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Integer Solutions of Some Diophantine Equations via Fibonacci and Lucas Numbers

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Abstract

We study the problem of finding all integer solutions of the Diophantine equations $x^2 - 5F_nxy - 5\left(-1\right)^n y^2 = \pm L_n^2$, $x^2 - L_nxy + \left(-1\right)^n y^2 = \pm 5F_n^2$, and $x^2 - L_nxy + \left(-1\right)^n y^2 = \pm F_n^2$. Using these equations, we also explore all integer solutions of some other Diophantine equations.

1 Introduction

In this paper we study some Diophantine equations involving the well-known Fibonacci and Lucas sequences, which are defined as follows; $F_0 = 0, F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

Both the Fibonacci and Lucas sequences may be extended backwards, i.e., $F_{-1} = F_1 - F_0$, $F_{-2} = F_0 - F_{-1}, \ldots$, $L_{-1} = L_1 - L_0$, $L_{-2} = L_0 - L_{-1}$ and so on. In general for $n > 0$, we set $F_{-n} = \left(-1\right)^{n+1} F_n$ and $L_{-n} = \left(-1\right)^n L_n$ [2, 12].

Fibonacci and Lucas numbers possess many interesting and important properties. To begin with, we shall give some of them. Perhaps the most important one is Binet's formula, which allows one to compute F_n directly without computing the previous Fibonacci numbers. Binet's formula is obtained by solving the following quadratic equation for x:

$$x^2 - x - 1 = 0. \tag{1}$$

The two solutions of (1) are $\alpha = \left(1 + \sqrt{5}\right)/2$ and $\beta = \left(1 - \sqrt{5}\right)/2$. Clearly $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$ and $\alpha\beta = -1$. So, for Fibonacci numbers Binet's formula is given by $F_n =$

$(\alpha^n - \beta^n) / (\alpha - \beta)$ with $n \in \mathbb{Z}$. Also the corresponding formula for Lucas numbers is given by $L_n = \alpha^n + \beta^n$ with $n \in \mathbb{Z}$. In addition to the above formulas, $\alpha^n = \alpha F_n + F_{n-1}$ and $\beta^n = \beta F_n + F_{n-1}$ for all $n \in \mathbb{Z}$.

Now we compile some identities and basic theorems involving Fibonacci and Lucas numbers from various sources to use in the following theorems [2, 6, 11].

The first identity is

$$F_n^2 - F_n F_{n-1} - F_{n-1}^2 = (-1)^{n+1} \text{ for all } n \in \mathbb{Z} \tag{2}$$

and is known as Cassini's identity. A similar identity is given for Lucas numbers as

$$L_n^2 - L_n L_{n-1} - L_{n-1}^2 = (-1)^n 5 \text{ for all } n \in \mathbb{Z}. \tag{3}$$

The other identities can be listed as follows:

$$L_m F_n - F_m L_n = 2 (-1)^m F_{n-m} \tag{4}$$

$$L_m L_n - 5 F_m F_n = 2 (-1)^m L_{n-m} \tag{5}$$

$$F_{m+1} L_n + L_{n-1} F_m = L_{n+m} \tag{6}$$

$$L_m L_n + 5 F_m F_n = 2 L_{n+m} \tag{7}$$

$$L_m F_n + F_m L_n = 2 F_{n+m} \tag{8}$$

$$F_{n-1} + F_{n+1} = L_n \tag{9}$$

$$L_{n-1} + L_{n+1} = 5 F_n \tag{10}$$

$$L_n^2 - 5 F_n^2 = (-1)^n 4 \tag{11}$$

for all $m, n \in \mathbb{Z}$.

We will give the following theorem without proof since it can be found in [3].

Theorem 1. *The set of units of the ring* $\mathbb{Z}[\alpha] = \{a\alpha + b : a, b \in \mathbb{Z}\}$ *is*

$$\{\pm \alpha^n : n \in \mathbb{Z}\}.$$

The proof of the following theorem can be found in [11], but for the sake of completeness we will give its proof.

Theorem 2. *All integer solutions of the equation* $x^2 - xy - y^2 = \pm 1$ *are given by* $(x, y) = \pm (F_n, F_{n-1})$ *with* $n \in \mathbb{Z}$.

Proof. If $(x, y) = \pm (F_n, F_{n-1})$, it is clear from (2) that $x^2 - xy - y^2 = \pm 1$. Assume that $x^2 - xy - y^2 = \pm 1$ for some integer x and y. Then it is seen that $\pm 1 = \alpha\beta x^2 + (\alpha + \beta) xy + y^2 = (\alpha x + y)(\beta x + y)$. Hence, it follows that $\alpha x + y$ is a unit in $\mathbb{Z}[\alpha]$. Thus from Theorem 1, we have $\alpha x + y = \pm \alpha^n$ for some $n \in \mathbb{Z}$. Since $\alpha^n = \alpha F_n + F_{n-1}$, we get $\alpha x + y = \pm (\alpha F_n + F_{n-1})$. Therefore it is seen that $(x, y) = \pm (F_n, F_{n-1})$. $\square$

Corollary 3. *All integer solutions of the equations $x^2 - xy - y^2 = -1$ and $x^2 - xy - y^2 = 1$ are given by $(x, y) = \pm (F_{2n}, F_{2n-1})$ and $(x, y) = \pm (F_{2n+1}, F_{2n})$ with $n \in \mathbb{Z}$, respectively.*

Since the proof of the following theorem can be seen easily we omit its proof.

Theorem 4. *All nonnegative integer solutions of the equation $x^2 - xy - y^2 = \pm 1$ are given by $(x, y) = (F_n, F_{n-1})$ with $n \geq 0$.*

Corollary 5. *All nonnegative integer solutions of the equations $x^2 - xy - y^2 = -1$ and $x^2 - xy - y^2 = 1$ are given by $(x, y) = (F_{2n}, F_{2n-1})$ and $(x, y) = (F_{2n+1}, F_{2n})$ with $n \geq 0$, respectively.*

Theorem 6. *All nonnegative integer solutions of the equation $u^2 - 5v^2 = \pm 4$ are given by $(u, v) = (L_n, F_n)$ with $n \geq 0$.*

Proof. It is clear that if $(u, v) = (L_n, F_n)$, then by (11) we get $u^2 - 5v^2 = \pm 4$. Assume that $u^2 - 5v^2 = \pm 4$. Then u and v have the same parity. Let $x = (u + v)/2$ and $y = v$. Then it follows that

$$x^2 - xy - y^2 = ((u + v)/2)^2 - ((u + v)/2)\, v - v^2 = \left(u^2 - 5v^2\right)/4 = \pm 1.$$

From Theorem 4, it is seen that $(x, y) = (F_{n+1}, F_n)$ for some $n \geq 0$. Thus, $(u + v)/2 = F_{n+1}$ and $y = F_n$. Therefore we get $(u, v) = (L_n, F_n)$. $\square$

Corollary 7. *All nonnegative integer solutions of the equations $u^2 - 5v^2 = -4$ and $u^2 - 5v^2 = 4$ are given by $(u, v) = (L_{2n+1}, F_{2n+1})$ and $(u, v) = (L_{2n}, F_{2n})$ with $n \geq 0$, respectively.*

2 Identities And Solutions of Some Diophantine Equations

There are various methods for deriving identities for Fibonacci and Lucas numbers, such as the use of Binet's formula, induction, matrices, etc. The usage of matrices enables us to obtain easily a large number of new identities [6].

In this section we introduce three kinds of matrices including Fibonacci and Lucas numbers. Also using the identities given in section 1, we derive some new identities for Fibonacci and Lucas numbers. Let us give them in the following theorems.

Theorem 8. *Let $k, m, n \in \mathbb{Z}$. Then*

$$L_{n+m}^2 - 5(-1)^{n+k+1} F_{k-n} L_{n+m} F_{m+k} - 5\,(-1)^{n+k} F_{m+k}^2 = (-1)^{m+k} L_{k-n}^2.$$

Proof. For the proof of the theorem, by using (7) and (8) we can consider the matrix multiplication given below. That is,

$$\left[\begin{array}{cc} L_n/2 & 5F_n/2 \\ F_k/2 & L_k/2 \end{array}\right]\left[\begin{array}{c} L_m \\ F_m \end{array}\right]=\left[\begin{array}{c} L_{n+m} \\ F_{m+k} \end{array}\right].$$

By (5),

$$\left|\begin{array}{cc} L_n/2 & 5F_n/2 \\ F_k/2 & L_k/2 \end{array}\right|=\frac{L_nL_k-5F_nF_k}{4}=\frac{(-1)^n L_{k-n}}{2}\neq 0,$$

and therefore we can write,

$$\left[\begin{array}{c} L_m \\ F_m \end{array}\right]=\left[\begin{array}{cc} L_n/2 & 5F_n/2 \\ F_k/2 & L_k/2 \end{array}\right]^{-1}\left[\begin{array}{c} L_{n+m} \\ F_{m+k} \end{array}\right].$$

From here we get,

$$\begin{aligned} L_m &= \frac{(-1)^n \left(L_kL_{n+m}-5F_nF_{m+k}\right)}{L_{k-n}} \\ F_m &= \frac{(-1)^n \left(L_nF_{m+k}-F_kL_{n+m}\right)}{L_{k-n}}. \end{aligned}$$

Since $L_m^2-5F_m^2=(-1)^m 4$, we get

$$\left(L_kL_{n+m}-5F_nF_{m+k}\right)^2-5\left(L_nF_{m+k}-F_kL_{n+m}\right)^2=(-1)^m 4L_{k-n}^2.$$

By using (4) and (11), we obtain

$$L_{n+m}^2-5(-1)^{n+k+1}F_{k-n}L_{n+m}F_{m+k}-5\,(-1)^{n+k}F_{m+k}^2=(-1)^{m+k}L_{k-n}^2. \tag{12}$$

□

Theorem 9. *Let $k,m,n\in\mathbb{Z}$ and $k\neq n$. Then*

$$L_{n+m}^2-(-1)^{k+n}L_{k-n}L_{n+m}L_{m+k}+(-1)^{n+k}L_{m+k}^2=(-1)^{m+k+1}5F_{k-n}^2.$$

Proof. By using (7) we can consider the following matrix multiplication for the proof of the theorem. That is,

$$\left[\begin{array}{cc} L_n/2 & 5F_n/2 \\ L_k/2 & 5F_k/2 \end{array}\right]\left[\begin{array}{c} L_m \\ F_m \end{array}\right]=\left[\begin{array}{c} L_{n+m} \\ L_{m+k} \end{array}\right].$$

By (4),

$$\left|\begin{array}{cc} L_n/2 & 5F_n/2 \\ L_k/2 & 5F_k/2 \end{array}\right|=\frac{5\left(L_nF_k-L_kF_n\right)}{4}=\frac{5\,(-1)^n F_{k-n}}{2}$$

and therefore for $k\neq n$, we get

$$\left[\begin{array}{c} L_m \\ F_m \end{array}\right]=\left[\begin{array}{cc} L_n/2 & 5F_n/2 \\ L_k/2 & 5F_k/2 \end{array}\right]^{-1}\left[\begin{array}{c} L_{n+m} \\ L_{m+k} \end{array}\right].$$

Hence we have

$$\begin{aligned} L_m &= \frac{(-1)^n (F_k L_{n+m} - F_n L_{m+k})}{F_{k-n}} \\ F_m &= \frac{(-1)^n (L_n L_{m+k} - L_k L_{n+m})}{5F_{k-n}}. \end{aligned}$$

Since $L_m^2 - 5F_m^2 = (-1)^m 4$, we get

$$5 (F_k L_{n+m} - F_n L_{m+k})^2 - (L_n L_{m+k} - L_k L_{n+m})^2 = (-1)^m 20F_{k-n}^2.$$

Using (5) and (11), we obtain

$$L_{n+m}^2 - (-1)^{k+n} L_{k-n} L_{n+m} L_{m+k} + (-1)^{n+k} L_{m+k}^2 = (-1)^{m+k+1} 5F_{k-n}^2. \tag{13}$$

□

Using

$$\begin{bmatrix} F_n/2 & L_n/2 \\ F_k/2 & L_k/2 \end{bmatrix} \begin{bmatrix} L_m \\ F_m \end{bmatrix} = \begin{bmatrix} F_{n+m} \\ F_{m+k} \end{bmatrix}$$

and (11) we can give the following theorem.

Theorem 10. *Let $k, m, n \in \mathbb{Z}$ and $k \neq n$. Then*

$$F_{n+m}^2 - L_{n-k} F_{n+m} F_{m+k} + (-1)^{n+k} F_{m+k}^2 = (-1)^{m+k} F_{n-k}^2. \tag{14}$$

The equations given in Theorem 8, Theorem 9, and Theorem 10 induced us to explore the solutions of Diophantine equations;

$$x^2 - 5F_n xy - 5 (-1)^n y^2 = \pm L_n^2$$

$$x^2 - L_n xy + (-1)^n y^2 = \pm 5F_n^2$$
$$x^2 - L_n xy + (-1)^n y^2 = \pm F_n^2,$$

where $n \geq 1$ is an integer. Our aim is to show that, the solutions of these equations are pairs of Fibonacci or Lucas numbers. Let us give the solutions of these equations in the following theorems. From now on we will assume that n is an integer greater than zero.

Theorem 11. *All integer solutions of the equations $x^2 - 5F_n xy - 5 (-1)^n y^2 = -L_n^2$ and $x^2 - 5F_n xy - 5 (-1)^n y^2 = L_n^2$ are given by $(x, y) = \pm (L_{n+m}, F_m)$ for some odd integer m and for some even integer m, respectively.*

Proof. Assume that $x^2 - 5F_nxy - 5(-1)^n y^2 = -L_n^2$. Then

$$(2x - 5F_ny)^2 - \left(25F_n^2 + 20(-1)^n\right) y^2 = -4L_n^2.$$

Using (11) we get $(2x - 5F_ny)^2 - 5L_n^2y^2 = -4L_n^2$. From here it follows that $L_n|2x - 5F_ny$. Therefore taking

$$u = (((2x - 5F_ny)/L_n) + y)/2 = (x - L_{n-1}y)/L_n \text{ and } v = y$$

we get,

$$\begin{aligned} u^2 - uv - v^2 &= ((x - L_{n-1}y)/L_n)^2 - ((x - L_{n-1}y)/L_n)y - y^2 \\ &= \left(x^2 - (L_{n-1} + L_{n+1})xy - y^2\left(L_n^2 - L_nL_{n-1} - L_{n-1}^2\right)\right)/L_n^2 \end{aligned}$$

and using the identities (3) and (11), we obtain

$$u^2 - uv - v^2 = \left(x^2 - 5F_nxy - 5(-1)^n y^2\right)/L_n^2 = -L_n^2/L_n^2 = -1.$$

Then from Corollary 3, it follows that $(u, v) = \pm(F_{m+1}, F_m)$ for some odd integer m. Hence we get

$$(x - L_{n-1}y)/L_n = \pm F_{m+1} \text{ and } y = \pm F_m.$$

Then we have $x = \pm(F_{m+1}L_n + L_{n-1}F_m)$ and $y = \pm F_m$. Using (6) we get

$$(x, y) = \pm(L_{n+m}, F_m)$$

for some odd integer m.

Now assume that $x^2 - 5F_nxy - 5(-1)^n y^2 = L_n^2$. Then similarly, taking

$$u = (x - L_{n-1}y)/L_n \text{ and } v = y,$$

we get $u^2 - uv - v^2 = 1$. From Corollary 3, it follows that $(u, v) = \pm(F_{m+1}, F_m)$ for some even integer m. Hence we get $(x - L_{n-1}y)/L_n = \pm F_{m+1}$ and $y = \pm F_m$. Using (6) we get $(x, y) = \pm(L_{n+m}, F_m)$ for some even integer m.

Conversely, if $(x, y) = \pm(L_{n+m}, F_m)$ for some odd integer m, then by (12) it follows that $x^2 - 5F_nxy - 5(-1)^n y^2 = -L_n^2$ and if $(x, y) = \pm(L_{n+m}, F_m)$ for some even integer m, then by (12) it follows that $x^2 - 5F_nxy - 5(-1)^n y^2 = L_n^2$. □

Theorem 12. *All integer solutions of both equations $x^2 - L_nxy + (-1)^n y^2 = -5F_n^2$ and $x^2 - L_nxy + (-1)^n y^2 = 5F_n^2$ are given by $(x, y) = \pm(L_{n+m}, L_m)$ for some even integer m and for some odd integer m, respectively.*

Proof. Assume that $x^2 - L_nxy + (-1)^n y^2 = -5F_n^2$. Then multiplying both sides of the equation by 4 and using the identity (11) we get $(2x - L_ny)^2 - 5F_n^2y^2 = -20F_n^2$. From here it follows that $5F_n|2x - L_ny$. Similarly it is seen that $5F_n|x + L_{n-1}y$. Then taking $u = (x + L_{n-1}y)/5F_n$ and $v = (2x - L_ny)/5F_n$, we see that

$$u^2 - uv - v^2 = -5\left(x^2 - L_nxy + (-1)^n y^2\right)/25F_n^2 = 1.$$

Then it follows from Corollary 3 that $(u,v) = \pm(F_{m+1}, F_m)$ for some even integer m. Thus, $(x + L_{n-1}y)/5F_n = \pm F_{m+1}$ and $(2x - L_n y)/5F_n = \pm F_m$. Using (6), (9), and (10), we get $(x,y) = \pm(L_{n+m}, L_m)$ for some even integer m.

Assume that $x^2 - L_n xy + (-1)^n y^2 = 5F_n^2$. Then, in a similar way it is seen that $(x,y) = \pm(L_{n+m}, L_m)$ for some odd integer m.

Conversely, if $(x,y) = \pm(L_{n+m}, L_m)$ for some even integer m, then by (13) it follows that $x^2 - L_n xy + (-1)^n y^2 = -5F_n^2$ and if $(x,y) = \pm(L_{n+m}, L_m)$ for some odd integer m, then by (13) it follows that $x^2 - L_n xy + (-1)^n y^2 = 5F_n^2$. □

Theorem 13. *All integer solutions of the equations* $x^2 - L_n xy + (-1)^n y^2 = -F_n^2$ *and* $x^2 - L_n xy + (-1)^n y^2 = F_n^2$ *are given by* $(x,y) = \pm(F_{n+m}, F_m)$ *for some odd integer* m *and for some even integer* m*, respectively.*

Proof. Assume that $x^2 - L_n xy + (-1)^n y^2 = -F_n^2$. Then we get $(2x - L_n y)^2 - 5F_n^2 y^2 = -4F_n^2$. It is seen that $F_n | 2x - L_n y$. It follows that $((2x - L_n y)/F_n)^2 - 5y^2 = -4$. Thus taking $u = (((2x - L_n y)/F_n) + y)/2 = (x - F_{n-1})\, y/F_n$ and $v = y$ we get $u^2 - uv - v^2 = -1$. Therefore from Corollary 3 we have $(u,v) = \pm(F_{m+1}, F_m)$ for some odd integer m. Then it follows that $(x,y) = \pm(F_{n+m}, F_m)$ for some odd integer m.

Assume that $x^2 - L_n xy + (-1)^n y^2 = F_n^2$. Following the same process with the solution of the above equation it can be seen that $(x,y) = \pm(F_{n+m}, F_m)$ for some even integer m.

Conversely, if $(x,y) = \pm(F_{n+m}, F_m)$ for some odd integer m, then by (14) it follows that $x^2 - L_n xy + (-1)^n y^2 = -F_n^2$ and if $(x,y) = \pm(F_{n+m}, F_m)$ for some even integer m, then by (14) it follows that $x^2 - L_n xy + (-1)^n y^2 = F_n^2$. □

Now we recall some divisibility properties of Fibonacci and Lucas numbers in our interjection of solutions of Diophantine equations. These divisibility properties are given in several sources such as [1, 6, 11]. Also we gave different proofs of the following theorems in [5]. So, now we give them without proof.

Theorem 14. *Let* $m, n \in \mathbb{Z}$ *and* $n \geq 3$*. Then* $F_n | F_m$ *if and only if* $n | m$*.*

Theorem 15. *Let* $m, n \in \mathbb{Z}$ *and* $n \geq 2$*. Then* $L_n | F_m$ *if and only if* $n | m$ *and* m/n *is an even integer.*

Theorem 16. *Let* $m, n \in \mathbb{Z}$ *and* $n \geq 2$*. Then* $L_n | L_m$ *if and only if* $n | m$ *and* m/n *is an odd integer.*

By these theorems the following identities can be obtained easily;

$$2 | L_n \text{ if and only if } 3 | n \tag{15}$$

$$3 | L_n \text{ if and only if } n = 4k + 2 \text{ for some } k \in \mathbb{Z}. \tag{16}$$

Now we turn to our problem of finding the solutions of different Diophantine equations benefiting from Theorem 11, Theorem 12, and Theorem 13.

Theorem 17. *If $n \geq 3$ is an odd integer, then all integer solutions of the equations $x^2 - L_n xy - y^2 = -1$ and $x^2 - L_n xy - y^2 = 1$ are given by $(x, y) = \pm\left(F_{(2k+2)n}/F_n, F_{(2k+1)n}/F_n\right)$ and $(x, y) = \pm\left(F_{(2k+1)n}/F_n, F_{2kn}/F_n\right)$ with $k \in \mathbb{Z}$, respectively. If n is an even integer, then all integer solutions of the equation $x^2 - L_n xy + y^2 = 1$ are given by $(x, y) = \pm\left(F_{(k+1)n}/F_n, F_{kn}/F_n\right)$ with $k \in \mathbb{Z}$.*

Proof. Assume that $x^2 - L_n xy - y^2 = -1$. Multiplying both sides of the equation by F_n^2, we get

$$(F_n x)^2 - L_n (F_n x)(F_n y) - (F_n y)^2 = -F_n^2.$$

From Theorem 13, we obtain $F_n x = \pm F_{n+m}$ and $F_n y = \pm F_m$ for some odd integer m. Then it follows that $x = \pm F_{n+m}/F_n$ and $y = \pm F_m/F_n$ for some odd integer m. By Theorem 14, it is known that for $n \geq 3$, $F_n | F_m$ if and only if $n|m$. Since $n|m$ and m is an odd integer we get $m = (2k+1)n$, for some $k \in \mathbb{Z}$. Thus we obtain

$$(x, y) = \pm\left(F_{(2k+2)n}/F_n, F_{(2k+1)n}/F_n\right).$$

If n is an odd integer, then similarly it can be seen that all integer solutions of the equation $x^2 - L_n xy - y^2 = 1$ are given by $(x, y) = \pm\left(F_{(2k+1)n}/F_n, F_{2kn}/F_n\right)$ with $k \in \mathbb{Z}$.

Assume that $x^2 - L_n xy + y^2 = 1$ and n is an even integer. From Theorem 13, we obtain $(x, y) = \pm(F_{n+m}/F_n, F_m/F_n)$ for some even integer m. If $n = 2$, then $F_2 = 1$ and by Theorem 13 we get $(x, y) = \pm(F_{m+2}, F_m)$ for some even integer m. If $n > 2$, then by Theorem 14, it follows that $n|m$. Since n and m are even, we get $m = kn$ for some $k \in \mathbb{Z}$. Therefore we obtain

$$(x, y) = \pm\left(F_{(k+1)n}/F_n, F_{kn}/F_n\right).$$

Conversely, if $n \geq 3$ is an odd integer and $(x, y) = \pm\left(F_{(2k+2)n}/F_n, F_{(2k+1)n}/F_n\right)$ for some $k \in \mathbb{Z}$, then by (14) it follows that $x^2 - L_n xy - y^2 = -1$ and if n is an odd integer and $(x, y) = \pm\left(F_{(2k+1)n}/F_n, F_{2kn}/F_n\right)$ for some $k \in \mathbb{Z}$, then by (14) it follows that $x^2 - L_n xy - y^2 = 1$. Furthermore, if $n > 2$ is an even integer and $(x, y) = \pm\left(F_{(k+1)n}/F_n, F_{kn}/F_n\right)$ for some $k \in \mathbb{Z}$, then by (14) it follows that $x^2 - L_n xy + y^2 = 1$. □

Corollary 18. *If n is an even integer greater than 2, then the equation $x^2 - L_n xy + y^2 = -1$ has no integer solutions.*

Proof. Let $x^2 - L_n xy + y^2 = -1$. Then we get $(x, y) = \pm(F_{n+m}/F_n, F_m/F_n)$ for some odd integer m. Using Theorem 14, it can be seen that (x, y) is a pair of integer if and only if $n|m$. But the fact that n is even and m is odd gives a contradiction. Therefore $x^2 - L_n xy + y^2 = -1$ has no integer solutions. □

Theorem 19. *All integer solutions of the equation $x^2 - 5F_n xy - 5(-1)^n y^2 = 1$ are given by $(x, y) = \pm\left(L_{(2k+1)n}/L_n, F_{2kn}/L_n\right)$ with $k \in \mathbb{Z}$.*

Proof. Assume that $x^2 - 5F_n xy - 5(-1)^n y^2 = 1$. Multiplying both sides of the equation by L_n^2, we get

$$(L_n x)^2 - 5F_n (L_n x)(L_n y) - 5(-1)^n (L_n y)^2 = L_n^2.$$

From Theorem 11, it is seen that $L_n x = \pm L_{n+m}$ and $L_n y = \pm F_m$ for some even integer m. That is, $(x, y) = \pm(L_{n+m}/L_n, F_m/L_n)$. By Theorem 15 and Theorem 16, it follows that m/n is an even integer. So that, $m = 2kn$, for some $k \in \mathbb{Z}$. Therefore we obtain $(x, y) = \pm \left(L_{(2k+1)n}/L_n, F_{2kn}/L_n\right)$.

Conversely, if $(x, y) = \pm \left(L_{(2k+1)n}/L_n, F_{2kn}/L_n\right)$ for some $k \in \mathbb{Z}$, then by (12) it follows that $x^2 - 5F_n xy - 5(-1)^n y^2 = 1$. $\square$

Corollary 20. *If $n > 1$, then the equation $x^2 - 5F_n xy - 5(-1)^n y^2 = -1$ has no integer solutions.*

Theorem 21. *All integer solutions of the equation $x^2 - L_n xy + (-1)^n y^2 = -5$ and $x^2 - L_n xy + (-1)^n y^2 = 5$ are given by $(x, y) = \pm (L_{n+m}/F_n, L_m/F_n)$ with even integer m and $(x, y) = \pm (L_{n+m}/F_n, L_m/F_n)$ with odd integer m, respectively, where $F_n \mid L_m$. In particular, $F_n \mid L_m$ if and only if $n = 1$, $F_n = 1$, $m = k$; $n = 2$, $F_n = 1$, $m = k$; $n = 3$, $F_n = 2$, $m = 3k$; $n = 4$, $F_n = 3$, $m = 4k + 2$, where k is an integer.*

Proof. The results concerning when $F_n \mid L_m$ are given in Theorem 1 of [4]. Assume that $x^2 - L_n xy + (-1)^n y^2 = -5$ for some integer x and y. Multiplying this equation by F_n^2, we get

$$(F_n x)^2 - L_n (F_n x)(F_n y) + (-1)^n (F_n y)^2 = -5F_n^2.$$

From Theorem 12, it follows that $(x, y) = \pm (L_{n+m}/F_n, L_m/F_n)$ for some even integer m. Similarly it can be seen that all integer solutions of the equation $x^2 - L_n xy + (-1)^n y^2 = 5$ are given by $(x, y) = \pm (L_{n+m}/F_n, L_m/F_n)$ for some odd integer m.

Conversely, if $(x, y) = \pm (L_{n+m}/F_n, L_m/F_n)$ for some even integer m, then by (13) it follows that $x^2 - L_n xy + (-1)^n y^2 = -5$ and if $(x, y) = \pm (L_{n+m}/F_n, L_m/F_n)$ for some odd integer m, then by (13) it follows that $x^2 - L_n xy + (-1)^n y^2 = 5$. $\square$

Theorem 22. *If n is an even integer, then all integer solutions of the equation $x^2 - L_{2n} xy + y^2 = -5F_n^2$ are given by $(x, y) = \pm \left(L_{(2k+3)n}/L_n, L_{(2k+1)n}/L_n\right)$ with $k \in \mathbb{Z}$, if n is an odd integer, then all integer solutions of the equation $x^2 - L_{2n} xy + y^2 = 5F_n^2$ are given by $(x, y) = \pm \left(L_{(2k+3)n}/L_n, L_{(2k+1)n}/L_n\right)$ with $k \in \mathbb{Z}$.*

Proof. Assume that n is an even integer and $x^2 - L_{2n} xy + y^2 = -5F_n^2$ for some integer x and y. Then multiplying this equation by L_n^2 and using Theorem 12 and noting that $F_{2n} = F_n L_n$, we get $L_n x = \pm L_{2n+m}$ and $L_n y = \pm L_m$, for some even integer m. Therefore it follows that $(x, y) = \pm (L_{2n+m}/L_n, L_m/L_n)$ for some even integer m. Furthermore by Theorem 16, it is seen that $L_n | L_m$ if and only if m/n is an odd integer. Therefore $m = (2k + 1) n$, for some $k \in \mathbb{Z}$. Thus $(x, y) = \pm \left(L_{(2k+3)n}/L_n, L_{(2k+1)n}/L_n\right)$.

Assume that n is an odd integer and $x^2 - L_{2n} xy + y^2 = 5F_n^2$ for some integer x and y. Then by Theorem 12, we get $(x, y) = \pm (L_{2n+m}/L_n, L_m/L_n)$ for some odd integer m. Also by Theorem 16 it follows that $m = (2k + 1) n$ for some $k \in \mathbb{Z}$. So that, $(x, y) = \pm \left(L_{(2k+3)n}/L_n, L_{(2k+1)n}/L_n\right)$ with $k \in \mathbb{Z}$. If $n = 1$, then all integer solutions of the equation $x^2 - 3xy + y^2 = 5$ are given by $(x, y) = \pm (L_{m+2}, L_m)$ with odd integer m.

Conversely, if n is an even integer and $(x, y) = \pm \left(L_{(2k+3)n}/L_n, L_{(2k+1)n}/L_n\right)$ for some $k \in \mathbb{Z}$, then by (13) it follows that $x^2 - L_{2n} xy + y^2 = -5F_n^2$ and if $n > 1$ is an odd integer and $(x, y) = \pm \left(L_{(2k+3)n}/L_n, L_{(2k+1)n}/L_n\right)$ for some $k \in \mathbb{Z}$, then by (13) it follows that $x^2 - L_{2n} xy + y^2 = 5F_n^2$. $\square$

Corollary 23. *If $n > 1$ is an odd integer, then the equation $x^2 - L_{2n}xy + y^2 = -5F_n^2$ has no integer solutions and if n is an even integer, then the equation $x^2 - L_{2n}xy + y^2 = 5F_n^2$ has no integer solutions.*

Theorem 24. *All integer solutions of the equation $x^2 - L_{2n}xy + y^2 = F_n^2$ are given by $(x, y) = \pm \left(F_{(2k+2)n}/L_n, F_{2kn}/L_n\right)$ with $k \in \mathbb{Z}$.*

Proof. Assume that $n \geq 2$ and $x^2 - L_{2n}xy + y^2 = F_n^2$ for some integer x and y. Multiplying this equation by L_n^2 we get

$$(L_n x)^2 - L_{2n} (L_n x) (L_n y) + (L_n y)^2 = F_{2n}^2.$$

Then by Theorem 13 it follows that $(x, y) = \pm (F_{2n+m}/L_n, F_m/L_n)$ for some even integer m. Hence using Theorem 15 it is seen that $L_n | F_m$ if and only if m/n is an even integer. Then we have $m = 2kn$ for some $k \in \mathbb{Z}$. Therefore $(x, y) = \pm \left(F_{(2k+2)n}/L_n, F_{2kn}/L_n\right)$. If $n = 1$, then it can be seen that all integer solutions of the equation $x^2 - 3xy + y^2 = 1$ are given by $(x, y) = \pm (F_{m+2}, F_m)$ with even integer m.

Conversely, if $(x, y) = \pm \left(F_{(2k+2)n}/L_n, F_{2kn}/L_n\right)$ for some $k \in \mathbb{Z}$, then by (14) it follows that $x^2 - L_{2n}xy + y^2 = F_n^2$. $\square$

Theorem 25. *For all $n \geq 2$, the equation $x^2 - L_{2n}xy + y^2 = -F_n^2$ has no integer solutions.*

Proof. Assume that $x^2 - L_{2n}xy + y^2 = -F_n^2$ for some integer x and y. Then by Theorem 13, we get $(x, y) = \pm (F_{2n+m}/L_n, F_m/L_n)$ for some odd integer m. Using Theorem 15 it is seen that $L_n | F_m$ if and only if $m = 2kn$ for some $k \in \mathbb{Z}$. Since m is odd, this is impossible. Therefore $x^2 - L_{2n}xy + y^2 = -F_n^2$ has no integer solutions. $\square$

Theorem 26. *Let $n \geq 3$ be an odd integer. Then all integer solutions of the equation $x^2 - L_{2n}xy + y^2 = -L_n^2$ are given by $(x, y) = \pm \left(F_{(2k+3)n}/F_n, F_{(2k+1)n}/F_n\right)$ with $k \in \mathbb{Z}$.*

Proof. Assume that $x^2 - L_{2n}xy + y^2 = -L_n^2$ for some integer x and y. Then by Theorem 13, it is seen that $(x, y) = \pm (F_{2n+m}/F_n, F_m/F_n)$ for some odd integer m. Furthermore by Theorem 14 it follows that $F_n | F_m$ if and only if $n | m$. Since both m and n are odd integers it is seen that $m = (2k + 1) n$ for some $k \in \mathbb{Z}$. Then this shows that $(x, y) = \pm \left(F_{(2k+3)n}/F_n, F_{(2k+1)n}/F_n\right)$.

Conversely, if $n \geq 3$ is an odd integer and $(x, y) = \pm \left(F_{(2k+3)n}/F_n, F_{(2k+1)n}/F_n\right)$ for some $k \in \mathbb{Z}$, then by (14) it follows that $x^2 - L_{2n}xy + y^2 = -L_n^2$. $\square$

We can give the following corollary easily.

Corollary 27. *If n is an even integer, then the equation $x^2 - L_{2n}xy + y^2 = -L_n^2$ has no integer solutions.*

Corollary 28. *Let n be an even integer. Then all integer solutions of the equation $x^2 - L_{2n}xy + y^2 = L_n^2$ are given by $(x, y) = \pm \left(F_{(k+2)n}/F_n, F_{kn}/F_n\right)$ with $k \in \mathbb{Z}$. Let n be an odd integer. Then all integer solutions of the equation $x^2 - L_{2n}xy + y^2 = L_n^2$ are given by $(x, y) = \pm \left(F_{(2k+2)n}/F_n, F_{2kn}/F_n\right)$ with $k \in \mathbb{Z}$.*

Proof. Assume that n is an even integer and $x^2 - L_{2n}xy + y^2 = L_n^2$ for some integer x and y. Then by Theorem 13, it is seen that $(x, y) = \pm (F_{2n+m}/F_n, F_m/F_n)$ for some even integer m. Also by Theorem 14 it follows that $n|m$. Since both m and n are even integers we have $m = kn$ for some $k \in \mathbb{Z}$. Then it is seen that $(x, y) = \pm \left(F_{(k+2)n}/F_n, F_{kn}/F_n\right)$.

Now assume that n is an odd integer and $x^2 - L_{2n}xy + y^2 = L_n^2$ for some integer x and y. Then $(x, y) = \pm (F_{2n+m}/F_n, F_m/F_n)$ for some even integer m and by Theorem 14, $n|m$. Thus we get $m = 2kn$ for some $k \in \mathbb{Z}$. Hence it follows that $(x, y) = \pm \left(F_{(2k+2)n}/F_n, F_{2kn}/F_n\right)$. □

Theorem 29. *All integer solutions of the equation* $x^2 - L_{2n}xy + y^2 = -5L_n^2$ *and* $x^2 - L_{2n}xy + y^2 = 5L_n^2$ *are given by* $(x, y) = \pm (L_{2n+m}/F_n, L_m/F_n)$ *with even integer* m *and* $(x, y) = \pm (L_{2n+m}/F_n, L_m/F_n)$ *with odd integer* m*, respectively, where* $F_n \mid L_m$*. In particular,* $F_n \mid L_m$ *if and only if* $n = 1$*,* $F_n = 1$*,* $m = k$*;* $n = 2$*,* $F_n = 1$*,* $m = k$*;* $n = 3$*,* $F_n = 2$*,* $m = 3k$*;* $n = 4$*,* $F_n = 3$*,* $m = 4k + 2$*, where* k *is an integer.*

Proof. The results concerning when $F_n \mid L_m$ are given in Theorem 1 of [4]. Assume that $x^2 - L_{2n}xy + y^2 = -5L_n^2$. Then by Theorem 12, it is seen that $(x, y) = \pm (L_{2n+m}/F_n, L_m/F_n)$ for some even integer m.

Assume that $x^2 - L_{2n}xy + y^2 = 5L_n^2$. It follows that $(x, y) = \pm (L_{2n+m}/F_n, L_m/F_n)$ for some odd integer m, by Theorem 12.

Conversely, if $(x, y) = \pm (L_{2n+m}/F_n, L_m/F_n)$ for some even integer m, then by (13) it follows that $x^2 - L_{2n}xy + y^2 = -5L_n^2$ and if $(x, y) = \pm (L_{2n+m}/F_n, L_m/F_n)$ for some odd integer m, then by (13) it follows that $x^2 - L_{2n}xy + y^2 = 5L_n^2$. □

Furthermore we explore some Diophantine equations different from the previous ones. Here are some of them.

Theorem 30. *Let* $k \geq 0$*. Then all nonnegative integer solutions of the equation* $u^2 - 5v^2 = 4^k$ *are given by* $(u, v) = \left(2^{k-1}L_{2m}, 2^{k-1}F_{2m}\right)$ *and all nonnegative integer solutions of the equation* $u^2 - 5v^2 = -4^k$ *are given by* $(u, v) = \left(2^{k-1}L_{2m+1}, 2^{k-1}F_{2m+1}\right)$ *with* $m \geq 0$*.*

Proof. Since the proof is obvious from mathematical induction, we omit it. □

Theorem 31. *All nonnegative integer solutions of the equation* $u^2 - 5v^2 = 1$ *are given by* $(u, v) = (L_{6m}/2, F_{6m}/2)$ *with* $m \geq 0$ *and all nonnegative integer solutions of the equation* $u^2 - 5v^2 = -1$ *are given by* $(u, v) = (L_{6m+3}/2, F_{6m+3}/2)$ *with* $m \geq 0$*.*

Proof. Assume that $u^2 - 5v^2 = 1$. Then $\left(2^k u\right)^2 - 5\left(2^k v\right)^2 = 4^k$. From Theorem 30, it is seen that $2^k u = 2^{k-1}L_{2n}$ and $2^k v = 2^{k-1}F_{2n}$. Therefore we get $u = L_{2n}/2$ and $v = F_{2n}/2$. Using Theorem 14, it is seen that $2|F_{2n}$ if and only if $3|2n$. So that, $3|n$ and that is $n = 3m$ for some $m \in \mathbb{N}$. Hence we get $(u, v) = (L_{6m}/2, F_{6m}/2)$.

Now assume that $u^2 - 5v^2 = -1$. Then we get $\left(2^k u\right)^2 - 5\left(2^k v\right)^2 = -4^k$. From Theorem 30, it follows that $2^k u = 2^{k-1}L_{2n+1}$ and $2^k v = 2^{k-1}F_{2n+1}$. Therefore we get $u = L_{2n+1}/2$ and $v = F_{2n+1}/2$. By Theorem 14, it follows that $2|F_{2n+1}$ if and only if $3|2n + 1$. That is $n = 3m + 1$ for some $m \in \mathbb{N}$. Hence we get $(u, v) = (L_{6m+3}/2, F_{6m+3}/2)$. □

Theorem 32. *Let $k \geq 0$. Then all nonnegative integer solutions of the equation $x^2 - xy - y^2 = -4^k$ are given by $(x, y) = \left(2^k F_{2m+2}, 2^k F_{2m+1}\right)$ and all integer solutions of the equation $x^2 - xy - y^2 = 4^k$ are given by $(x, y) = \left(2^k F_{2m+1}, 2^k F_{2m}\right)$ with $m \geq 0$.*

Proof. Proof follows from mathematical induction. $\square$

Theorem 33. *Let $k \geq 0$. Then all nonnegative integer solutions of the equation $u^2 - 5v^2 = 4 \cdot 5^k$ are given by*

$$(u, v) = \begin{cases} \left(5^{(k+1)/2} F_{2m+1}, 5^{(k-1)/2} L_{2m+1}\right), & k \text{ is an odd integer;} \\ \left(5^{k/2} L_{2m}, 5^{k/2} F_{2m}\right), & k \text{ is an even integer} \end{cases}$$

and all nonnegative integer solutions of the equation $u^2 - 5v^2 = -4 \cdot 5^k$ are given by

$$(u, v) = \begin{cases} \left(5^{(k+1)/2} F_{2m}, 5^{(k-1)/2} L_{2m}\right), & k \text{ is an odd integer;} \\ \left(5^{k/2} L_{2m+1}, 5^{k/2} F_{2m+1}\right), & k \text{ is an even integer} \end{cases}$$

where $m \geq 0$.

Proof. Proof is obvious from mathematical induction. $\square$

Theorem 34. *Let $k \geq 0$. Then all nonnegative integer solutions of the equation $x^2 - xy - y^2 = 5^k$ are given by*

$$(x, y) = \begin{cases} \left(5^{(k-1)/2} L_{2m+2}, 5^{(k-1)/2} L_{2m+1}\right), & k \text{ is an odd integer;} \\ \left(5^{k/2} F_{2m+1}, 5^{k/2} F_{2m}\right), & k \text{ is an even integer} \end{cases}$$

and all nonnegative integer solutions of the equation $x^2 - xy - y^2 = -5^k$ are given by

$$(x, y) = \begin{cases} \left(5^{(k-1)/2} L_{2m+1}, 5^{(k-1)/2} L_{2m}\right), & k \text{ is an odd integer;} \\ \left(5^{k/2} F_{2m+2}, 5^{k/2} F_{2m+1}\right), & k \text{ is an even integer} \end{cases}$$

where $m \geq 0$.

Proof. Assume that $x^2 - xy - y^2 = 5^k$. Then $x > y$ and we get $(2x - y)^2 - 5y^2 = 4 \cdot 5^k$. By Theorem 33, we obtain

$$(2x - y, y) = \begin{cases} \left(5^{(k+1)/2} F_{2m+1}, 5^{(k-1)/2} L_{2m+1}\right), & k \text{ is an odd integer;} \\ \left(5^{k/2} L_{2m}, 5^{k/2} F_{2m}\right), & k \text{ is an even integer.} \end{cases}$$

Thus it follows that

$$(x, y) = \begin{cases} \left(5^{(k-1)/2} L_{2m+2}, 5^{(k-1)/2} L_{2m+1}\right), & k \text{ is an odd integer;} \\ \left(5^{k/2} F_{2m+1}, 5^{k/2} F_{2m}\right), & k \text{ is an even integer.} \end{cases}$$

In a similar way, it can be shown that all nonnegative integer solutions of the equation $x^2 - xy - y^2 = -5^k$ are given by

$$(x, y) = \begin{cases} \left(5^{(k-1)/2} L_{2m+1}, 5^{(k-1)/2} L_{2m}\right), & k \text{ is an odd integer;} \\ \left(5^{k/2} F_{2m+2}, 5^{k/2} F_{2m+1}\right), & k \text{ is an even integer.} \end{cases}$$

$\square$

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