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Notes on a Family of Riordan Arrays and Associated Integer Hankel Transforms

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Abstract

We examine a set of special Riordan arrays, their inverses and associated Hankel transforms.

1 Introduction

In this note we explore the properties of a simply defined family of Riordan arrays [9]. The inverses of these arrays are closely related to well-known Catalan-defined matrices. This motivates us to study the Hankel transforms [6] of the images of some well-known families of sequences under the inverse matrices. This follows a general principle which states that the Hankel transform of the images of "simple" sequences under certain Catalan-defined matrices can themselves be "simple" in structure. We give several examples of this phenomenon in this note.

Special sequences will be referred to by their A-number in the On-Line Encyclopedia of Integer Sequences, [10].

2 The matrix $\mathbf{M}$

Throughout this note, we let the matrix $\mathbf{M}$ be the Riordan array A158454

$$\mathbf{M} = \left(\frac{1}{1-x^2}, \frac{x}{(1+x)^2}\right).$$

We shall also consider the related matrices

$$\tilde{\mathbf{M}} = \left(\frac{1}{1-x^2}, \frac{-x}{(1-x)^2}\right) \quad \text{and} \quad \mathbf{M}^+ = \left(\frac{1}{1-x^2}, \frac{x}{(1-x)^2}\right).$$

The matrix $\mathbf{M}$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 4 & -4 & 1 & 0 & 0 & \dots \\ 1 & -6 & 11 & -6 & 1 & 0 & \dots \\ 0 & 9 & -24 & 22 & -8 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

while $\tilde{\mathbf{M}}$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & -4 & 4 & -1 & 0 & 0 & \dots \\ 1 & -6 & 11 & -6 & 1 & 0 & \dots \\ 0 & -9 & 24 & -22 & 8 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The entries of $\mathbf{M}^+$ are the absolute value of entries of both these matrices. We can calculate the general term $M_{n,k}$ of $\mathbf{M}$ as follows.

$$\begin{aligned} M_{n,k} &= [x^n]\frac{1}{1-x^2}\frac{x^k}{(1+x^2)^k} \\ &= [x^{n-k}](1-x)^{-1}(1+x)^{-(2k+1)} \\ &= [x^{n-k}]\sum_{j=0}^{\infty} x^j \sum_{i=0}^{\infty}\binom{-(2k+1)}{i}x^i \\ &= [x^{n-k}]\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{2k+i}{i}(-1)^i x^{i+j} \\ &= \sum_{j=0}^{n-k}\binom{n+k-j}{n-k-j}(-1)^{n-k-j} \\ &= \sum_{j=0}^{n-k}\binom{n+k-j}{2k}(-1)^{n-k-j}. \end{aligned}$$

An alternative expression for $M_{n,k}$ can be obtained by noticing that

$$\mathbf{M} = \left(\frac{1}{1-x^2}, \frac{x}{(1+x)^2}\right) = \left(\frac{1}{1-x}, x\right) \cdot \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right).$$

The general term of $\left(\frac{1}{1-x}, x\right)$ is $[k \le n] \cdot 1$, while that of $\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$ is $(-1)^{n-k}\binom{n+k}{2k}$. Thus we obtain the alternative expression

$$M_{n,k} = \sum_{j=0}^{n} (-1)^{j-k}\binom{k+j}{2k}.$$

This translates the combinatorial identity

$$\sum_{j=0}^{n} (-1)^{j-k}\binom{k+j}{2k} = \sum_{j=0}^{n-k} \binom{n+k-j}{2k}(-1)^{n-k-j}.$$

Yet another expression for $M_{n,k}$ can be obtained by observing that we have the factorization

$$\mathbf{M} = \left(\frac{1}{1-x^2}, x\right) \cdot \left(1, \frac{x}{(1+x)^2}\right).$$

This leads to the expression

$$M_{n,k} = \sum_{j=0}^{n} \binom{j+k-1}{j-k}(-1)^{j-k}\frac{1+(-1)^{n-j}}{2}.$$

The row sums of $\mathbf{M}$ are the periodic sequence

$$1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$$

with generating function $\frac{1+x}{1-x^3}$, while the diagonal sums are the alternating sign version of A078008 given by

$$1, 0, 2, -2, 6, -10, 22, \ldots$$

with generating function $\frac{1+x}{(1-x)(1+2x)}$.

Of particular note are the images of the Catalan numbers C_n A000108 and the central binomial numbers $\binom{2n}{n}$ A000984 under this matrix. Letting

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}$$

be the generating function of the Catalan numbers C_n, we have

$$\left(\frac{1}{1-x^2}, \frac{x}{(1+x)^2}\right) \cdot c(x) = \frac{1}{1-x^2}\frac{1-\sqrt{1-4\frac{x}{(1+x)^2}}}{2\frac{x}{(1+x)^2}} = \frac{1}{1-x}.$$

Thus the image of C_n under the matrix $\mathbf{M}$ is the all 1's sequence

$$1, 1, 1, \ldots$$

The image of C_{n+1} is also interesting. We get the sequence $1, 2, 2, 2, \ldots$ with generating function $\frac{1+x}{1-x}$. This can be generalized to the following result: The image of C_{n+k} under the matrix $\mathbf{M}$ has generating function $\frac{\sum_{j=0}^{k} a_{k,j} x^j}{1-x}$, where $a_{n,k}$ is the general term of the matrix $(c(x), xc(x)^2)$. We have $a_{n,k} = \binom{2n}{n-k} \frac{2k+1}{n+k+1}$. A consequence of this is the fact that the image by $\mathbf{M}$ of the Hankel matrix with general term C_{n+k} is a matrix whose rows tend to $\binom{2n}{n}$, where the first row is C_n :

$$\begin{pmatrix} 1 & 1 & 2 & 5 & 14 & 42 & \ldots \\ 1 & 2 & 5 & 14 & 42 & 132 & \ldots \\ 1 & 2 & 6 & 19 & 62 & 207 & \ldots \\ 1 & 2 & 6 & 20 & 69 & 242 & \ldots \\ 1 & 2 & 6 & 20 & 70 & 251 & \ldots \\ 1 & 2 & 6 & 20 & 70 & 252 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix is in fact equal to

$$\mathbf{M} \cdot (c(x), xc(x)^2) \cdot (c(x), xc(x)^2)^T = \left(\frac{1}{1-x}, x\right) \cdot (c(x), xc(x)^2)^T$$

since $(C_{n+k}) = \mathbf{LDL}^T$ [7] where $\mathbf{L} = (c(x), xc(x)^2)$ and $\mathbf{D} = \mathbf{I}$ in the case of the Catalan numbers.
Now taking $\binom{2n}{n}$ with generating function $\frac{1}{\sqrt{1-4x}}$, we obtain

$$\left(\frac{1}{1-x^2}, \frac{x}{(1+x)^2}\right) \cdot \frac{1}{\sqrt{1-4x}} = \frac{1}{1-x^2} \frac{1}{\sqrt{1-4\frac{x}{(1+x)^2}}} = \frac{1}{(1-x)^2}.$$

Thus the image of $\binom{2n}{n}$ under $\mathbf{M}$ is $n+1$ or the counting numbers A000027

$$1, 2, 3, 4, 5, \ldots$$

3 The inverse matrix $\mathbf{M}^{-1}$

Since

$$\mathbf{M} = \left(\frac{1}{1-x}, x\right) \cdot \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$$

we see that

$$\mathbf{M}^{-1} = \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)^{-1} (1-x, x) = (c(x), xc(x)^2)(1-x, x).$$

Thus

$$\mathbf{M}^{-1} = (c(x)(1 - xc(x)^2), xc(x)^2) = (1 - x^2c(x)^4, xc(x)^2).$$

The general term of $\mathbf{M}^{-1}$ is given by

$$M^{-1}_{n,k} = \sum_{j=0}^{n} (-1)^{j+k} \binom{1}{j-k} \binom{2n}{n-j} \frac{2j+1}{n+j+1}.$$

We observe that

$$\mathbf{M}^{-1} = (c(x)(1 - xc(x)^2), xc(x)^2) = (c(x), xc(x)^2) - (xc(x)^3, xc(x)^2)$$

and hence

$$\begin{aligned} M^{-1}_{n,k} &= \frac{2k+1}{n+k+1}\binom{2n}{n-k} - \frac{2k+3}{n+k+2}\binom{2n}{n-k-1} \\ &= \frac{2k+1}{n+k+1}\binom{2n}{n-k} - \frac{2k+3}{n+k+2}\frac{n-k}{n+k+1}\binom{2n}{n-k} \\ &= \frac{2(2k^2+4k-n+1)}{(k+n+1)(k+n+2)}\binom{2n}{n-k}. \end{aligned}$$

Thus if we apply the matrix $\mathbf{M}^{-1}$ to the sequence a_n we obtain the image sequence b_n given by

$$\begin{aligned} b_n &= \sum_{k=0}^{n}\sum_{j=0}^{n} (-1)^{j+k} \binom{1}{j-k} \binom{2n}{n-j} \frac{2j+1}{n+j+1} a_k \\ &= \sum_{k=0}^{n} \frac{2k+1}{n+k+1}\binom{2n}{n-k} a_k - \sum_{k=0}^{n} \frac{2k+3}{n+k+2}\binom{2n}{n-k-1} a_k \\ &= \sum_{k=0}^{n} \frac{2(2k^2+4k-n+1)}{(k+n+1)(k+n+2)}\binom{2n}{n-k} a_k. \end{aligned}$$

We can obtain a further expression for this image by observing that

$$\mathbf{M}^{-1} = \left(1, \frac{x}{(1+x)^2}\right)^{-1} \cdot (1 - x^2, x) = (1, xc(x)^2) \cdot (1 - x^2, x).$$

We let $T_{n,k}$ be the general term of the matrix $(1, xc(x)^2)$. We have $T_{0,0} = 1$, and

$$T_{n,k} = \binom{2n-1}{n-k} \frac{2k}{n+k}$$

otherwise. Then

$$b_n = \sum_{k=0}^{n} T_{n,k}(a(k) - a^{(2)}(k)),$$

where if $k < 2$, $a^{(2)}(k) = 0$, otherwise $a^{(2)}(k) = a(k-2)$.

The production matrix [2] of $\mathbf{M}^{-1}$ begins

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ -2 & 1 & 2 & 1 & 0 & 0 & \ldots \\ -2 & 0 & 1 & 2 & 1 & 0 & \ldots \\ -2 & 0 & 0 & 1 & 2 & 1 & \ldots \\ -2 & 0 & 0 & 0 & 1 & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We note that the production matrix of the array

$$\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)^{-1} = (c(x), xc(x)^2)$$

is given by the related tri-diagonal matrix

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 2 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 2 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

which is the Jacobi-Stieltjes matrix for the orthogonal signed Morgan-Voyce polynomials $b_n(x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n+k}{2k} x^k$ (see A085478).

4 $\mathbf{M}^{-1}$ and Hankel transforms

We can characterize the image of the family of sequences $rn+1$ under the transformation matrix $\mathbf{M}^{-1}$ as follows. Here, $r \in \mathbf{Z}$.

Proposition 1. *The image of $rn+1$ under the matrix $\mathbf{M}^{-1}$ is*

$$(rn+1)C_n$$

with Hankel transform given by

$$[x^n]\frac{1-(r-1)^2 x}{1-2x+(r-1)^2 x^2}.$$

Proof. The image of $rn+1$ has generating function given by

$$\begin{aligned} g(r,x) &= \mathbf{M}^{-1}\frac{(r-1)x+1}{(1-x)^2} \\ &= (c(x)(1-xc(x)^2), xc(x)^2)\cdot\frac{(r-1)x+1}{(1-x)^2} \\ &= c(x)(1-xc(x)^2)\frac{(r-1)xc(x)^2+1}{(1-xc(x)^2)^2} \\ &= c(x)\frac{(r-1)xc(x)^2+1}{1-xc(x)^2} \\ &= \frac{(1-r)(1-4x)-(1-r+2x(r-2))\sqrt{1-4x}}{2x(1-4x)} \end{aligned}$$

which is the generating function of $(rn+1)C_n$. Then the orthogonal measure associated to $g(x,r)$ (i.e., the measure for which the numbers $(rn+1)C_n$ are the moments) is given by

$$w(x,r)=\frac{1}{2\pi}\frac{x(r-1)-2(r-2)}{\sqrt{x(4-x)}}.$$

This leads to a modified Jacobi polynomial. Using the techniques of [3] (see also [1, 8]) we see that the β coefficients corresponding to the relevant three term recurrence are given by

$$-r^2+2r+1, \frac{-3r^{+}6r+1}{(r^2-2r-1)^2}, -\frac{-(r^2-27r-1)7(r^4-47r^3-27r^2+127r+1)}{(37r^2-67r-1)^2}, \ldots$$

Thus the Hankel transform is given by [4, 5]

$$1, -r^2+27r+1, -37r^2+67r+1, r^4-47r^3-27r^2+127r+1, \ldots$$

which is the expansion of

$$\frac{1-(r-1)^2x}{1-2x+(r-1)^2x^2}$$

as required. □

Similarly, we can look at the image of the power sequences $n \to r^n$ under $\mathbf{M}^{-1}$. We find

Proposition 2. *The Hankel transform of the image of r^n under the matrix $\mathbf{M}^{-1}$ is given by*

$$[x^n]\frac{1-(r-1)^2x}{1-(r^2-r+2)x+(r-1)^2x^2}.$$

5 The matrix $\tilde{\mathbf{M}}$

The general term $\tilde{M}_{n,k}$ of the matrix $\tilde{\mathbf{M}}$ is given by

$$\tilde{M}_{n,k}=\sum_{j=0}^{n}(-1)^{n-j+k}\binom{k+j}{2k}=\sum_{j=0}^{n-k}(-1)^{j-k}\binom{n+k-j}{2k}.$$

The row sums of this matrix are the periodic sequence $1, -1, 0, 1, -1, 0, \ldots$ with generating function $\frac{1-x}{1-x^3}$. The diagonal sums are the sequence A078014 with generating function $\frac{1-x}{1-x+2x^3}$. The image of C_n under $\tilde{\mathbf{M}}$ is

$$1, -1, 1, -1, \ldots$$

or $(-1)^n$. Similarly the image of $\binom{2n}{n}$ is $(-1)^n(n+1)$. The image of the Catalan Hankel matrix (C_{n+k}) under $\tilde{\mathbf{M}}$ begins

$$\begin{pmatrix} 1 & 1 & 2 & 5 & 14 & 42 & \ldots \\ -1 & -2 & -5 & -14 & -42 & -132 & \ldots \\ 1 & 2 & 6 & 19 & 62 & 207 & \ldots \\ -1 & -2 & -6 & -20 & -69 & -242 & \ldots \\ 1 & 2 & 6 & 20 & 70 & 251 & \ldots \\ -1 & -2 & -6 & -20 & -70 & -252 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The inverse matrix $\tilde{\mathbf{M}}^{-1}$ has general term

$$\sum_{j=0}^{n} (-1)^j \binom{1}{j-k} \binom{2n}{n-j} \frac{2j+1}{n+j+1}.$$

We have the following proposition concerning the image of r^n under the inverse of the matrix $\tilde{\mathbf{M}}$.

Proposition 3. *The Hankel transform of the image of the sequence $n \to r^n$ under the matrix $\tilde{\mathbf{M}}^{-1}$ is given by*

$$[x^n] \frac{1-x}{1+2rx+x^2}.$$

6 The positive matrix $\mathbf{M}^{+} = \left(\frac{1}{1-x^2}, \frac{x}{(1-x)^2}\right)$

The matrix $\mathbf{M}^{+}$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 4 & 4 & 1 & 0 & 0 & \ldots \\ 1 & 6 & 11 & 6 & 1 & 0 & \ldots \\ 0 & 9 & 24 & 22 & 8 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It has general term

$$M^{+}_{n,k} = \sum_{j=0}^{n} (-1)^{n-j} \binom{k+j}{2k} = \sum_{j=0}^{n-k} \binom{n+k-j}{2k} (-1)^j = \sum_{j=0}^{n} \binom{j+k-1}{j-k} \frac{1+(-1)^{n-j}}{2}.$$

The row sums of this matrix are easily seen to have generating function $\frac{1-x}{(1+x)(1-3x+x^2)}$. These are the squared Fibonacci numbers $F(n+1)^2$, A007598. Thus we have

$$F(n+1)^2=\sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^{n-j}\binom{k+j}{2k},$$

for instance. The diagonal sums are the Jacobsthal numbers variant A078008, with general term $\frac{2^n}{3}+2\frac{(-1)^n}{3}$. The image of the Catalan numbers C_n by this matrix are an alternating sum of the large Schröder numbers S_n (A006318) given by

$$\sum_{k=0}^{n}(-1)^{n-k}S_k.$$

This has generating function

$$\frac{1-x-\sqrt{1-6x+x^2}}{2x(1+x)}.$$

Likewise, the image of the central binomial coefficients $\binom{2n}{n}$ is the sequence starting $1, 2, 11, 52, 269, \ldots$ A026933 with generating function

$$\frac{1}{(1+x)\sqrt{1-6x+x^2}}.$$

This is therefore

$$\sum_{k=0}^{n}(-1)^{n-k}D_k$$

where D_n is the sequence of central Delannoy numbers A001850.

Again, for appropriate families of sequences, the images under the inverse of this matrix can have interesting Hankel transforms. For instance, we have the following result.

Proposition 4. *The Hankel transform of the image under* $\mathbf{M}^{+^{-1}}$ *of the generalized Fibonacci numbers with general term*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-k}{k}r^k$$

is given by

$$[x^n]\frac{1+(r-1)x}{1+2x+(r-1)^2x^2}.$$

Finally, we note that the production matrix of the inverse $\mathbf{M}^{+^{-1}}$ takes the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & -2 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 1 & -2 & 1 & 0 & 0 & \ldots \\ -2 & 0 & 1 & -2 & 1 & 0 & \ldots \\ 2 & 0 & 0 & 1 & -2 & 1 & \ldots \\ -2 & 0 & 0 & 0 & 1 & -2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

7 A one-parameter family of Riordan arrays

We note that

$$\mathbf{M} = \left(\frac{1}{1-x^2}, \frac{x}{(1+x)^2}\right) = \left(\frac{1}{(1-x)(1+x)}, \frac{x}{(1+x)^2}\right)$$

is the element $\mathbf{M}_1$ of the family of Riordan arrays $\mathbf{M}_r$ defined by

$$\mathbf{M}_r = \left(\frac{1}{(1-x)(1+rx)}, \frac{x}{(1+rx)^2}\right).$$

The general term of $\mathbf{M}_r$ is given by

$$\sum_{j=0}^{n-k} \binom{n+k-j}{2k} (-r)^{n-k-j}.$$

We can generalize the results of the foregoing to this family. For instance, the row sums of $\mathbf{M}_r^{-1}$ are precisely $r^n C_n$, while the image of the sequence $r^n\binom{2n}{n}$ under the matrix $\mathbf{M}_r$ has general term $\frac{1-r^{n+1}}{1-r} = \sum_{k=0}^n r^k$.
The production matrix of $\mathbf{M}_r^{-1}$ begins

$$\begin{pmatrix}
r-1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
r^2-r-1 & 2r & 1 & 0 & 0 & 0 & \ldots \\
-r-1 & r^2 & 2r & 1 & 0 & 0 & \ldots \\
-r-1 & 0 & r^2 & 2r & 1 & 0 & \ldots \\
-r-1 & 0 & 0 & r^2 & 2r & 1 & \ldots \\
-r-1 & 0 & 0 & 0 & r^2 & 2r & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

8 Acknowledgements

We are grateful to an anonymous reviewer whose careful reading of the original manuscript has resulted in some important clarifications. Some are incorporated in the above text. We single out two further results here. Firstly, it can be noted that

$$\begin{aligned}
M_{n,k}^{-1} &= [x^n](1-x^2c(x)^2)(xc(x)^2)^k \\
&= \frac{k}{n}\binom{2n}{n-k} - \frac{k+2}{n}\binom{2n}{n-k-2}.
\end{aligned}$$

Thus for $n, k \geq 1$ we obtain the identity

$$\frac{2k+1}{n+k+1}\binom{2n}{n-k} - \frac{2k+3}{n+k+2}\binom{2n}{n-k-1} = \frac{k}{n}\binom{2n}{n-k} - \frac{k+2}{n}\binom{2n}{n-k-2}.$$

Secondly, Proposition 1 can be extended as follows:

Proposition 5. *The image of the sequence* $(rn+s)_{n=0}^{\infty}$ *under the matrix* M^{-1} *is*

$$(rn+s)C_n,$$

with Hankel transform given by

$$[x^n]\frac{s-(r-s)^2x}{1-2sx+(r-s)^2x^2}.$$

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