Journal of Integer Sequences, Vol. 12 (2009),
Article 09.8.4

Some Trigonometric Identities Involving Fibonacci and Lucas Numbers

Kh. Bibak and M. H. Shirdareh Haghighi
Department of Mathematics
Shiraz University
Shiraz 71454
Iran
khmath@gmail.com
shirdareh@susc.ac.ir

Abstract

In this paper, using the number of spanning trees in some classes of graphs, we prove the identities:

$$F_n = \frac{2^{n-1}}{n}\sqrt{\prod_{k=1}^{n-1}(1-\cos\frac{k\pi}{n}\cos\frac{3k\pi}{n})}, \quad n \geq 2,$$

$$\prod_{k=0}^{n-1}(1+4\sin^2\frac{k\pi}{n}) = L_{2n} - 2 = F_{2n+2} - F_{2n-2} - 2, \quad n \geq 1,$$

where F_n and L_n denote the Fibonacci and Lucas numbers, respectively. Also, we give a new proof for the identity:

$$F_n = \prod_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(1+4\sin^2\frac{k\pi}{n}) = \prod_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(1+4\cos^2\frac{k\pi}{n}), \quad n \geq 4.$$

1 Introduction

Let F_n and L_n denote the Fibonacci and Lucas numbers respectively. That is, $F_{n+2} = F_{n+1} + F_n$, for $n \geq 1$ with $F_1 = F_2 = 1$, and $L_{n+2} = L_{n+1} + L_n$, for $n \geq 1$ with $L_1 = 1$ and $L_2 = 3$.

In this paper, we derive the identities:

$$F_n = \frac{2^{n-1}}{n}\sqrt{\prod_{k=1}^{n-1}(1-\cos\frac{k\pi}{n}\cos\frac{3k\pi}{n})}, \quad n \geq 2, \tag{1}$$

$$\prod_{k=0}^{n-1}(1+4\sin^2\frac{k\pi}{n}) = L_{2n}-2 = F_{2n+2}-F_{2n-2}-2, \quad n \geq 1. \tag{2}$$

To prove identity (1), we apply the number of spanning trees in a special class of graphs known as circulant graphs. Identity (2) is derived from the number of spanning trees in a wheel.

Applying the same technique to a graph known as fan gives us a new proof for the following identity:

$$F_n = \prod_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(1+4\sin^2\frac{k\pi}{n}) = \prod_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(1+4\cos^2\frac{k\pi}{n}), \quad n \geq 4, \tag{3}$$

appeared in [6] and its corresponding references.

Also, applying this technique to the path P_n and the cycle C_n gives us a new proof for the well-known identities:

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}, \qquad n \geq 2, \tag{4}$$

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}, \qquad n \geq 2. \tag{5}$$

2 Techniques and Proofs

For a graph G, a *spanning tree* in G is a tree which has the same vertex set as G. The number of spanning trees in a graph (network) G, denoted by $t(G)$, is an important invariant of the graph (network). It is also an important measure of reliability of a network. In the sequel, we assume our graphs are loopless but multiple edges are allowed.

A famous and classic result on the study of $t(G)$ is the following theorem, known as the *Matrix-tree Theorem*. The *Laplacian matrix* of a graph G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ and $A(G)$ are the degree matrix and the adjacency matrix of G, respectively. Since this theorem is first proved by Kirchhoff [7], $L(G)$ is also known as the *Kirchhoff matrix* of the graph G.

Theorem 1. *For every connected graph G, $t(G)$ is equal to any cofactor of $L(G)$.*

The number of spanning trees of a connected graph G can be expressed in terms of the eigenvalues of $L(G)$. Since by definition, $L(G)$ is a real symmetric matrix, it therefore has n non-negative real eigenvalues, where n is the number of vertices of G. Anderson and Morley

[1, Theorem 1] proved that the multiplicity of 0 as an eigenvalue of $L(G)$ equals the number of components of G. Therefore, the Laplacian matrix of a connected graph G has 0 as an eigenvalue with multiplicity one.

Theorem 2. *([5]) Suppose G is a connected graph with n vertices. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $L(G)$, with $\lambda_n = 0$. Then $t(G) = \frac{1}{n}\lambda_1 \cdots \lambda_{n-1}$.*

As the first example, we prove identity (4).

Proof of identity (4). Consider the path P_n. It is known that the eigenvalues of the Laplacian matrix of P_n are $2 - 2\cos\frac{k\pi}{n}$ $(0 \le k \le n-1)$ (see, e.g., [4]). On the other hand, we know that $t(P_n) = 1$, therefore by using Theorem (2) we obtain (4). □

Now, we state some more definitions and theorems.

Definition 3. An $n \times n$ matrix $C = (c_{ij})$ is called a *circulant matrix* if its entries satisfy $c_{ij} = c_{1,\, j-i+1}$, where subscripts are reduced modulo n and lie in the set $\{1, 2, \ldots, n\}$.

Definition 4. Let $1 \le s_1 < s_2 < \cdots < s_k < \frac{n}{2}$, where n and s_i $(1 \le i \le k)$ are positive integers. An *undirected circulant graph* $C_n(s_1, s_2, \ldots, s_k)$ is a $2k$-regular graph with vertex set $V = \{0, 1, \ldots, n-1\}$ and edge set $E = \{\{i, i + s_j \pmod n\} \mid i = 0, 1, \ldots, n-1,\ j = 1, 2, \ldots, k\}$.

The Laplacian matrix of $C_n(s_1, s_2, \ldots, s_k)$ is clearly a circulant matrix. By a direct using of Theorem 4.8 of [12], we obtain the following lemma:

Lemma 5. *The nonzero eigenvalues of $L(C_n(s_1, s_2, \ldots, s_k))$ are*

$$2k - \omega^{s_1 j} - \cdots - \omega^{s_k j} - \omega^{-s_1 j} - \cdots - \omega^{-s_k j}, \;\; 1 \le j \le n-1,$$

where $\omega = e^{\frac{2\pi i}{n}}$.

With combining Theorem 2 and the lemma above, we obtain the following corollary:

Corollary 6. *The number of spanning trees in $G = C_n(s_1, s_2, \ldots, s_k)$ is equal to:*

$$t(G) = \frac{1}{n}\prod_{j=1}^{n-1}\Big(\sum_{i=1}^{k}(2 - 2\cos\frac{2js_i\pi}{n})\Big).$$

Proof of identity (1). Consider the *square cycle* $C_n(1,2)$. We can use Corollary 6 to obtain the number of spanning trees of $C_n(1,2)$. On the other hand, Kleitman and Golden [8] proved that $t(C_n(1,2)) = nF_n^2$. Now, with a little additional algebraic manipulation, identity (1) follows. □

Proof of identity (5). Look at the cycle $C_n(1) = C_n$. We know that $t(C_n) = n$, therefore by applying Corollary 6 to it, (5) follows. □

Definition 7. The join $W_n = C_n \bigvee K_1$ of a cycle C_n and a single vertex is referred to as a *wheel* with n *spokes*. Similarly, the join $\mathcal{F}_n = P_n \bigvee K_1$ of a path P_n and a single vertex is called a *fan*.

Sedlacek [11] and later Myers [10] showed that $t(W_n) = L_{2n} - 2 = F_{2n+2} - F_{2n-2} - 2,\ n \geq 1$. Also, Bibak and Shirdareh Haghighi [2, 3] proved that $t(\mathcal{F}_n) = F_{2n},\ n \geq 1$.

Now, we find the number of spanning trees in W_n and $\mathcal{F}_n$ by applying Theorem 2. We first need to determine the eigenvalues of $L(W_n)$ and $L(\mathcal{F}_n)$.

Theorem 8. *([9]) Let G_1 and G_2 be simple graphs on disjoint sets of r and s vertices, respectively. If $S(G_1) = (\mu_1, \ldots, \mu_r)$ and $S(G_2) = (\nu_1, \ldots, \nu_s)$ are the eigenvalues of $L(G_1)$ and $L(G_2)$ arranged in nonincreasing order, then the eigenvalues of $L(G_1 \bigvee G_2)$ are $n = r+s$; $\mu_1 + s, \ldots, \mu_{r-1} + s$; $\nu_1 + r, \ldots, \nu_{s-1} + r$; and 0.*

Since the eigenvalues of $L(C_n)$ are $2 - 2\cos\frac{2k\pi}{n}$ $(0 \leq k \leq n-1)$ (by Lemma 5), and the eigenvalues of $L(P_n)$ are $2 - 2\cos\frac{k\pi}{n}$ $(0 \leq k \leq n-1)$, therefore, by Theorem 8 we can determine the eigenvalues of $L(W_n)$ and $L(\mathcal{F}_n)$.

Theorem 9. *The eigenvalues of $L(W_n)$ are $n+1$, 0 and $1 + 4\sin^2\frac{k\pi}{n}$ $(1 \leq k \leq n-1)$, and the eigenvalues of $L(\mathcal{F}_n)$ are $n+1$, 0 and $1 + 4\sin^2\frac{k\pi}{2n}$ $(1 \leq k \leq n-1)$ (or $n+1$, 0 and $1 + 4\cos^2\frac{k\pi}{2n}$ $(1 \leq k \leq n-1)$).*

Proofs of the identities (2) and (3). By Theorems 2 and 9, the number of spanning trees of W_n and $\mathcal{F}_n$ are, respectively,

$$t(W_n) = \prod_{k=0}^{n-1}(1 + 4\sin^2\frac{k\pi}{n}), \quad n \geq 1,$$

$$t(\mathcal{F}_n) = \prod_{k=1}^{n-1}(1 + 4\sin^2\frac{k\pi}{2n}) = \prod_{k=1}^{n-1}(1 + 4\cos^2\frac{k\pi}{2n}), \quad n \geq 2.$$

On the other hand, as we already referred, $t(W_n) = L_{2n} - 2 = F_{2n+2} - F_{2n-2} - 2,\ n \geq 1$ and $t(\mathcal{F}_n) = F_{2n},\ n \geq 1$. Therefore, we obtain (2) and (3). $\square$

References

[1] W. N. Anderson and T. D. Morley, Eigenvalues of the Laplacian of a graph, *Linear Multilinear Algebra* **18** (1985), 141–145.

[2] Kh. Bibak and M. H. Shirdareh Haghighi, Recursive relations for the number of spanning trees, *Appl. Math. Sci.* **3** (2009), 2263–2269.

[3] Kh. Bibak and M. H. Shirdareh Haghighi, The number of spanning trees in some classes of graphs, *Rocky Mountain J. Math.*, to appear.

[4] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, 1989.

[5] D. Cvetkovič, M. Doob and H. Sachs, *Spectra of Graphs: Theory and Applications*, third ed., Johann Ambrosius Barth, 1995.

[6] N. Garnier and O. Ramaré, Fibonacci numbers and trigonometric identities, *Fibonacci Quart.* **46** (2008), 1–7.

[7] G. Kirchhoff, Über die Auflösung der gleichungen auf, welche man bei der untersuchung der linearen verteilung galvanischer Ströme geführt wird, *Ann. Phy. Chem.* **72** (1847), 497–508.

[8] D. J. Kleitman and B. Golden, Counting trees in a certain class of graphs, *Amer. Math. Monthly* **82** (1975), 40–44.

[9] R. Merris, Laplacian graph eigenvectors, *Linear Algebra Appl.* **278** (1998), 221–236.

[10] B. R. Myers, Number of spanning trees in a wheel, *IEEE Trans. Circuit Theory* **18** (1971), 280–282.

[11] J. Sedlacek, On the skeletons of a graph or digraph, *In Proc. Calgary International Conference on Combinatorial Structures and their Applications*, Gordon and Breach, 1970, pp. 387–391.

[12] F. Zhang, *Matrix Theory: Basic Results and Techniques*, Springer-Verlag, 1999.

2000 *Mathematics Subject Classification*: Primary 11B39, Secondary 05C05, 15A18.
Keywords: Fibonacci numbers, Lucas numbers, spanning tree, trigonometric identity.

(Concerned with sequences A000032 and A000045.)

Received November 16 2009; revised version received November 26 2009. Published in *Journal of Integer Sequences*, November 29 2009.

Return to Journal of Integer Sequences home page.