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A New Identity for Complete Bell Polynomials Based on a Formula of Ramanujan

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Abstract

Let $p(n)$ be the number of partitions of n. In this paper, we give a new identity for complete Bell polynomials based on a sequence related to the generating function of $p(5n+4)$ established by Srinivasa Ramanujan.

1 Introduction

Let us first present some necessary definitions related to the Bell polynomials, which are quite general and have numerous applications in combinatorics. For a more complete exposition, the reader is referred to the excellent books of Comtet [4], Riordan [6] and Stanley [9].

Let $(a_1, a_2, \ldots)$ be a sequence of real or complex numbers. Its partial (exponential) Bell polynomial $B_{n,k}(a_1, a_2, \ldots)$, is defined as follows:

$$\sum_{n=k}^{\infty} B_{n,k}\left(a_1, a_2, \ldots\right) \frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{m=1}^{\infty} a_m \frac{t^m}{m!}\right)^k.$$

Their exact expression is

$$B_{n,k}\left(a_1, a_2, \ldots\right) = \sum_{\pi(n,k)} \frac{n!}{k_1!k_2!\cdots}\left(\frac{a_1}{1!}\right)^{k_1}\left(\frac{a_2}{2!}\right)^{k_2}\cdots,$$

where $\pi\left(n,k\right)$ denotes the set of all integer solutions $(k_1, k_2, \ldots)$ of the system

$$\begin{cases} k_1 + \cdots + k_j + \cdots = k; \\ k_1 + \cdots + jk_j + \cdots = n. \end{cases}$$

The (exponential) complete Bell polynomials are given by

$$\exp\left(\sum_{m=1}^{\infty} a_m \frac{t^m}{m!}\right) = \sum_{n=0}^{\infty} A_n(a_1, a_2, \ldots)\frac{t^n}{n!}.$$

In other words,

$$A_0(a_1, a_2, \ldots) = 1 \quad \text{and} \quad A_n(a_1, a_2, \ldots) = \sum_{k=1}^{n} B_{n,k}(a_1, a_2, \ldots), \forall n \geq 1.$$

Hence

$$A_n\left(a_1, a_2, \ldots\right) = \sum_{k_1+\ldots+jk_j+\ldots=n} \frac{n!}{k_1!k_2!\cdots}\left(\frac{a_1}{1!}\right)^{k_1}\left(\frac{a_2}{2!}\right)^{k_2}\cdots.$$

The main tool used to prove our main result in the next section is the following formula of Ramanujan, about which G. H. Hardy [5] said: "... but here Ramanujan must take second place to Prof. Rogers; and if I had to select one formula from all of Ramanujan's work, I would agree with Major MacMahon in selecting ...

$$\sum_{n=0}^{\infty} p(5n+4)\, x^n = \frac{5\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^6}, \tag{1}$$

where $p(n)$ is the number of partitions of n."

2 Some basic properties of the divisor function

Let $\sigma(n)$ be the sum of the positive divisors of n. It is clear that $\sigma(p) = 1 + p$ for any prime number p, since the only positive divisors of p are 1 and p. Also the only divisors of p^2 are 1, p and p^2. Thus

$$\sigma(p^2) = 1 + p + p^2 = \frac{p^3 - 1}{p - 1}\cdot$$

It is now easy to prove [8]

$$\sigma(p^k) = \frac{p^{k+1} - 1}{p - 1}\cdot \tag{2}$$

It is well known in number theory [8] that $\sigma(n)$ is a multiplicative function, that is, if n and m are relatively prime, then

$$\sigma(nm) = \sigma(n)\ \sigma(m). \tag{3}$$

An immediate consequence of these facts is the following Lemma:

Lemma 1. *If $5 \mid n$, then it exists $\alpha \geq 1$, so that*

$$\sigma(n) = \frac{5^{\alpha+1} - 1}{5^\alpha - 1}\ \sigma\left(\frac{n}{5}\right). \tag{4}$$

where α is the power to which 5 occur in the decomposition of n into prime factors.

Proof. From (2) and (3), we have

$$\begin{aligned} \sigma(n) &= \frac{5^{\alpha+1} - 1}{4}\ \sigma\left(\frac{n}{5^\alpha}\right), \text{ and} \\ \sigma\left(\frac{n}{5}\right) &= \frac{5^\alpha - 1}{4}\ \sigma\left(\frac{n}{5^\alpha}\right). \end{aligned}$$

Hence the result follows. □

3 Main result

Henceforth, let us express n by $n = 5^\alpha \cdot p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdots p_r^{\alpha_r}$, where the $p's$ are distinct primes different from 5, and the $\alpha's$ are the powers to which they occur.

The present theorem is the main result of this work.

Theorem 2. *Let $a(n)$ be the real number defined as follows:*

$$a_n = \left(1 + \frac{20}{5^{\alpha+1} - 1}\right) \frac{\sigma(n)}{n}\cdot$$

Then we have

$$A_n(1!a_1, 2!a_2, \ldots, n!a_n) = \frac{n!}{5}\ p(5n + 4)\cdot$$

Proof. Put

$$g(x) = \frac{5\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^6}.$$

Then

$$\begin{aligned} \ln(g(x)) &= \ln 5 + 5\ln \prod_{i=1}^{\infty}(1-x^{5i}) - 6\ln\prod_{i=1}^{\infty}(1-x^i) \\ &= \ln 5 + 5\sum_{i=1}^{\infty}\ln(1-x^{5i}) - 6\sum_{i=1}^{\infty}\ln(1-x^i) \\ &= \ln 5 - 5\sum_{i,j=1}^{\infty}\frac{x^{5ij}}{j} + 6\sum_{i,j=1}^{\infty}\frac{x^{ij}}{j} \\ &= \ln 5 + \sum_{n=1}^{\infty} a_n x^n, \end{aligned} \tag{5}$$

where

$$a_n = \begin{cases} \dfrac{6\ \sigma(n) - 25\ \sigma\left(\dfrac{n}{5}\right)}{n}, & if\ 5 \mid n; \\ \dfrac{6}{n}\ \sigma(n), & otherwise. \end{cases}$$

If $\alpha \geq 1$, i.e., $5 \mid n$, then we get by (4)

$$\sigma\left(\frac{n}{5}\right) = \frac{5^{\alpha}-1}{5^{\alpha+1}-1}\ \sigma(n).$$

Thus

$$a_n = \left(1 + \frac{20}{5^{\alpha+1}-1}\right)\frac{\sigma(n)}{n}, \forall \alpha \geq 0.$$

Hence, we obtain from (5)

$$\begin{aligned}
g(x) &= 5.\exp\left(\sum_{n=1}^{\infty} a_n x^n\right)\\
&= 5\left(1+\sum_{k=1}^{\infty}\frac{\left(\sum_{n=1}^{\infty} n!a_n\frac{x^n}{n!}\right)^k}{k!}\right)\\
&= 5+5\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} B_{n,k}(1!a_1,2!a_2,\ldots)\frac{x^n}{n!}\right)\\
&= 5+5\sum_{n=1}^{\infty} A_n(1!a_1,2!a_2,\ldots,n!a_n)\frac{x^n}{n!}\\
&= 5\sum_{n=0}^{\infty} A_n(1!a_1,\ldots,n!a_n)\frac{x^n}{n!}.
\end{aligned}$$

Therefore, by comparing coefficients of the two power series in (1), we finally get

$$A_n(1!a_1,2!a_2,\ldots,n!a_n)=\frac{n!}{5}\,p(5n+4),\quad \text{for}\quad n\geq 0.$$

□

Theorem 2 has the following Corollary.

Corollary 3. *For $n \geq 1$, we have*

$$\sigma(n)=\frac{5^{\alpha+1}-1}{5^{\alpha+1}+19}\cdot\frac{1}{(n-1)!}\ \sum_{j=1}^{n}(-1)^{j-1}(j-1)!\ B_{n,j}\left(\frac{1!}{5}p(9),\frac{2!}{5}p(14),\ldots\right).$$

Proof. This follows from the following inversion relation of Chaou and al [3]:

$$y_n=\sum_{k=1}^{n} B_{n,k}(x_1,x_2,\ldots)\Leftrightarrow x_n=\sum_{k=1}^{n}(-1)^{k-1}(k-1)!\ B_{n,k}(y_1,y_2,\ldots).$$

□

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