Journal of Integer Sequences, Vol. 12 (2009),
Article 09.8.3

Some Classes of Numbers and Derivatives

Milan Janjić
Department of Mathematics and Informatics
University of Banja Luka
Republic of Srpska, Bosnia and Herzegovina
agnus@blic.net

Abstract

We prove that three classes of numbers – the non-central Stirling numbers of the first kind, generalized factorial coefficients, and Gould-Hopper numbers – may be defined by the use of derivatives. We derive several properties of these numbers from their definitions. We also prove a result for harmonic numbers. The coefficients of Hermite and Bessel polynomials are a particular case of generalized factorial coefficients, The coefficients of the associated Laguerre polynomials are a particular case of Gould-Hopper numbers. So we obtain some properties of these polynomials. In particular, we derive an orthogonality relation for the coefficients of Hermite and Bessel polynomials.

1 Introduction

The purpose of this paper is to investigate properties of the non-central Stirling numbers of the first kind, the generalized factorial coefficients, and Gould-Hopper numbers by the use of derivatives.

We use the following notation throughout the paper:

- $(a)_n$ denotes the falling factorial of a, that is, $(a)_n = a(a-1)\cdots(a-n+1)$, $(a)_0 = 1$;
- $s(n,k)$ and $\mathbf{s}(n,k)$ denote the signed and the unsigned Stirling numbers of the first kind respectively;
- H_n denotes the harmonic number $\sum_{1\le i\le n} 1/i$;
- $s(n,k,a)$ denotes the non-central Stirling number of the first kind;
- $C(n,k,a)$ denotes the generalized factorial coefficient;

- $C(n,k,b,a)$ denotes the non-central generalized factorial coefficient or Gould-Hopper number.

The notation and the terminology are taken from Charalambides' book [4]. Sloane [6] calls the $s(n,k,a)$ the generalized Stirling numbers. Further,

- $H_n(x)$ denotes the Hermite polynomial;
- $p_n(x)$ denotes the (reverse) Bessel polynomial, and
- $L_n^k(x)$ denotes the associated Laguerre polynomial, where $L_n^0(x) = L_n(x)$ is a Laguerre polynomial.

The paper is organized as follows. The first section is an introduction.

In the second section we prove that the non-central Stirling numbers $s(n,k,a)$ of the first kind naturally appear in the expansion of derivatives of the function $x^{-a}\ln^b x$, where a and b are arbitrary real numbers. We first obtain a recurrence relation for $s(n,k,a)$ and then, using Leibnitz rule, we obtain an explicit formula. We then consider a particular formula for $s(n,1,a)$ and derive some combinatorial identities. The results are related to a number of sequences from Sloane's *Encyclopedia* [6].

In the third section we first prove that the generalized factorial coefficients appear as coefficients in the expansion of the nth derivative of the function $f(x^a)$, where a is arbitrary real number, and $f \in C^\infty(0,+\infty)$ is arbitrary function. Choosing suitable functions f we derive some properties of generalized factorial coefficients. We are particularly concerned with some properties of coefficients of Hermite and Bessel polynomials. The results of this section are also related to a number of sequences from [6].

In the fourth section we first show that Gould-Hopper numbers are coefficients in the expansion of the nth derivative of the function $x^a f(x^b)$, where a,b are arbitrary real numbers, and $f \in C^\infty(0,+\infty)$ is arbitrary function. The coefficients of associated Laguerre polynomials are particular case of Gould-Hopper numbers. Using similar methods as in the third section we prove a number of properties which describe connections between Gould-Hopper numbers, generalized factorial coefficients, powers, factorials, binomial coefficients, and Stirling numbers. The results are also concerned with some sequences from [6].

Note that these considerations are related with Bell polynomials which naturally appear in derivatives of composition functions [2, 5, 7] .

2 Non-central Stirling numbers of the first kind

We shall first derive a formula for the nth derivative of the function

$$f(x) = x^{-a}\ln^b x,\ (a,b\in\mathbb{R}).$$

Theorem 1. *Let a be a real number, and let n be a nonnegative integer. Then*

$$\frac{d^n}{dx^n}f(x) = x^{-a-n}\sum_{i=0}^{n} p(n,i,a)(b)_i \ln^{b-i} x. \tag{1}$$

where $p(n,i,a)$, $(0\le i\le n)$ are polynomials of a with integer coefficients.

Proof. Theorem 1 is true for $n = 0$, if we define $p(0,0,a) = 1$.

If we define
$$p(1,0,a) = -a,\ p(1,1,a) = 1$$
then Theorem 1 is also true for $n = 1$.

Assume Theorem 1 is valid for $n \geq 1$.

Taking derivative in (1) we find that
$$\frac{d^{n+1}}{dx^{n+1}} f(x) = x^{-a-n-1}\left[(-a-n)\sum_{i=0}^{n} p(n,i,a)(b)_i \ln^{b-i} x + \sum_{i=0}^{n} p(n,i,a)(b)_{i+1} \ln^{b-i-1} x\right].$$
Replacing $i+1$ by i in the second sum on the right side yields
$$\frac{d^{n+1}}{dx^{n+1}} f(x) = x^{a-n-1}(-a-n)p(n,0,a) + s(n,n,a)(b)_{n+1} \ln^{b-n-1} x +$$
$$+x^{a-n-1}\sum_{i=1}^{n}\left[(-a-n)p(n,i,a) + s(n,i-1,a)\ln^{b-i} x\right](b)_i.$$

It follows that Theorem 1 is true if we define
$$p(n+1,0,a) = -(a+n)p(n,0,a),\ p(n+1,n+1,a) = p(n,n,a),$$
$$p(n+1,i,a) = -(a+n)p(n,i,a) + p(n,i-1,a),\ (i = 1,\ldots,n).$$
$\square$

The preceding equations are the recurrence relations for non-central Stirling numbers of the first kind $s(n,i,a)$, [4, p. 316]. In what follows we shall denote $p(n,i,a)$ by $s(n,i,a)$.

It is easy to see that the following equations hold
$$s(n,0,a) = (-a)_n,\ (n = 0,1,2,\ldots),$$
and
$$s(n,n,a) = 1,\ (n = 0,1,2,\ldots).$$
By Leibnitz rule we get
$$\frac{d^n}{dx^n} f(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} x^{-a} \frac{d^{n-k}}{dx^{n-k}} \ln^b x. \tag{2}$$
From the well-known formulas
$$\frac{d^k}{dx^k} x^{-a} = (-a)_k x^{a-k},$$
and
$$\frac{d^{n-k}}{dx^{n-k}} \ln^b x = x^{-n+k} \sum_{i=1}^{n-k} s(n-k,i)(b)_i \ln^{b-i} x,$$
by comparing (1) and (2) we obtain the following:

Proposition 2. *Let a be a real number, and let n, i, $(i \leq n)$ be nonnegative integers. Then*

$$s(n,i,a) = \sum_{k=0}^{n-i} \binom{n}{k} (-a)_k s(n-k,i). \tag{3}$$

Remark 3. Proposition 2 is true in the case $a = 0$ with the convention that $(0)_0 = 1$.

Taking $i = 1$ in (3) we have the following:

Proposition 4. *Let a be a real number, and n be a positive integer. Then*

$$s(n,1,a) = n! \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{\binom{-a}{k}}{n-k}.$$

For $s(n,1,a)$ we have the following recurrence relation:

$$s(1,1,a) = 1,\ s(n,1,a) = (-a-n+1)s(n-1,1,a) + (-a)_{n-1},\ (n \geq 2). \tag{4}$$

We shall now prove that polynomials $r(n,a),\ (n = 1,2,\ldots)$ defined by

$$r(n,a) = \sum_{k=0}^{n-1} (k+1)s(n,k+1)(-a)^k$$

satisfy (4). For $n = 1$ this is obviously true.

Using the two terms recurrence relation for Stirling numbers of the first kind, for $n > 1$ we have

$$r(n,a) = \sum_{k=0}^{n-1} (k+1)s(n-1,k)(-a)^k - (n-1) \sum_{k=0}^{n-2} (k+1)s(n-1,k+1)(-a)^k.$$

Since $s(n-1,0) = 0$, by replacing $k+1$ instead of k in the first sum on the right side we obtain

$$r(n,a) = (-a-n+1)r(n-1,a) + \sum_{k=0}^{n-2} s(n-1,k+1)(-a)^{k+1}.$$

Furthermore, a well known property of Stirling numbers of the first kind implies

$$\sum_{k=0}^{n-2} s(n-1,k+1)(-a)^{k+1} = (-a)_{n-1},$$

which means that $r(n,a)$ satisfies (4). We have proved the following:

Proposition 5. *Let a be a real number, and let $n \geq 1$ be an integer. Then*

$$n! \sum_{k=0}^{n-1} (-1)^k \frac{\binom{-a}{k}}{n-k} = \sum_{k=0}^{n-1} (k+1)\mathbf{s}(n,k+1)a^k. \tag{5}$$

Remark 6. Proposition 5 is true for $a = 0$ with the convention that $0^0 = 1$.

In the case that a is a negative integer and $n \leq -a$, the identity (5) is related to the harmonic numbers.

Proposition 7. *Define $h(n, m)$ such that*

$$h(n, m) = (H_m - H_{m-n})\frac{m!}{(m-n)!}, \; (m = 1, 2, \ldots; n = 1, 2, \ldots, m).$$

Then $h(n, m)$ satisfies (4).

Proof. The proof goes by induction with respect to n. For $n = 1$ we have

$$h(1, m) = \frac{(m)!}{(m-1)!}(H_m - H_{m-1}) = 1.$$

Furthermore, for $n > 1$ we have

$$(m - n + 1)h(n - 1, m) + (m)_{n-1} = \frac{(m)!}{(m-n)!}(H_m - H_{m-n+1}) + (m)_{n-1} =$$

$$= \frac{(m)!}{(m-n)!}(H_m - H_{m-n}),$$

since $\frac{(m)!}{(m-n)!(m-n+1)} = (m)_{n-1}$. It follows that

$$h(n, m) = (m - n + 1)h(n - 1, m) + (m)_{n-1},$$

and the result is proved. $\square$

As an immediate consequence of Proposition 7 we obtain

Proposition 8. *Let m be a positive integer and let n, $(1 \leq n \leq m)$ be any integers. Then*

$$H_m - H_{m-n} = \frac{(-1)^{n+1}}{\binom{m}{n}} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{m}{k}}{n-k}.$$

Remark 9. The results of this section are concerned with the following sequences in [6]: A001701, A001702, A001705, A001706, A001707, A001708, A001709, A001711, A001712, A001713, A001716, A001717, A001718, A001722, A001723, A001724, A049444, A049458, A049459, A049600, A051338, A051339, A051379, A051523, A051524, A051525, A051545, A051546, A051560, A051561, A051562, A051563, A051564, A051565.

3 Generalized factorial coefficients

The first result in this section is a closed formula for the nth derivative of the function $f(x^a)$, where $f \in C^\infty(0, +\infty)$, and a is a real number. Such one formula may be obtained as a particular case of Faá di Bruno's formula. We obtain here the formula which is easily proved by induction. In addition, we obtain a recurrence relation for coefficients.

Theorem 10. *Let $n > 0$ be an integer, and let a be a real number. Then*

$$\frac{d^n}{dx^n} f(x^a) = \sum_{k=1}^{n} q(n,k,a) x^{ak-n} \frac{d^k}{dx^k} f(t), \tag{6}$$

where $t = x^a$, and $q(n,k,a)$ is a polynomials of a with integer coefficients. The degree of $q(n,k,a)$ is n, and it does not depend on f.

Proof. The result is true for $n = 1$ if we take $q(1,1,a) = a$. Assume that the result is true for $n \geq 1$. Taking derivative in (6) we obtain

$$\frac{d^{n+1}}{dx^{n+1}} f(x^a) = \sum_{k=1}^{n} (ka-n) q(n,k,a) x^{ka-n-1} \frac{d^k}{dx^k} f(t) +$$

$$+ a \sum_{k=1}^{n} q(n,k,a) x^{ka-n+a-1} \frac{d^{k+1}}{dx^{k+1}} f(t) =$$

$$= \sum_{k=2}^{n} \left[(ka-n) q(n,k,a) + a q(n,k-1,a) \right] x^{ka-n-1} \frac{d^k}{dx^k} f(t) +$$

$$+ (a-n) q(n,1,a) x^{a-n-1} \frac{d}{dx} f(t) + a q(n,n,a) x^{(n+1)(a-1)} \frac{d^{n+1}}{dx^{n+1}} f(t).$$

Define

$$q(n,0,a) = 0, \; q(n,k,a) = 0, \; (k > n),$$

and

$$q(n+1,k,a) = (ka-n) q(n,k,a) + a q(n,k-1,a), \; (k = 1, \ldots, n+1), \tag{7}$$

to obtain

$$x^{n+1} \frac{d^{n+1}}{dx^{n+1}} f(x^a) = \sum_{k=1}^{n+1} q(n+1,k,a) t^k \frac{d^k}{dx^k} f(t),$$

and the result is proved. □

If, additionally, we define $q(0,0,a) = 1$ then the formula (6) may be written in the form

$$\frac{d^n}{dx^n} f(x^a) = \sum_{k=0}^{n} q(n,k,a) \frac{d^k}{dx^k} f(t) x^{ak-n}, \; (n = 0, 1, \ldots). \tag{8}$$

The equations (7) shows that the polynomials $q(n,k,a)$ are in fact the generalized factorial coefficients $C(n,k,a)$ ([4, p. 309]).

Proposition 11. *Generalized factorial coefficients satisfy the following equations:*

$$C(n,1,a)=(a)_n,\ C(n,n,a)=a^n,\ C(n,k,1)=0,\ (k<n)(n=1,2,\ldots).$$

Proof. For the first equation it is enough to take $f(t)\equiv t$ in (8).

The second equation follows immediately from (7).

If $a=1$ then (8) takes the form

$$x^n\frac{d^n}{dx^n}f(x)=\sum_{k=0}^{n}C(n,k,1)x^k\frac{d^k}{dx^k}f(t).$$

and since f is arbitrary function the third equation is also true. □

The generalized factorial coefficients are related with Hermite and Bessel polynomials.

Taking $f(t)=e^{bt}$ in (8) we obtain

$$\frac{d^n}{dx^n}e^{bx^a}=e^{bx^a}\sum_{k=1}^{n}C(n,k,a)b^kx^{ak-n}. \tag{9}$$

Proposition 12. *Let n and m be positive integers. Then*

$$\frac{d^n}{dx^n}e^{-x^m}=e^{-x^m}\sum_{k=\lceil\frac{n}{m}\rceil}^{n}(-1)^kC(n,k,m)x^{mk-n}. \tag{10}$$

Proof. Take $b=-1$ in (9), hence

$$\frac{d^n}{dx^n}e^{-x^m}=e^{-x^m}\sum_{k=0}^{n}C(n,k,m)x^{mk-n},\ (n\geq 0).$$

It is clear that taking derivatives on the left-hand side of this equation can not produce negative powers of x. This means that $C(n,k,m)=0$ if $km-n<0$, and Proposition 12 is proved. □

Remark 13. The equation (10) defines generalized Hermite polynomials, [3].

Proposition 14. *If $H_n(x),\ n=1,\ldots$ are Hermite polynomials then*

$$H_n(x)=\sum_{k=\lceil\frac{n}{2}\rceil}^{n}(-1)^{n+k}C(n,k,2)x^{2k-n}. \tag{11}$$

It is easy to check that functions $f(n,k),\ (n=1,\ldots;\ k=1,\ldots,n)$ defined by

$$f(n,k)=\frac{(-1)^{n-k}(2n-k-1)!}{2^{2n-k}(n-k)!(k-1)!}$$

fulfill the recurrence relation (7) for $a=\frac{1}{2}$. We thus obtain the following:

Proposition 15. *Bessel polynomials $p_n(x)$ satisfy the following equation:*

$$p_n(x) = 2^n \sum_{k=1}^{n} (-1)^{n-k} C\left(n, k, \frac{1}{2}\right) x^k. \tag{12}$$

The next result shows that generalized factorial coefficients are coefficients in the expansion of falling factorials of b in terms of falling factorials of a, where a and b are arbitrary real numbers.

Proposition 16. *Let n be a nonnegative integer, and let a, b be arbitrary real numbers. Then*

$$(b)_n = \sum_{k=0}^{n} C\left(n, k, \frac{b}{a}\right) (a)_k. \tag{13}$$

Proof. Replacing a by $\frac{b}{a}$ in (8) we have

$$x^n \frac{d^n}{dx^n} f(x^{\frac{b}{a}}) = \sum_{k=1}^{n} C\left(n, k, \frac{b}{a}\right) t^k \frac{d^k}{dx^k} f(t),$$

where $t = x^{\frac{b}{a}}$.

Choosing $f(t) = t^a$ implies $f(x^{\frac{b}{a}}) = x^b$, hence

$$x^n \frac{d^n}{dx^n} x^b = \sum_{k=1}^{n} C\left(n, k, \frac{b}{a}\right) t^k \frac{d^k}{dx^k} t^a,$$

that is,

$$(b)_n x^b = \sum_{k=1}^{n} C\left(n, k, \frac{b}{a}\right) (a)_k t^a.$$

Since $x^b = t^a$ the result follows. $\square$

Remark 17. The equation (13) serves as the definition of generalized factorial coefficients in [4, Definition 8.2].

Choosing $b = -a$ implies $(-a)_n = (-1)^n a(a+1)\cdots(a+n-1)$. We thus obtain the expression in which rising factorials are given in terms of falling factorials.

Proposition 18. *Let a be a real number. Then*

$$a(a+1)\cdots(a+n-1) = (-1)^n \sum_{k=1}^{n} C(n, k, -1)(a)_k.$$

Remark 19. The preceding equation means that $C(n, k, -1)$ are Lah numbers.

From the equation (13) we shall derive some properties of coefficients of Hermite and Bessel polynomials. Denote by $b(n,k)$ the coefficient by x^k in the expansion of $P_n(x)$ in (12). Then

$$C\left(n,k,\frac{1}{2}\right) = (-1)^{n-k}2^{-n}b(n,k),\ (n=1,2,\ldots,\ k=1,2,\ldots,n).$$

Next, denote by $h(n,k)$ the coefficient by x^k in the expansion of $H_n(x)$ in (11). It follows that

$$C(n,k,2) = (-1)^{n+k}h(n,2k-n),$$

where $h(n,2k-n)=0$ if $2k-n<0$. We have thus proved the following:

Proposition 20. *Let a be a real number, and let n be a positive integer. Then the following equations hold*

$$(2a)_n = \sum_{k=1}^{n}(-1)^{n+k}h(n,2k-n)(a)_k,$$

and

$$(a)_n = \sum_{k=1}^{n}(-1)^{n-k}2^{-n}b(n,k)(2a)_k.$$

The following proposition gives a known property of generalized factorial coefficients, [4, Theorem 8.18].

Proposition 21. *Let $n \geq k$ be integers, and let a, b be real numbers. Then*

$$C(n,k,a_1a_2) = \sum_{j=k}^{n} C(n,j,a_2)C(j,k,a_1). \tag{14}$$

Proof. Take $f_1(t)=t^{a_1}$ and $f_2(t)=t^{a_2}$, hence

$$f(x^{a_1a_2}) = (f\circ f_1)(x^{a_2}).$$

Firstly, it follows from (6) that

$$x^n\frac{d^n}{dx^n}f(x^{a_1a_2}) = \sum_{k=1}^{n} C(n,k,a_1a_2)x^{a_1a_2k}\frac{d^k}{dx^k}f(t),\ (t=x^{a_1a_2}). \tag{15}$$

On the other hand, (6) also implies

$$x^n\frac{d^n}{dx^n}(f\circ f_1)(x^{a_2}) = \sum_{j=1}^{n} C(n,j,a_2)x^{a_2j}\frac{d^j}{dx^j}(f\circ f_1)(u),\ (u=x^{a_2}).$$

Applying (6) once more yields

$$x^n\frac{d^n}{dx^n}(f\circ f_1)(x^{a_2}) = \sum_{j=1}^{n}\sum_{k=1}^{j} C(n,j,a_2)C(j,k,a_1)x^{a_2j}u^{-j}v^k\frac{d^k}{dx^k}f(v),\ (v=u^{a_1}).$$

Changing the order of summation and taking into account that $v = u^{a_2} = x^{a_1 a_2} = t$ we obtain

$$x^n \frac{d^n}{dx^n}(f \circ f_1)(x^{a_2}) = \sum_{k=1}^{n} \left(\sum_{j=k}^{n} C(n, j, a_2) C(j, k, a_1) \right) x^{a_1 a_2 k} \frac{d^k}{dx^k} f(t).$$

Comparing (15) and the preceding equation shows that Proposition 21 is true. ∎

From Proposition 21 we derive an orthogonality relation between coefficients of Hermite and Bessel polynomials.

Proposition 22. *If $h(n, k)$ and $b(n, k)$ are the coefficients of Hermite and Bessel polynomials respectively, then*

$$\sum_{k=1}^{n} b(n, k) h(n, 2k - n) = 0.$$

Proof. Since $C(n, k, 1) = 0$ for $k < n$ the result follows from (11) and (12). ∎

Remark 23. The results of this section are related to the following sequences in [6]: A000369, A001497, A001801, A004747, A008297, A013988, A035342, A035469, A049029, A049385, A059343, A092082, A105278, A111596, A122850, A132056, A132062, A136656.

4 Gould-Hopper numbers

In the first result of this section we prove that Gould-Hopper numbers are coefficients in the expansion of the nth derivative of $x^a f(x^b)$, where a, b are arbitrary real numbers, and $f \in C^\infty(0, +\infty)$ is arbitrary function.

Theorem 24. *Let n be a positive integer, and let a, b be real numbers. Then*

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{a-n} \sum_{k=0}^{n} p(n, k, b, a) x^{bk} \frac{d^k}{dx^k} f(t), \tag{16}$$

where $t = x^b$, and $p(n, k, b, a)$ are polynomials of a and b with integer coefficients, which do not depend on f.

Proof. Using Leibnitz rule and (6) we easily obtain

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{a-n} \left[(a)_n f(x^b) + \sum_{j=1}^{n} \sum_{k=1}^{j} \binom{n}{j} C(j, k, b) \frac{d^k}{dx^k} f(t) (a)_{n-j} x^{bk} \right].$$

Changing the order of summation implies

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{a-n} \left[(a)_n f(x^b) + \sum_{k=1}^{n} \left[\sum_{j=k}^{n} \binom{n}{j} C(j, k, b) (a)_{n-j} \right] \frac{d^k}{dx^k} f(t) x^{bk} \right].$$

Theorem 24 is true if we define

$$p(n,0,b,a)=(a)_n,\ p(n,k,b,a)=\sum_{j=k}^{n}\binom{n}{j}C(j,k,b)(a)_{n-j},\ (k=1,\dots,n).$$

□

Remark 25. According to [4, p. 318] we see that $p(n,k,b,a)$ are Gould-Hopper numbers or non-central generalized factorial coefficients and will be dented by $C(n,k;b,a)$.

Gould-Hopper numbers generalize coefficients of associated Laguerre polynomials $L_n^k(x)$, [1, p. 726].

Namely, $L_n^k(x)$ are defined to be

$$L_n^k(x)=\frac{e^x x^{-k}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+k}),\ L_n^0(x)=L_n(x).$$

Take $a=n+k, b=1, f(x)=e^{-x}$ in (16) to obtain

Proposition 26. *Let $L_n^k(x),\ (n=0,1,\dots,\ k=0,1,\dots)$ be associated Laguerre polynomials. Then*

$$L_n^k(x)=\frac{1}{n!}\sum_{i=0}^{n}(-1)^i C(n,i,1,n+k)x^i.$$

Proposition 27. *Let n be a nonnegative integer, and let a,b,c be nonzero real numbers. Then*

$$(a+bc)_n=\sum_{k=0}^{n}C(n,k;b,a)(c)_k.$$

Proof. Take $f(t)=t^c$ in (16) to obtain $x^af(x^b)=x^{a+bc}$, and the resul follows. □

Remark 28. The equation from the preceding proposition serves as the definition of Gould-Hopper numbers in [4, p. 317].

The following result shows that Gould-Hopper numbers, with a suitable chosen sign, are coefficients in the expression of falling factorial of a in terms rising factorial of b.

Proposition 29. *Let n be a positive integer, and let a,b be nonzero real numbers. Then*

$$(a)_n=\sum_{k=1}^{n}(-1)^{k-1}C\big(n,k;\frac{a}{b},a\big)\cdot b\cdot(b+1)\cdots(b+k-1).$$

Proof. Take $f(t)=t^{-\frac{a}{c}}$, where $c\neq 0$. Then $x^af(x^c)=1$, hence $\frac{d^n}{dx^n}(x^af(x^c))=0,\ (n>0)$. Applying (16) we obtain

$$\sum_{k=0}^{n}C\left(n,k,\frac{a}{b},a\right)(-b)_k=0,$$

where $b=\frac{a}{c}$, and the result holds. □

The next result is an explicit formula for $C(n,k;b,a)$ in terms of generalized factorial coefficients.

Proposition 30. *Let $m \le n$ be nonnegative integers, and let a,b be nonzero real numbers. Then*

$$C(n,m;b,a) = \sum_{k=m}^{n} C\left(k,m,\frac{b}{a}\right)\Big[C(n,k,a)+(k+1)C(n,k+1,a)\Big].$$

Proof. Let us choose $f_1(t)=t,\ f_2(t)=f(t^{\frac{b}{a}})$, where f is arbitrary function. Then

$$f_1(x^a)f_2(x^a) = x^a f(x^b).$$

Using (8) and Leibnitz rule we obtain

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{-n}\sum_{j=0}^{n}\sum_{k=0}^{j} C(n,k,a)\binom{m}{j}\frac{d^j}{dt^j}t\frac{d^{k-j}}{dt^{k-j}}\left[f(t^{\frac{b}{a}})\right]x^{ak},$$

where $t=x^a$. On the right side of this equation only terms obtained for $j=0$ and $j=1$ remain. It follows that

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{-n}\sum_{k=0}^{n} C(n,k,a)\frac{d^k}{dt^k}\left[f(t^{\frac{b}{a}})\right]x^{(k+1)a} + \sum_{k=1}^{n} kC(n,k,a)\frac{d^{k-1}}{dt^{k-1}}\left[f(t^{\frac{b}{a}})\right]x^{ak} =$$

$$= \sum_{k=0}^{n}[C(n,k,a)+(k+1)C(n,k+1,a)]\frac{d^k}{dt^k}\left[f(t^{\frac{b}{a}})\right]x^{(k+1)a}.$$

According to (8) we have

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{a-n}\sum_{k=0}^{n}\sum_{m=0}^{k} C\left(k,m,\frac{b}{a}\right)\left[C(n,k,a)+(k+1)C(n,k+1,a)\right]f^{(m)}(u)x^{bm},$$

where $u=t^{\frac{b}{a}}=x^b$.

Interchanging the order of summation gives

$$\left[x^a f(x^b)\right]^{(n)} = x^{a-n}\sum_{m=0}^{n}\left[\sum_{k=m}^{n} C\left(k,m,\frac{b}{a}\right)\left[C(n,k,a)+(k+1)C(n,k+1,a)\right]\right]f^{(m)}(t)x^{bm}.$$

Comparing this equation with (16) implies

$$C(n,m,b,a) = \sum_{k=m}^{n} C\left(k,m,\frac{b}{a}\right)\left[C(n,k,a)+(k+1)C(n,k+1,a)\right],\ (m=0,1,\ldots,n)],$$

and Proposition 30 is proved. □

We finish with a result connecting Gould-Hopper numbers, Stirling numbers of the first kind, powers, binomial coefficients, and falling factorials.

Proposition 31. *Let $j \le n$ be nonnegative integers and let a, b be nonzero real numbers. Then*

$$\sum_{k=j}^{n} C(n,k,b,a)s(k,j) = b^j \sum_{k=j}^{n} \binom{n}{k} (a)_{n-k} s(k,j).$$

Proof. Take $f(t) = \ln^c t$, where c is a real number such that $(c)_i \neq 0,\ (i = 1, 2, \ldots)$. It follows that $x^a f(x^b) = x^a b^c \ln^c x$. From (1) and (3) we conclude that

$$b^c \frac{d^n}{dx^n}(x^a \ln^c x) = b^c (a)_n x^{a-n} \ln x + b x^{a-n} \sum_{k=1}^{n} \sum_{j=1}^{k} \binom{n}{k} (a)_{n-k} s(k,j)(c)_j \ln^{c-j} x.$$

Using (16) yields

$$b^c \left[x^a \ln^c x\right]^{(n)} = b^c (a)_n x^{a-n} \ln^c x + x^{a-n} b^c \sum_{k=1}^{n} \sum_{j=1}^{k} C(n,k,b,a) s(k,j) b^{-j} (c)_j \ln^{c-j} x.$$

Changing the order of summation in both sums leads to the following equation:

$$\sum_{j=1}^{n} \left[\sum_{k=j}^{n} \binom{n}{k} (a)_{n-k} s(k,j) \right] (c)_j \ln^{c-j} x =$$

$$= \sum_{j=1}^{n} \left[\sum_{k=j}^{n} C(n,k,b,a) s(k,j) b^{-j} \right] (c)_j \ln^{c-j} x.$$

Comparing terms by the same $\ln^{c-j} x$, and then dividing by $(c)_j \neq 0$ proves the result. $\square$

Remark 32. The results of this section are concerned with the following sequences in [6]: A000522, A021009, A035342, A035469, A049029, A049385, A072019, A072020, A084358, A092082, A094587, A105278, A111596, A132013, A132014, A132056, A132159, A132681, A132710, A132792, A136215, A136656.

References

[1] G. Arfken, *Laguerre Functions, Mathematical Methods for Physicists*, 3rd ed., Academic Press, 1985, 721–731.

[2] E. T. Bell, Exponential polynomials, *Ann. Math.* **35** (1934), 258–277.

[3] A. Bernardini and P. E. Ricci, Bell polynomials and differential equations of Freud-type polynomials, *Math. Comput. Modelling,* **36** (2002), 1115–1119.

[4] Ch. A. Charalambides, *Enumerative Combinatorics,* Chapman & Hall/CRC, 2002.

[5] P. Natalini and P. E. Ricci, Bell polynomials and some of their applications, *Cubo Mat. Educ.,* **5** (2003), 263–274.

[6] N. J. Sloane, The Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences/

[7] A. Xu and C. Wang, On the divided difference form of Faá di Bruno's formula, *J. Comput. Math.* **25** (2007), 697–704.

2000 *Mathematics Subject Classification*: Primary 05A10; Secondary 11C08.
Keywords: non-central Stirling numbers of the first kind, generalized factorial coefficients, Gould-Hopper numbers, generalized Stirling numbers.

(Concerned with sequences A000369, A000522, A001497, A001701, A001702, A001705, A001706, A001707, A001708, A001709, A001711, A001712, A001713, A001716, A001717, A001718, A001722, A001723, A001724, A001801, A004747, A008297, A013988, A021009, A035342, A035469, A049029, A049385, A049444, A049458, A049459, A049600, A051338, A051339, A051379, A051523, A051524, A051525, A051545, A051546, A051560, A051561, A051562, A051563, A051564, A051565, A059343, A072019, A072020, A084358, A092082, A094587, A105278, A111596, A122850, A132013, A132014, A132056, A132062, A132159, A132681, A132710, A132792, A136215, A136656.)

Received August 4 2009; revised version received November 19 2009. Published in *Journal of Integer Sequences*, November 25 2009. Minor correction, January 29 2010.

Return to Journal of Integer Sequences home page.