,s}^{X,Y} = s!(x_n - s)!(x_n^c)!\binom{x_n}{s}\binom{y_n}{s}\binom{y_n^c}{x_n - s}. \tag{1}$$

Proof. To count the number of permutations of length n with s occurrences of $\mathrm{val}_{X,Y}$, we can first pick s positions from X_n in $\binom{x_n}{s}$ ways for the places where we will have values of Y occurring in the places corresponding to X_n. Then we pick s values from Y_n in $\binom{y_n}{s}$ ways, and permute the values in $s!$ ways to arrange the s occurrences of values in Y_n in the places in X_n. In the remaining $x_n - s$ places in X_n, we must choose values from Y_n^c. We thus have $\binom{y_n^c}{x_n-s}$ ways to choose those values and $(x_n - s)!$ ways to rearrange them. Finally we have $x_n^c!$ ways to arrange the elements in places outside of X_n. $\square$

Similarly, it is easy to count $A_{n,s}^{X,X}$ for any set $X \subseteq \mathbb{N}$. That is, we have the following theorem.

Stat.	**Description, related polynomial, and enumeration**
S_1 S_2 S_3	# of even descent-tops ($D^{\mathbb{E},\mathbb{N}}_{n,k}$). E.g., $S_1(\mathbf{2}15\mathbf{4}36) = 2$. # of even excedance values ($E^{\mathbb{N},\mathbb{E}}_{n,k}$). E.g., $S_2(\mathbf{2}15436) = 1$. # of even entries in even positions ($V^{\mathbb{E},\mathbb{E}}_{n,k}$). E.g., $S_3(215\mathbf{436}) = 2$. $a_{2n,k} = \left[n!\binom{n}{k}\right]^2$; $a_{2n+1,k} = n!(n+1)!\binom{n}{k}\binom{n+1}{k+1}$.
S_4 S_5 S_6 S_7 S_8 S_9	# of odd descent-bottoms ($D^{\mathbb{N},\mathbb{O}}_{n,k}$). E.g., $S_4(2\mathbf{1}54\mathbf{3}6) = 2$. # of odd excedance positions ($E^{\mathbb{O},\mathbb{N}}_{n,k}$). E.g., $S_5(\mathbf{215}436) = 2$. # of even entries in odd positions ($V^{\mathbb{O},\mathbb{E}}_{n,k}$). E.g., $S_6(\mathbf{2}15436) = 1$. # of odd entries in even positions ($V^{\mathbb{E},\mathbb{O}}_{n,k}$). E.g., $S_7(2\mathbf{1}5436) = 1$. # of (odd,even) pairs ($A^{\mathbb{O},\mathbb{E}}_{n,k}$). E.g., $S_8(21\underline{54}\ \underline{36}) = 2$. # of (even, odd) pairs ($A^{\mathbb{E},\mathbb{O}}_{n,k}$). E.g., $S_9(\underline{21}5\underline{43}6) = 2$. $a_{2n,k} = \left[n!\binom{n}{k}\right]^2$; $a_{2n+1,k} = n!(n+1)!\binom{n}{k}\binom{n+1}{k}$.
S_{10} S_{11} S_{12}	# of odd descent-tops ($D^{\mathbb{O},\mathbb{N}}_{n,k}$). E.g., $S_{10}(21\mathbf{5}436) = 1$. # of odd excedance values ($E^{\mathbb{N},\mathbb{O}}_{n,k}$). E.g., $S_{11}(21\mathbf{5}436) = 1$. # of (odd,odd) pairs ($A^{\mathbb{O},\mathbb{O}}_{n,k}$). E.g., $S_{12}(2\underline{15}436) = 1$. $a_{2n,k} = (n!)^2\binom{n-1}{k}\binom{n+1}{k+1}$; $a_{2n+1,k} = n!(n+1)!\binom{n}{k}\binom{n+1}{k}$.
S_{13} S_{14}	# of even descent-bottoms ($D^{\mathbb{N},\mathbb{E}}_{n,k}$). E.g., $S_{13}(215\mathbf{436}) = \mathbf{1}$. # of even excedance positions ($E^{\mathbb{E},\mathbb{N}}_{n,k}$). E.g., $S_{14}(215436) = 0$. $a_{2n,k} = (n!)^2\binom{n-1}{k}\binom{n+1}{k+1}$; $a_{2n+1,k} = n!(n+1)!\binom{n}{k}\binom{n+1}{k+1}$.
S_{15}	# of odd entries in odd positions ($V^{\mathbb{O},\mathbb{O}}_{n,k}$). E.g., $S_{15}(21\mathbf{5}4\mathbf{3}6) = 2$. $a_{2n,k} = \left[n!\binom{n}{k}\right]^2$; $a_{2n+1,k} = n!(n+1)!\binom{n}{k-1}\binom{n+1}{k}$.
S_{16}	# of (even,even) pairs ($A^{\mathbb{E},\mathbb{E}}_{n,k}$). E.g., $S_{16}(215436) = 0$. $a_{2n,k} = (n!)^2\binom{n-1}{k}\binom{n+1}{k+1}$; $a_{2n+1,k} = n!(n+1)!\binom{n-1}{k}\binom{n+2}{k+2}$.

Table 1: 16 statistics under consideration classified into 6 statistic groups.

Theorem 3. *For any $X \subseteq \mathbb{N}$, $n \geq 1$, and $0 \leq s \leq n-1$,*

$$A_{n,s}^{X,X} = (x_n)!(x_n^c)!\binom{x_n - 1}{s}\binom{x_n^c + 1}{x_n - s}. \tag{2}$$

Proof. Fix $n \geq 1$. First we pick a permutation σ of $X \cap [n]$ and a permutation τ of $[n] - X$. Clearly, we have $(x_n)!(x_n^c)!$ ways to pick σ and τ. We are now interested in finding the number of permutations of γ of S_n such that γ restricted to the elements in $X \cap [n]$ yields the permutation σ, γ restricted to the elements in $[n] - X$ yields the permutation τ, and $\mathrm{adj}_{X,X}(\gamma) = s$. Next in $\sigma_1\sigma_2\cdots\sigma_{x_n}$, we think of choosing s spaces from the $x_n - 1$ spaces between the elements of σ to create the adjacencies that will appear in such a γ. For example, if $n = 12$, $s = 2$, $X = \mathbb{E}$, $\sigma = 4\ 2\ 10\ 8\ 6\ 12$, and we pick spaces 2 and 5, then our choice partitions σ into four blocks, 4, $2-10$, 8 and $6-12$. Our idea is to insert these blocks into the spaces that either lie immediately before an element of τ or immediately after the last element of τ. We label these spaces from left to right. For example, suppose $\tau = 5\ 1\ 7\ 9\ 3\ 11$ and we pick spaces 2, 4, 5, and 7. Then we would insert the block 4 immediately before 1, the block $2-10$ immediately before 9, the block 8 immediately before 3, $6-12$ immediately after 11 to obtain the permutation

$$5\ 4\ 1\ 7\ 2\ 10\ 9\ 8\ 3\ 11\ 6\ 12.$$

Clearly there are $\binom{x_n-1}{s}$ ways to choose the spaces to obtain our s adjacencies. This will leave us with $x_n - s$ blocks. Then there are $\binom{x_n^c+1}{x_n-s}$ to choose the spaces for τ where we insert the blocks. $\square$

Hall and Remmel [2] gave direct combinatorial proofs of a pair of formulas for $D_{n,s}^{X,Y}$ which combined with our equidistribution results, gives formulas for the other polynomials under consideration. We state these results here together with an example of using them.

Theorem 4. *For any $X, Y \subseteq \mathbb{N}$, $n \geq 1$, and $0 \leq s \leq n-1$,*

$$D_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{s} (-1)^{s-r} \binom{|X_n^c| + r}{r} \binom{n+1}{s-r} \prod_{x \in X_n} (1 + r + \alpha_{X,n,x} + \beta_{Y,n,x}), \tag{3}$$

Theorem 5. *For any $X, Y \subseteq \mathbb{N}$, $n \geq 1$, and $0 \leq s \leq n-1$,*

$$D_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{|X_n|-s} (-1)^{|X_n|-s-r} \binom{|X_n^c| + r}{r} \binom{n+1}{|X_n| - s - r} \prod_{x \in X_n} (r + \beta_{X,n,x} - \beta_{Y,n,x}), \tag{4}$$

where for any set A and any $j, 1 \leq j \leq n$, we define

$$\begin{aligned} \alpha_{A,n,j} &= |A^c \cap \{j+1, j+2, \ldots, n\}| = |\{x : j < x \leq n \ \&\ x \notin A\}|, \text{ and} \\ \beta_{A,n,j} &= |A^c \cap \{1, 2, \ldots, j-1\}| = |\{x : 1 \leq x < j \ \&\ x \notin A\}|. \end{aligned}$$

Example 1. Suppose $X = \{2, 3, 4, 6, 7, 9\}$, $Y = \{1, 4, 8\}$, and $n = 6$. Thus $X_6 = \{2, 3, 4, 6\}$, $X_6^c = \{1, 5\}$, $Y_6 = \{1, 4\}$, $Y_6^c = \{2, 3, 5, 6\}$, and we have the following table of values of $\alpha_{X,6,x}$, $\beta_{Y,6,x}$, and $\beta_{X,6,x}$.

x	2	3	4	6
$\alpha_{X,6,x}$	1	1	1	0
$\beta_{Y,6,x}$	0	1	2	3
$\beta_{X,6,x}$	1	1	1	2

Equation (3) gives

$$\begin{aligned} D_{6,2}^{X,Y} &= 2!\sum_{r=0}^{2}(-1)^{2-r}\binom{2+r}{r}\binom{7}{2-r}(2+r)(3+r)(4+r)(4+r) \\ &= 2\left(1\cdot 21\cdot 2\cdot 3\cdot 4\cdot 4-3\cdot 7\cdot 3\cdot 4\cdot 5\cdot 5+6\cdot 1\cdot 4\cdot 5\cdot 6\cdot 6\right) \\ &= 2(2016-6300+4320) \\ &= 72, \end{aligned}$$

while (4) gives

$$\begin{aligned} D_{6,2}^{X,Y} &= 2!\sum_{r=0}^{2}(-1)^{2-r}\binom{2+r}{r}\binom{7}{2-r}(1+r)(0+r)(-1+r)(-1+r) \\ &= 2\left(1\cdot 21\cdot 1\cdot 0\cdot (-1)\cdot (-1)-3\cdot 7\cdot 2\cdot 1\cdot 0\cdot 0+6\cdot 1\cdot 3\cdot 2\cdot 1\cdot 1\right) \\ &= 2(0-0+36) \\ &= 72. \end{aligned}$$

The paper is organized as follows. In Section 2 we find general recurrence relations for $D_{n,k}^{X,Y}$, $A_{n,k}^{X,Y}$, and $V_{n,k}^{X,Y}$, and use them to explain the facts in Table 1. In Section 3 we generalize several of the results that appear in Table 1, and use this to obtain combinatorial proofs of several remarkable identities. Finally, in Section 4, we discuss some directions for further research.

2 Recurrence relations for $D_{n,k}^{X,Y}$, $A_{n,k}^{X,Y}$, and $V_{n,k}^{X,Y}$

In this section, we derive recurrence relations for $D_{n,k}^{X,Y}$, $A_{n,k}^{X,Y}$, and $V_{n,k}^{X,Y}$. We notice that the recurrences we get for $A_{n,k}^{X,Y}$ and $V_{n,k}^{X,Y}$ are almost identical, except for the case when the element $n+1 \in X\cap Y$ — the recurrences differ by "1+." However, assuming $X \cap Y = \emptyset$, we do not have this case, leading, in particular, to the explanation of all of the equidistributions in Table 1, and to many more results for other choices of X and Y, $X \cap Y = \emptyset$.

Another thing to observe is that in the case of the same recurrence relations, we naturally get bijective proofs for the corresponding equidistributed statistics. Indeed, one can label positions in a permutation, say from left to right, in which we insert the largest element, $n+1$, or do the other insertion procedure (see Subsection 2.3); then, it is enough to match insertions in the positions having the same labels. However, such straightforward approach is not necessarily the best one, as labeling positions differently, rather than just from left to right, one may preserve extra statistics in bijections (see Section 4 for conjectures, which should be possible to prove using our approach with different labeling).

2.1 Recurrences for $D_{n,k}^{X,Y}$

A recursion for $D_{n,k}^{X,Y}$ is derived in [2]:

$$D_{n+1,k}^{X,Y} = \begin{cases} (k+1)D_{n,k+1}^{X,Y} + (n+1-k)D_{n,k}^{X,Y}, & \text{if } n+1 \notin X; \\ (y_n-(k-1))D_{n,k-1}^{X,Y} + (n+1-(y_n-k))D_{n,k}^{X,Y}, & \text{if } n+1 \in X. \end{cases}$$

An argument for deriving the recursion is as follows. We are thinking of inserting the element $n+1$ in a permutation $\sigma = \sigma_1 \cdots \sigma_n$, and we consider which of the obtained permutations are counted by $D_{n,k}^{X,Y}$. If $n+1 \notin X$ then one never increases the number of (X,Y)-descents by inserting $n+1$. More precisely, the number of (X,Y)-descents is either unchanged, or it is decreased by 1, when $n+1$ is inserted between $\sigma_i \in X$ and $\sigma_{i+1} \in Y$ where $\sigma_i > \sigma_{i+1}$. The corresponding recursion case follows.

For the second case, notice that if $n+1 \in X$, then the number of (X,Y)-descents is unchanged if $n+1$ is inserted at the end of the permutation, in front of $\sigma_j \notin Y$, or between $\sigma_i \in X$ and $\sigma_{i+1} \in Y$ where $\sigma_i > \sigma_{i+1}$, and it is increased by 1 in other cases (that is, when $n+1$ is inserted in front of $\sigma_j \in Y$ not involved in an (X,Y)-descent). The second recursion case follows.

We use a similar approach to derive recurrence relations for $A_{n,k}^{X,Y}$. Our derivations for $V_{n,k}^{X,Y}$ use a different insertion procedure.

2.2 Recurrences for $A_{n,k}^{X,Y}$

We consider 4 cases.

Case 1. $n+1 \notin X \cup Y$. The number of (X,Y)-adjacent pairs is decreased by 1 when $n+1$ is inserted between $\sigma_i \in X$ and $\sigma_{i+1} \in Y$ and it is unchanged otherwise. Thus, in this case

$$A_{n+1,k}^{X,Y} = (k+1)A_{n,k+1}^{X,Y} + (n+1-k)A_{n,k}^{X,Y}.$$

Case 2. $n+1 \in X \cap Y$. Adding $n+1$ after a $\sigma_i \in X$ or before a $\sigma_j \in Y$ increases $\mathrm{adj}_{X,Y}$ by 1, while it keeps $\mathrm{adj}_{X,Y}(\sigma)$ unchanged otherwise. However, we note that the place between $\sigma_i \in X$ and $\sigma_{i+1} \in Y$ is after a $\sigma_i \in X$ *and* before a $\sigma_{i+1} \in Y$. Thus, in this case

$$A_{n+1,k}^{X,Y} = (x_n+y_n-(k-1))A_{n,k-1}^{X,Y} + (n+1-(x_n+y_n-k))A_{n,k}^{X,Y}.$$

Case 3. $n+1 \in X-Y$. Inserting $n+1$ to the left of a $\sigma_i \notin Y$ does not change $\mathrm{adj}_{X,Y}(\sigma)$, which is also the case if $n+1$ is inserted between $\sigma_i \in X$ and $\sigma_{i+1} \in Y$, or $n+1$ is inserted at the very end. On the other hand, if $n+1$ is inserted between $\sigma_i \notin X$ and $\sigma_{i+1} \in Y$, the number of (X,Y)-adjacent pairs is increased by 1. Thus, in this case

$$A_{n+1,k}^{X,Y} = (y_n-(k-1))A_{n,k-1}^{X,Y} + (n+1-(y_n-k))A_{n,k}^{X,Y}.$$

Case 4. $n+1 \in Y-X$. Inserting $n+1$ to the right of a $\sigma_i \notin X$ does not change $\mathrm{adj}_{X,Y}(\sigma)$, which is also the case if $n+1$ is inserted between $\sigma_i \in X$ and $\sigma_{i+1} \in Y$, or $n+1$ is inserted at the very beginning. On the other hand, if $n+1$ is inserted between $\sigma_i \in X$ and $\sigma_{i+1} \notin Y$, the number of (X,Y)-adjacent pairs is increased by 1. Thus, in this case

$$A_{n+1,k}^{X,Y} = (x_n-(k-1))A_{n,k-1}^{X,Y} + (n+1-(x_n-k))A_{n,k}^{X,Y}.$$

2.3 Recurrences for $V_{n,k}^{X,Y}$

Instead of inserting the largest element, $n+1$, in all possible places, we use another insertion procedure $I_n^{(i)}(\sigma)$ that generates $\mathcal{S}_{n+1}$ from $\mathcal{S}_n$. For $\sigma = \sigma_1 \cdots \sigma_n$, let $I_{n+1}^{(n+1)}(\sigma) = \sigma(n+1) = \sigma_1 \cdots \sigma_n(n+1)$, and for $1 \leq i \leq n$, let $I_{n+1}^{(i)}(\sigma) = \sigma_1 \cdots \sigma_{i-1}(n+1)\sigma_{i+1} \cdots \sigma_n \sigma_i$ (that is, in the last case we replace σ_i in σ by $n+1$ and move σ_i to the very end).

We now consider 4 cases.

Case 1. $n+1 \notin X \cup Y$. In this case, one can only decrease the number of (X,Y)-place-value pairs. This happens when $n+1$ occupies position $i \in X$ in $I_{n+1}^{(i)}(\sigma)$ for some σ, such that $\sigma_i \in Y$ (σ_i is in position $n+1$ in $I_{n+1}^{(i)}(\sigma)$). Thus, in this case

$$V_{n+1,k}^{X,Y} = (k+1)V_{n,k+1}^{X,Y} + (n+1-k)V_{n,k}^{X,Y}.$$

Case 2. $n+1 \in X \cap Y$. It is straightforward to see that the number of (X,Y)-place-value pairs is unchanged if $i \notin X$ and $\sigma_i \notin Y$, and it increases by 1 in each of the following three cases: $i \in X$ and $\sigma_i \in Y$, $i \in X$ and $\sigma_i \notin Y$, and $i \notin X$ and $\sigma_i \in Y$. Note that we add 1 for each $i \in X$ and 1 for each $\sigma_i \in Y$, so we count $i \in X$ and $\sigma_i \in Y$ twice. Moreover, having $n+1$ in position $n+1$ gives one more (X,Y)-place-value pair. Thus, in this case

$$V_{n+1,k}^{X,Y} = (1 + x_n + y_n - (k-1))V_{n,k-1}^{X,Y} + (n+1-(1+(x_n+y_n-k)))V_{n,k}^{X,Y}.$$

Case 3. $n+1 \in X - Y$. One can check that in this case, the number of (X,Y)-place-value pairs increases by 1 if $i \notin X$ and $\sigma_i \in Y$, and it is unchanged otherwise. Thus, in this case

$$V_{n+1,k}^{X,Y} = (y_n - (k-1))V_{n,k-1}^{X,Y} + (n+1-(y_n-k))V_{n,k}^{X,Y}.$$

Case 4. $n+1 \in Y - X$. One can check that in this case, the number of (X,Y)-place-value pairs increases by 1 if $i \in X$ and $\sigma_i \notin Y$, and it is unchanged otherwise. Thus, in this case

$$V_{n+1,k}^{X,Y} = (x_n - (k-1))V_{n,k-1}^{X,Y} + (n+1-(x_n-k))V_{n,k}^{X,Y}.$$

There are a number of cases where the recursions for $V_{n,k}^{A,B}$, $A_{n,k}^{C,D}$, and $D_{n,k}^{E,F}$ coincide so that we immediately have equality between the various pairs of statistics. For example, comparing the recursions for $A_{n,k}^{X,Y}$ and $V_{n,k}^{X,Y}$, we immediately have the following theorem.

Theorem 6. *For all $X, Y \subseteq \mathbb{N}$ such that $X \cap Y = \emptyset$, $n \geq 1$, and $0 \leq k \leq n$, $V_{n,k}^{X,Y} = A_{n,k}^{X,Y}$.*

In fact, it is easy to see that our proofs of the recursions can be used to give an inductive proof that there exists a bijection from S_n onto S_n for all n that will witness this equality. That is, our proofs of the recursions immediately allow us to construct inductively bijections $\Theta_n : S_n \to S_n$ for all n such that for all $\sigma \in S_n$,

$$\mathrm{adj}_{X,Y}(\sigma) = \mathrm{val}_{X,Y}(\Theta_n(\sigma)).$$

For example, suppose that we have constructed Θ_n and $n+1 \notin X \cup Y$. First consider our insertion procedure to prove the recursions for $A_{n,s}^{X,Y}$. If $\sigma \in S_n$, then we consider the places where we can insert $n+1$ to σ. We first label the spaces between the elements $\sigma_i \in X$ and $\sigma_{i+1} \in Y$ from left to right with $1, \ldots, \mathrm{adj}_{X,Y}(\sigma)$ and then label the rest of the spaces from left to right with $\mathrm{adj}_{X,Y}(\sigma)+1, \ldots, n+1$. For example, if $X = \mathbb{E}$, $Y = \mathbb{O}$, and $\sigma = 1\ 4\ 3\ 2\ 5$, the spaces would be labeled by

$$\underset{3}{\_}1\underset{4}{\_}4\underset{1}{\_}3\underset{5}{\_}2\underset{2}{\_}5\underset{6}{\_}.$$

We then let $\sigma^{(i)}$ be the permutation that results by inserting $n+1$ into the space labeled i. For example, in our example, $\sigma^{(4)} = 1\ 6\ 4\ 3\ 2\ 5$. Next we consider our insertion procedure for proving the recursions for $V_{n,s}^{X,Y}$. Now if $\tau \in S_n$, then we label the positions of τ by first labeling the positions i such that $i \in X$ and $\tau_i \in Y$ from left to right with $1, \ldots, \mathrm{val}_{X,Y}(\tau)$ and then label the remaining positions from left to right with $\mathrm{val}_{X,Y}(\tau)+1, \ldots, n$. For example, if $X = \mathbb{O}$ and $Y = \mathbb{E}$ and $\tau = 1\ 4\ 2\ 5\ 3$, then we would label the positions

$$\frac{1}{\mathbf{2}}\ \frac{4}{\mathbf{3}}\ \frac{2}{\mathbf{1}}\ \frac{5}{\mathbf{4}}\ \frac{3}{\mathbf{5}}$$

where we have indicated the labels in boldface. If label j is in position i, then we let $\tau^{(j)} = I_n^{(i)}(\tau)$ and we let $\tau^{(n+1)} = I_n^{(n+1)}(\tau)$. For example, in our case, $\tau^{(2)} = 6\ 4\ 2\ 5\ 3\ 1$. Then for any $\sigma \in S_n$ and $i \in \{1, \ldots, n+1\}$, we can define

$$\Theta_{n+1}(\sigma^{(i)}) = \Theta_n(\sigma)^{(i)}.$$

We can extend Θ_n to Θ_{n+1} in the other cases of the recursions in a similar manner.

Similarly, comparing the recursions for the $V_{n,k}^{A,B}$, $A_{n,k}^{C,D}$, and $D_{n,k}^{E,F}$, we can also derive bijective proofs of the following theorems.

Theorem 7. *If X and Y are subsets of $\mathbb{N}$, $A = X \cup Y$ and there exists a $B \subseteq \mathbb{N}$ such that $b_n = |B \cap [n]|$ satisfies*

$$b_n = \begin{cases} x_n + y_n = |X \cap [n]| + |Y \cap [n]|, & \text{if } n+1 \in X \cap Y; \\ y_n = |Y \cap [n]|, & \text{if } n+1 \in X - Y; \\ x_n = |X \cap [n]|, & \text{if } n+1 \in Y - X, \end{cases}$$

then $D_{n,k}^{A,B} = A_{n,k}^{X,Y}$.

Theorem 8. *If X and Y are subsets of $\mathbb{N}$, $A = X \cup Y$ and there exists a $B \subseteq \mathbb{N}$ such that $b_n = |B \cap [n]|$ satisfies*

$$b_n = \begin{cases} 1 + x_n + y_n = 1 + |X \cap [n]| + |Y \cap [n]|, & \text{if } n+1 \in X \cap Y; \\ y_n = |Y \cap [n]|, & \text{if } n+1 \in X - Y; \\ x_n = |X \cap [n]|, & \text{if } n+1 \in Y - X, \end{cases}$$

then $D_{n,k}^{A,B} = V_{n,k}^{X,Y}$.

2.4 Explanation of Table 1 using our general results

1. **The first group of statistics.** $D_{n,k}^{\mathbb{E},\mathbb{N}} = E_{n,k}^{\mathbb{N},\mathbb{E}}$ by Foata's first transformation. Also, $D_{n,k}^{\mathbb{E},\mathbb{N}} = V_{n,k}^{\mathbb{E},\mathbb{E}}$ by Theorem 8. Indeed, in this case $A = X = Y = \mathbb{E}$ and $B = \mathbb{N}$ leading to $A = X \cup Y$, $X - Y = Y - X = \emptyset$, and $b_n = n = 1 + 2|\mathbb{E} \cap [n]|$ if $n+1 \in \mathbb{E}$. As for the formulas, we can apply Theorem 2 with $X = Y = \mathbb{E}$:

$$a_{2n,k} = V_{2n,k}^{X,Y} = k!(n-k)!n!\binom{n}{k}\binom{n}{k}\binom{n}{n-k} = \left[n!\binom{n}{k}\right]^2;$$

$$a_{2n+1,k} = V_{2n+1,k}^{X,Y} = k!(n-k)!(n+1)!\binom{n}{k}\binom{n}{k}\binom{n+1}{n-k} = n!(n+1)!\binom{n}{k}\binom{n+1}{k+1}.$$

2. **The second group.** $D_{n,k}^{\mathbb{N},\mathbb{O}} = E_{n,k}^{\mathbb{O},\mathbb{N}}$ by Foata's first transformation. Applying the reverse operation to each permutation, one sees that $A_{n,k}^{\mathbb{O},\mathbb{E}} = A_{n,k}^{\mathbb{E},\mathbb{O}}$. Applying the inverse operation to each permutation, one gets $V_{n,k}^{\mathbb{O},\mathbb{E}} = V_{n,k}^{\mathbb{E},\mathbb{O}}$. By Theorem 6, $V_{n,k}^{\mathbb{O},\mathbb{E}} = A_{n,k}^{\mathbb{O},\mathbb{E}}$ as $\mathbb{O} \cap \mathbb{E} = \emptyset$. Finally, by Theorem 7, $D_{n,k}^{\mathbb{N},\mathbb{O}} = A_{n,k}^{\mathbb{O},\mathbb{E}}$. Indeed, in this case $A = \mathbb{N}$, $B = X = \mathbb{O}$, and $Y = \mathbb{E}$ leading to $A = X \cup Y$, and

$$b_n = \#\text{ of odd numbers in } [n] = \begin{cases} \mathbb{E} \cap [n], & \text{if } n+1 \notin \mathbb{O}; \\ \mathbb{O} \cap [n], & \text{if } n+1 \in \mathbb{E}. \end{cases}$$

As for the formulas, we can apply Theorem 2 with $X = \mathbb{E}$ and $Y = \mathbb{O}$:

$$a_{2n,k} = V_{2n,k}^{\mathbb{O},\mathbb{E}} = k!(n-k)!n!\binom{n}{k}\binom{n}{k}\binom{n}{n-k} = \left[n!\binom{n}{k}\right]^2;$$

$$a_{2n+1,k} = V_{2n+1,k}^{\mathbb{O},\mathbb{E}} = k!(n-k)!(n+1)!\binom{n}{k}\binom{n+1}{k}\binom{n}{n-k} = n!(n+1)!\binom{n}{k}\binom{n+1}{k}.$$

3. **The third group.** Again, $D_{n,k}^{\mathbb{O},\mathbb{N}} = E_{n,k}^{\mathbb{N},\mathbb{O}}$ by Foata's first transformation. Moreover, by Theorem 7, $D_{n,k}^{\mathbb{O},\mathbb{N}} = A_{n,k}^{\mathbb{O},\mathbb{O}}$. Indeed, in this case $A = X = Y = \mathbb{O}$ and $B = \mathbb{N}$ leading to $A = X \cup Y$, $X - Y = Y - X = \emptyset$, and $b_n = n = 2|\mathbb{O} \cap [n]|$ if $n+1 \in \mathbb{O}$. As for the formulas, we can apply Theorem 3 with $X = \mathbb{O}$:

$$a_{2n,k} = A_{2n,k}^{\mathbb{O},\mathbb{O}} = (n!)^2\binom{n-1}{k}\binom{n+1}{k+1};$$

$$a_{2n+1,k} = A_{2n+1,k}^{\mathbb{O},\mathbb{O}} = n!(n+1)!\binom{n}{k}\binom{n+1}{k}.$$

4. **The fourth group.** $D_{n,k}^{\mathbb{N},\mathbb{E}} = E_{n,k}^{\mathbb{E},\mathbb{N}}$ by Foata's first transformation. The formulas for $D_{n,k}^{\mathbb{N},\mathbb{E}}$ are proved in [4, Section 4].

5. **The fifth group.** We use Theorem 2 with $X = Y = \mathbb{O}$, to get

$$a_{2n,k} = V_{2n,k}^{\mathbb{O},\mathbb{O}} = k!n!n!\binom{n}{k}\binom{n}{k}\binom{n}{n-k} = \left[n!\binom{n}{k}\right]^2;$$

$$a_{2n+1,k} = V_{2n+1,k}^{\mathbb{O},\mathbb{O}} = k!(n+1-k)!n!\binom{n+1}{k}\binom{n+1}{k}\binom{n}{k-1} = n!(n+1)!\binom{n}{k-1}\binom{n+1}{k}.$$

6. **The sixth group.** We use Theorem 3 with $X = \mathbb{E}$, to get

$$a_{2n,k} = A_{2n,k}^{\mathbb{E},\mathbb{E}} = (n!)^2\binom{n-1}{k}\binom{n+1}{k+1};$$

$$a_{2n+1,k} = A_{2n+1,k}^{\mathbb{E},\mathbb{E}} = n!(n+1)!\binom{n-1}{k}\binom{n+2}{n-k}.$$

3 Applications

In this section, we shall generalize several of the results that appear in Table 1. That is, in Table 1, we consider the parity of the elements in a descent, adjacency, or place-value pair. We shall show that we can get similar formulas when we consider the equivalence class modulo k of the elements in a descent, adjacency, or place-value pair. See [3] for related research on descents generalizing results of [4]. For any $k \geq 2$ and $0 \leq i \leq k-1$, we let $i + k\mathbb{N} = \{i + kn : n \geq 0\}$.

First, we shall consider $V_{n,k}^{X,Y}$ and $A_{n,k}^{X,Y}$ where $X = i + k\mathbb{N}$ and $Y = j + k\mathbb{N}$ and $0 \leq i < j \leq k-1$. It follows from Theorem 6 that $V_{n,k}^{X,Y} = A_{n,k}^{X,Y}$ in this case. Suppose that $A = i + k\mathbb{N} \cup j + k\mathbb{N}$ and $B = i + k\mathbb{N}$. Note that when $m+1 = kn+i \in X - Y$, then $y_m = n = b_m = |B \cap [m]|$ and when $m+1 = kn+j \in Y - X$, then $x_m = n+1 = b_m$. Thus it follows from Theorems 7 and 8 that $D_{n,k}^{A,B} = V_{n,s}^{X,Y} = A_{n,s}^{X,Y}$ for all n and s. We then have three cases.

Case 1. $m = kn + t$ where $0 \leq t < i$. In this case, $x_m = |X \cap [m]| = y_m = |Y \cap [m]| = n$ and $x_m^c = |[m] - X| = y_m^c = |[m] - Y| = (k-1)n + t$. Thus it follows from Theorem 2 that

$$V_{m,s}^{X,Y} = \binom{n}{s}^2\binom{(k-1)n+t}{n-s}s!(n-s)!((k-1)n+t)!. \tag{5}$$

On the other hand, it follows from Theorem 4 that

$$D_{m,s}^{A,B} = |A_m^c|!\sum_{r=0}^{s}(-1)^{s-r}\binom{|A_m^c|+r}{r}\binom{m+1}{s-r}\prod_{x\in A_m}(1 + r + \alpha_{A,m,x} + \beta_{B,m,x}). \tag{6}$$

In this case $|A_m^c| = kn + t - 2n = (k-2)n + t$. For any x, it is easy to see that

$$\alpha_{A,m,x} + \beta_{B,m,x} = kn + t - 1 - |A \cap [x+1, kn+t] - |B \cap [x-1]|$$

where $[x+1, kn+t] = \{r : x+1 \leq r \leq kn+t\}$. Thus for any $0 \leq a \leq n-1$,

$$\begin{aligned}\alpha_{A,m,ak+i} + \beta_{B,m,ak+i} &= kn+t-1-(2n-(2a+1))-a \\ &= (k-2)n+t+a \end{aligned} \tag{7}$$

and

$$\begin{aligned}\alpha_{A,m,ak+j} + \beta_{B,m,ak+j} &= kn+t-1-(2n-(2a+2))-(a+1) \\ &= (k-2)n+t+a. \end{aligned} \tag{8}$$

Thus

$$\begin{aligned}&\prod_{x \in A_m} (1+r+\alpha_{A,m,x}+\beta_{B,m,x}) = \\ &\left(\prod_{a=0}^{n-1}(1+r+\alpha_{A,m,ak+i}+\beta_{B,m,ak+i})\right)\left(\prod_{a=0}^{n-1}(1+r+\alpha_{A,m,ak+j}+\beta_{B,m,ak+j})\right) = \\ &\left(\prod_{a=0}^{n-1}(1+r+(k-2)n+t+a)\right)^2 = \\ &(1+r+(k-2)n+t)_n(1+r+(k-2)n+t)_n\end{aligned}$$

where we define $(a)_n$ by $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \geq 1$. Since we have a combinatorial proof of the fact that $V_{m,s}^{X,Y} = D_{m,s}^{A,B}$ in this case and the proof of Theorem 4 is also completely combinatorial, it follows that we have a combinatorial proof of the following identity:

$$\begin{aligned}&\binom{n}{s}^2\binom{(k-1)n+t}{n-s}s!(n-s)!((k-1)n+t)! = \\ &((k-2)n+t)!\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-2)n+t+r}{r}\binom{kn+t+1}{s-r} \times \\ &\qquad (1+r+(k-2)n+t)_n(1+r+(k-2)n+t)_n.\end{aligned} \tag{9}$$

Case 2. $m = kn+t$ where $i \leq t < j$. In this case, $x_m = |X \cap [m]| = n+1$ and $y_m = |Y \cap [m]| = n$ and $x_m^c = |[m]-X| = (k-1)n+t-1$ and $y_m^c = |[m]-Y| = (k-1)n+t$. Thus it follows from Theorem 2 that

$$V_{m,s}^{X,Y} = \binom{n+1}{s}\binom{n}{s}\binom{(k-1)n+t}{n+1-s}s!(n+1-s)!((k-1)n+t-1)!. \tag{10}$$

On the other hand, we can obtain a formula for $V_{m,s}^{X,Y} = D_{m,s}^{A,B}$ from equation (6). In this case $|A_m^c| = kn+t-(2n+1) = (k-2)n+t-1$. For any $0 \leq a \leq n$,

$$\begin{aligned}\alpha_{A,m,ak+i} + \beta_{B,m,ak+i} &= kn+t-1-(2n+1-(2a+1))-a \\ &= (k-2)n+t-1+a \end{aligned} \tag{11}$$

and, for any $0 \leq a \leq n-1$

$$\begin{aligned} \alpha_{A,m,ak+j} + \beta_{B,m,ak+j} &= kn+t-1-(2n+1-(2a+2))-(a+1) \\ &= (k-2)n+t-1+a. \end{aligned} \tag{12}$$

Thus

$$\begin{aligned} &\prod_{x \in A_m} (1+r+\alpha_{A,m,x}+\beta_{B,m,x}) = \\ &\left(\prod_{a=0}^{n}(1+r+\alpha_{A,m,ak+i}+\beta_{B,m,ak+i})\right)\left(\prod_{a=0}^{n-1}(1+r+\alpha_{A,m,ak+j}+\beta_{B,m,ak+j})\right) = \\ &(1+r+(k-2)n+t-1)_{n+1}(1+r+(k-2)n+t-1)_n = \\ &(r+(k-2)n+t)_{n+1}(r+(k-2)n+t)_n. \end{aligned}$$

As in Case 1, it follows that we have a combinatorial proof of the following identity:

$$\begin{aligned} &\binom{n+1}{s}\binom{n}{s}\binom{(k-1)n+t}{n+1-s}s!(n-s)!((k-1)n+t-1)! = \\ &((k-2)n+t-1)!\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-2)n+t-1+r}{r}\binom{kn+t+1}{s-r} \times \\ &\qquad (r+(k-2)n+t)_{n+1}(r+(k-2)n+t)_n. \end{aligned} \tag{13}$$

Case 3. $m = kn+t$ where $j \leq t \leq k-1$. In this case, $x_m = |X \cap [m]| = y_m = |Y \cap [m]| = n+1$ and $x_m^c = |[m]-X| = y_m^c = |[m]-Y| = (k-1)n+t-1$. Thus it follows from Theorem 2 that

$$V_{m,s}^{X,Y} = \binom{n+1}{s}^2\binom{(k-1)n+t-1}{n+1-s}s!(n+1-s)!((k-1)n+t-1)!. \tag{14}$$

On the other hand, we can obtain a formula for $V_{m,s}^{X,Y} = D_{m,s}^{A,B}$ from equation (6). In this case $|A_m^c| = kn+t-(2n+2) = (k-2)n+t-2$. For any $0 \leq a \leq n$,

$$\begin{aligned} \alpha_{A,m,ak+i} + \beta_{B,m,ak+i} &= kn+t-1-(2n+2-(2a+1))-a \\ &= (k-2)n+t-2+a \end{aligned} \tag{15}$$

and, for any $0 \leq a \leq n$

$$\begin{aligned} \alpha_{A,m,ak+j} + \beta_{B,m,ak+j} &= kn+t-1-(2n+2-(2a+2))-(a+1) \\ &= (k-2)n+t-2+a. \end{aligned} \tag{16}$$

Thus

$$\begin{aligned} &\prod_{x \in A_m} (1+r+\alpha_{A,m,x}+\beta_{B,m,x}) = \\ &\left(\prod_{a=0}^{n}(1+r+\alpha_{A,m,ak+i}+\beta_{B,m,ak+i})\right)\left(\prod_{a=0}^{n}(1+r+\alpha_{A,m,ak+j}+\beta_{B,m,ak+j})\right) = \\ &(1+r+(k-2)n+t-2)_{n+1}(1+r+(k-2)n+t-2)_{n+1} = \\ &(r+(k-2)n+t-1)_{n+1}(r+(k-2)n+t-1)_{n+1}. \end{aligned}$$

It follows that we have a combinatorial proof of the following identity:

$$\binom{n+1}{s}^2 \binom{(k-1)n+t-1}{n+1-s} s!(n+1-s)!((k-1)n+t-1)! = \tag{17}$$
$$((k-2)n+t-1)! \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-2)n+t-1+r}{r} \binom{kn+t+1}{s-r} \times$$
$$(r+(k-2)n+t-1)_{n+1}(r+(k-2)n+t-1)_{n+1}.$$

Next we shall consider $V_{n,k}^{X,Y}$ and $A_{n,k}^{X,Y}$ where $X = Y = i + k\mathbb{N}$ for $0 \leq i \leq k-1$ and $k \geq 2$. In this case, it is no longer the case that $A_{m,s}^{X,X} = V_{m,s}^{X,X}$ so we will handle the cases of $A_{m,s}^{X,X}$ and $V_{m,s}^{X,X}$ separately.

First we shall consider $V_{n,s}^{X,Y}$. Note that if $A = i+k\mathbb{N}$ and $B = i+k\mathbb{N} \cup i+1+k\mathbb{N}$, then for $m+1 = kn+i \in X \cap Y$, then $x_n = |X \cap [m]| = n = y_m = |Y \cap [m]|$ and $b_m = |B \cap [m]| = 2n$. Thus it follows from Theorem 8 that $V_{n,s}^{X,X} = D_{n,s}^{A,B}$ for all n and s. We then have two cases.

Case I. $m = kn + t$ where $0 \leq t < i$. In this case, $x_m = |X \cap [m]| = n$ and $x_n^c = |[m] - X| = (k-1)n + t$. Then it follows from Theorem 2 that

$$V_{m,s}^{X,X} = \binom{n}{s}^2 \binom{(k-1)n+t}{n-s} s!(n-s)!((k-1)n+t)!. \tag{18}$$

On the other hand, we can obtain a formula for $V_{m,s}^{X,X} = D_{m,s}^{A,B}$ from equation (6). In this case $|A_m^c| = kn + t - n = (k-1)n + t$. For any $0 \leq a \leq n$,

$$\begin{aligned} \alpha_{A,m,ak+i} + \beta_{B,m,ak+i} &= kn+t-1-(n-(a+1))-2a \\ &= (k-1)n+t-a. \end{aligned} \tag{19}$$

Thus

$$\prod_{x \in A_m} (1 + r + \alpha_{A,m,x} + \beta_{B,m,x}) =$$
$$\prod_{a=0}^{n-1} (1 + r + \alpha_{A,m,ak+i} + \beta_{B,m,ak+i}) =$$
$$\prod_{a=0}^{n-1} (1 + r + (k-1)n + t - a) =$$
$$(1 + r + (k-1)n + t) \downarrow_n$$

where $(a) \downarrow_n$ is defined by $(a) \downarrow_0 = 1$ and $(a) \downarrow_n = a(a-1)\cdots(a-n+1)$ for $n \geq 1$. Thus it follows that

$$\binom{n}{s}^2 \binom{(k-1)n+t}{n-s} s!(n-s)!((k-1)n+t)! = \tag{20}$$
$$((k-1)n+t)! \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+t+r}{r} \binom{kn+t+1}{s-r} (1+r+(k-1)n+t) \downarrow_n .$$

Case II. $m = kn + t$ where $i \leq t \leq k-1$. In this case, $x_m = |X \cap [m]| = n+1$ and $x_n^c = |[m] - X| = (k-1)n + t - 1$. Then it follows from Theorem 2 that

$$V_{m,s}^{X,X} = \binom{n+1}{s}^2 \binom{(k-1)n+t-1}{n+1-s} s!(n+1-s)!((k-1)n+t-1)!. \tag{21}$$

On the other hand, we can obtain a formula for $V_{m,s}^{X,X} = D_{m,s}^{A,B}$ from equation (6). In this case $|A_m^c| = kn + t - (n+1) = (k-1)n + t - 1$. For any $0 \leq a \leq n$,

$$\begin{aligned} \alpha_{A,m,ak+i} + \beta_{B,m,ak+i} &= kn + t - 1 - (n + 1 - (a+1)) - (2a+1) \\ &= (k-1)n + t - 1 - a. \end{aligned} \tag{22}$$

Thus

$$\begin{aligned} &\prod_{x \in A_m} (1 + r + \alpha_{A,m,x} + \beta_{B,m,x}) = \\ &\prod_{a=0}^{n} (1 + r + \alpha_{A,m,ak+i} + \beta_{B,m,ak+i}) = \\ &\prod_{a=0}^{n-1} (1 + r + (k-1)n + t - 1 - a) = \\ &(r + (k-1)n + t) \downarrow_{n+1}. \end{aligned}$$

Thus it follows that

$$\begin{aligned} &\binom{n+1}{s}^2 \binom{(k-1)n+t-1}{n+1-s} s!(n+1-s)!((k-1)n+t-1)! = \\ &((k-1)n+t-1)! \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+t-1+r}{r} \binom{kn+t+1}{s-r} (r + (k-1)n + t) \downarrow_{n+1}. \end{aligned} \tag{23}$$

Next we consider the case of computing $A_{n,s}^{X,Y}$ where $X = Y = i + k\mathbb{N}$ where $k \geq 2$ and $0 \leq i \leq k-1$. Let $A = i + k\mathbb{N}$ and $B = i - 1 + k\mathbb{N}$. Then it is easy to see that if $m + 1 = kn + i \in X \cap Y = X$, then $x_m = y_m = n$ and $b_m = 2n+1$. Thus it follows from Theorem 7 that $A_{n,k}^{X,Y} = D_{n,s}^{A,B}$ for all n and s in this case. We then have two cases.

Case A. $m = kn + t$ where $0 \leq t < i - 1$. In this case, $x_m = n$ and $x_m^c = (k-1)n + t$. Thus if follows from Theorem 3 that

$$A_{m,s}^{X,X} = n!((k-1)n+t)! \binom{n-1}{s} \binom{(k-1)n+t+1}{n-s}. \tag{24}$$

On the other hand, we can obtain a formula for $A_{m,s}^{X,X} = D_{m,s}^{A,B}$ from equation (6). In this case $|A_m^c| = (k-1)n + t$. For any $0 \leq a \leq n$,

$$\begin{aligned} \alpha_{A,m,ak+i} + \beta_{B,m,ak+i} &= kn + t - 1 - (n - (a+1)) - (2a+1) \\ &= (k-1)n + t - 1 - a. \end{aligned} \tag{25}$$

Thus

$$\begin{aligned}
&\prod_{x\in A_m}(1+r+\alpha_{A,m,x}+\beta_{B,m,x}) = \\
&\prod_{a=0}^{n-1}(1+r+\alpha_{A,m,ak+i}+\beta_{B,m,ak+i}) = \\
&\prod_{a=0}^{n-1}(1+r+(k-1)n+t-1-a) = \\
&(r+(k-1)n+t)\downarrow_n .
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
&n!((k-1)n+t)!\binom{n-1}{s}\binom{(k-1)n+t+1}{n-s} = \qquad (26)\\
&((k-1)n+t)!\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+t+r}{r}\binom{kn+t+1}{s-r}(r+(k-1)n+t)\downarrow_n .
\end{aligned}$$

Case B. $m = kn+t$ where $i \leq t < k-1$. In this case, $x_m = n+1$ and $x_m^c = (k-1)n+t-1$. Thus if follows from Theorem 3 that

$$A_{m,s}^{X,X} = (n+1)!((k-1)n+t-1)!\binom{n}{s}\binom{(k-1)n+t}{n+1-s}. \qquad (27)$$

On the other hand, we can obtain a formula for $A_{m,s}^{X,X} = D_{m,s}^{A,B}$ from equation (6). In this case $|A_m^c| = (k-1)n+t-1$. For any $0 \leq a \leq n$,

$$\begin{aligned}
\alpha_{A,m,ak+i}+\beta_{B,m,ak+i} &= kn+t-1-(n+1-(a+1))-(2a+1)\\
&= (k-1)n+t-2-a. \qquad (28)
\end{aligned}$$

Thus

$$\begin{aligned}
&\prod_{x\in A_m}(1+r+\alpha_{A,m,x}+\beta_{B,m,x}) = \\
&\prod_{a=0}^{n+1}(1+r+\alpha_{A,m,ak+i}+\beta_{B,m,ak+i}) = \\
&\prod_{a=0}^{n+1}(1+r+(k-1)n+t-2-a) = \\
&(r+(k-1)n+t-1)\downarrow_{n+1} .
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
&n!((k-1)n+t-1)!\binom{n+1}{s}\binom{(k-1)n+t}{n+1-s} = \qquad (29)\\
&((k-1)n+t-1)!\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+t-1+r}{r}\binom{kn+t+1}{s-r}(r+(k-1)n+t-1)\downarrow_{n+1} .
\end{aligned}$$

4 Direction for future research

In this section, we shall describe some problems for further research that naturally arise from the work in this paper.

There are other statistics which are closely related to the statistics that we consider in this paper. For example, suppose that $X, Y \subseteq \mathbb{N}$ and define

$$\gamma_{X,Y}(\sigma) = |\{i \in X : \sigma_i \in X\} \cup \{i \in Y : \sigma_i \in Y\}|.$$

Let $\Gamma^{X,Y}_{n,s} = |\{\sigma \in \mathcal{S}_n : \gamma_{X,Y}(\sigma) = s\}|$. Then we have the following theorem.

Theorem 9. *For any X and Y such that $X \cup Y = \mathbb{N}$ and $X \cap Y = \emptyset$, we have $\Gamma^{X,Y}_{n,s} = 0$ unless $s = 2k + y_n - x_n$ for some k, in which case*

$$\Gamma^{X,Y}_{n,2k+y_n-x_n} = (x_n)!(y_n)!\binom{x_n}{k}\binom{y_n}{x_n - k}.$$

Proof. Suppose we pick k positions in X_n to contain elements in X_n in $\binom{x_n}{k}$ ways and we pick $x_n - k$ positions Y_n to put the other elements of X_n in $\binom{y_n}{x_n-k}$ ways. The remaining positions in the permutation must be filled with elements of Y_n. Next arrange elements in X_n in $(x_n)!$ ways and we arrange elements in Y_n in $(y_n)!$. Clearly the number of permutations σ that can be constructed in this way is $(x_n)!(y_n)!\binom{x_n}{k}\binom{y_n}{x_n-k}$. Note that our construction forces $y_n - (x_n - k)$ elements of Y_n to be in positions in Y_n so that for any σ constructed in this way $\gamma_{X,Y}(\sigma) = 2k + y_n - x_n$. $\square$

Note that in the special case where $X = \mathbb{E}$ and $Y = \mathbb{O}$, we have that $\Gamma^{\mathbb{E},\mathbb{O}}_{2s,2n} = (n!)^2\binom{n}{s}^2$ and $\Gamma^{\mathbb{E},\mathbb{O}}_{2s+1,2n+1} = n!(n+1)!\binom{n}{s}\binom{n+1}{s+1}$ which agrees with other formulas in our table in these special cases. However, for general X and Y, we get quite different recursions. For example, suppose that $n+1 \in X \cap Y$, $\sigma \in S_n$, $i \in X - Y$, and $\sigma_i \in Y - X$. Then it is easy to see that

$$\gamma_{X,Y}(I^{(i)}_{n+1}(\sigma)) = \gamma_{X,Y}(\sigma) + 2$$

so that the value of $\gamma_{X,Y}$ can jump by 2 with a single insertion. This type of phenomenon does not happen with any of the other statistics studied in this paper. Thus it would be interesting to further study $\Gamma^{X,Y}_{n,s}$ for arbitrary $X, Y \subseteq \mathbb{N}$ to see if one can prove explicit formulas for $\Gamma^{X,Y}_{n,s}$ which are similar to the formulas for $D^{X,Y}_{n,s}$ given in Theorems 4 and 5.

Even though we found solutions to all the bijective questions related to the objects in our table, in some cases, one should be able to modify our bijections (find new ones) to preserve more than one statistic.

Recall that the statistic S_{10} is the number of odd descent-tops, and S_{12} is the number of (odd,odd) pairs. In this section, we will use the following statistics as well.

- S_{17} — the length of the maximal subsequence of the form $12 \cdots i$ in a permutation. E.g., $S_{17}(34152) = 2$ while the increasing permutation of length n gives the maximum value of S_{17} in $\mathcal{S}_n$. A modification of this statistic was studied by Zeilberger [5] in connection with *2-stack sortable permutations*.

- $T_1 = S_{10}$ but *not* S_{12}.
- $T_2 = S_{12}$ but *not* S_{10}.
- $T_3 = S_{10}$ *and* S_{12}.

Our first conjecture is the following joint equidistribution:

Conjecture 10. The following should be true: $(S_{10}, S_{12}, S_{17}) \sim (S_{12}, S_{10}, S_{17})$.

Notice, that Conjecture 10 suggests existence of an involution turning S_{10} to S_{12} and vice versa. This conjecture can be refined as follows.

Conjecture 11. $(T_1, T_2, T_3, S_{17}) \sim (T_2, T_1, T_3, S_{17})$. That is, if the involution mentioned above exists, it is likely to leave pairs that are both S_{10} and S_{12} untouched.

Observe that to preserve statistic S_{17} in Conjectures 10 and 11, we need to require the increasing n-permutation to go to itself, and this is the only thing we need to worry about in our recursive construction of the bijection regarding S_{17} as otherwise it is not changed and thus preserved by induction no matter where we stick the largest element. So, it seems like we should be able to have the increasing permutation as a fixed point.

Here is how a proof of Conjecture 11 could be arranged for even n assuming the rest is constructed by induction. For odd n's things seem to be much more complicated.

For $n = 1$, 1 is mapped to 1. Suppose we have constructed a bijection from $\mathcal{S}_{n-1}$ to $\mathcal{S}_{n-1}$ (n is even) such that it sends k (resp. ℓ, s) occurrences of T_1 (resp. T_2, T_3) to k (resp. ℓ, s) occurrences of T_2 (resp. T_1, T_3). Inserting n in T_1 (resp. T_2, T_3) pair decreases the number of T_1 (resp. T_2, T_3) by 1 keeping all other statistics unchanged. Clearly, we can manage the corresponding insertion on the other side that would decrease by 1 the number of occurrences of the corresponding statistic. Inserting n at any other position does *not* change a thing in either side and can be matched to each other. In particular, inserting n at the end corresponds to inserting n at the end and this guarantees that the statistic S_{17} is preserved (either it is unchanged in both cases, or assuming we deal with the increasing permutation $12 \cdots (n-1)$ going to itself, S_{17} is increased by 1 in both cases).

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