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A Note on the Relation Between Fixed Point and Orbit Count Sequences

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Abstract

The relation between fixed point and orbit count sequences is investigated from the point of view of linear mappings on the space of arithmetic functions. Spectral and asymptotic properties are derived and several quantities are explicitly given in terms of Gaussian binomial coefficients.

1 Introduction and general setting

Each map of an arbitrary set into itself gives rise to sequences $a = (a_n)_{n\in\mathbb{N}}$ and $c = (c_n)_{n\in\mathbb{N}}$ that count the number of periodic points with period n and the orbits of length n, respectively. In many interesting cases, a_n and c_n are finite for all $n \in \mathbb{N}$, turning a and c into sequences of non-negative integers. They are then related by Möbius inversion as

$$a_n = \sum_{d|n} d\, c_d \quad \text{and} \quad c_n = \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right) a_d. \tag{1}$$

Here, μ denotes the Möbius function from elementary number theory [1]. Puri and Ward [10] study these relations as paired transformations between certain non-negative integer sequences, with special emphasis on arithmetic and asymptotic aspects. It is the purpose of this short note, which is based on [8], to extend these transformations to the space of arithmetic functions, on which they act as linear operators, and to investigate their spectral properties and their asymptotic behaviour under iteration, thus generalising results from [10].

Let $\mathcal{A} = \mathbb{C}^{\mathbb{N}}$ denote the set of arithmetic functions. Its elements can be thought of either as functions $f\colon \mathbb{N} \to \mathbb{C}, n \mapsto f(n)$, or as sequences of complex numbers $(f_n)_{n\in\mathbb{N}}$, hence the notations $f(n)$ and f_n are used in parallel; see [1] for details and general notation. Clearly, $\mathcal{A}$ is a complex vector space with respect to component-wise addition and the usual scalar multiplication. Additionally, it is equipped with the structure of a commutative algebra $(\mathcal{A}, +, *)$ with unit element, where $*$ denotes *Dirichlet convolution*, defined by

$$\big(f * g\big)(n) = \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right)$$

for all $n \in \mathbb{N}$. The unit element with regard to this product is $\delta_1 = (1, 0, 0, \ldots)$. When $f \in \mathcal{A}$ is invertible, the notation f^{-1} refers to its *Dirichlet inverse*, defined by $f * f^{-1} = f^{-1} * f = \delta_1$. Furthermore, $f \cdot g$ denotes the pointwise product where $\big(f \cdot g\big)(n) = f(n)g(n)$ for all $n \in \mathbb{N}$. Motivated by the transformations (1), we define 'fix' and 'orb' as mappings from $\mathcal{A}$ into itself by $a \mapsto a'$, with

$$a'_n = \mathrm{fix}(a)_n = \sum_{d|n} d\, a_d \quad \text{and} \quad a'_n = \mathrm{orb}(a)_n = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) a_d, \tag{2}$$

respectively, for all $n \in \mathbb{N}$. Denoting by N the arithmetic function $n \mapsto n$, fix and orb can be written as

$$\mathrm{fix}(f) = N \cdot \left(\frac{1}{N} * f\right) \quad \text{and} \quad \mathrm{orb}(f) = \frac{1}{N} \cdot \big(\mu * f\big). \tag{3}$$

This is particularly useful when considering the Dirichlet series of $\mathrm{fix}(f)$ and $\mathrm{orb}(f)$. In general, the Dirichlet series D_f of $f \in \mathcal{A}$ is defined by

$$D_f(s) = \sum_{m=1}^{\infty} \frac{f(m)}{m^s},$$

which is viewed as a formal series at this stage. This generating function gives rise to the (formal) identities [1]

$$D_{f+g} = D_f + D_g \quad \text{and} \quad D_{f*g} = D_f \cdot D_g.$$

Using these in connection with (3), a straightforward calculation shows

Lemma 1. *Let $D_a(s)$ denote the Dirichlet series of $a \in \mathcal{A}$. Then,*

$$D_{\mathrm{fix}(a)}(s) = \zeta\,(s)\, D_a(s-1) \quad \textit{and} \quad D_{\mathrm{orb}(a)}(s) = \frac{1}{\zeta\,(s+1)}\, D_a(s+1).$$

where ζ is Riemann's zeta function. □

For a discussion of the identities from Lemma 1 in a different context, and for interesting recent development on Dirichlet series as orbit counting generating functions, we refer the reader to [3, Secs. 2 and 10]. We now turn our attention to the structure of fix and orb as linear mappings on $\mathcal{A}$.

2 Spectral properties

When studying fix on $\mathcal{A}$, it is natural to ask for eigenvalues and eigenvectors. Let x be an arithmetic function that satisfies the fixed point condition $\mathrm{fix}(x)_n = \sum_{d|n} d\, x_d = x_n$ for all $n \in \mathbb{N}$. Solving for x_n yields the recursive relation

$$x_n = \frac{1}{1-n} \sum_{n>d|n} d\, x_d \tag{4}$$

for $n > 1$, so that all x_n are uniquely determined from x_1. The x_n depend linearly on x_1, which is the only degree of freedom in solving (4), wherefore the eigenspace for the eigenvalue 1 is one-dimensional. Setting $x_1 = a_1 = 1$, we denote the resulting solution of (4) by $a = (a_1, a_2, a_3, \ldots)$. Since fix maps multiplicative sequences to multiplicative sequences, it is an interesting question to what extent this is reflected by a.

Lemma 2. *Let a satisfy $\mathrm{fix}(a) = a$, with $a_1 = 1$. The eigenvector a is multiplicative, so that $a_{nm} = a_n \cdot a_m$ for all $n, m \geq 1$ with $(m, n) = 1$.*

Proof. In order to show multiplicativity by induction on the index, we first note that the claim is true for trivial decompositions such as $n \cdot 1$, because $a_1 = 1$. In particular, it is then true for all natural numbers $n \leq 5 = 2 \cdot 3 - 1$. Let $(m, n) = 1$ with $mn > 1$ and suppose the a_k satisfy the multiplicativity property for all $k \leq mn - 1$ (so that $a_k = a_d\, a_{d'}$ for all d, d' with $(d, d') = 1$ and $d\,d' = k$). This implies

$$\begin{aligned} a_{mn} &= \frac{1}{1-mn} \sum_{\substack{d|mn \\ d<mn}} d\, a_d = \frac{1}{1-mn} \sum_{\substack{d=d_m d_n \\ d_m|m,\ d_n|n \\ d_m d_n < mn}} (d_m a_{d_m})\,(d_n a_{d_n}) \\ &= \frac{1}{1-mn} \Bigg(\Big(\underbrace{\sum_{d|m} d a_d}_{=a_m} \underbrace{\sum_{d'|n} d' a_{d'}}_{=a_n} \Big) - mn\, a_m a_n \Bigg) = a_m\, a_n . \end{aligned}$$

This argument shows the induction step from $mn - 1$ to mn. In general, the step from N to $N + 1$ is trivial when $N + 1$ is prime, and of the above form otherwise, which completes the induction. $\square$

For the remainder of this section, a always denotes the unique solution of $\mathrm{fix}(a) = a$ with $a_1 = 1$. The recursion in (4) runs over all proper divisors of n and yields an explicit expression for a_{p^r} with p prime and $r \in \mathbb{N}$. By multiplicativity, this extends to a closed formula for a_n.

Proposition 3. *Let $n > 1$ with prime decomposition $n = p_1^{r_1} \cdot \ldots \cdot p_s^{r_s}$. Then, the n-th entry of the eigenvector a of Lemma 2 reads*

$$a_n = \prod_{k=1}^{s} \prod_{\ell=1}^{r_k} \frac{1}{1 - p_k^{\ell}} .$$

Proof. Observing that $a_p = \frac{a_1}{1-p}$, induction on the exponent r leads to $a_{p^r} = \prod_{\ell=1}^{r}(1-p^\ell)^{-1}$, from which the statement follows by Lemma 2. $\square$

The sequence of denominators is now entry A153038 of the Online Encyclopedia of Integer Sequences (OEIS) [11]. In particular, $|a_n| \le 1$ for all n, and Proposition 3 suggests that the a_n become 'asymptotically small' in some sense, as $n \to \infty$. Indeed, with

$$\ell^q = \big\{ f \in \mathbb{C}^{\mathbb{N}} \,\big|\, \sum_{n=1}^{\infty} |f_n|^q < \infty \big\},$$

one obtains the following result.

Proposition 4. *The eigenvector a is an element of $\ell^{1+\varepsilon}$ for all $\varepsilon > 0$.*

Proof. Substituting $p^k - 1 = (p-1)\sum_{j=0}^{k-1} p^j$ in the denominator of $|a_{p^r}|$ leads to

$$\prod_{k=1}^{r}(p^k - 1) = (p-1)^r \prod_{k=1}^{r}\sum_{j=0}^{k-1} p^j \ge (p-1)^r p^{\sum_{k=1}^{r}(k-1)} = (p-1)^r p^{r(r-1)/2}.$$

Since $(p-1)^r p^{r(r-1)/2} \ge p^r$ for $r > 2$ or $p > 2$, the inequality $|a_{p^r}| \le \frac{1}{p^r}$ holds in these cases. Besides, one has $|a_p| = \frac{1}{p-1} = \frac{p}{p-1} \cdot \frac{1}{p}$ for every prime p. The remaining term $a_{2^2} = \frac{4}{3}\frac{1}{2^2}$ introduces an additional factor of $c = \frac{4}{3}$ in the estimate of $|a_n|$, to account for the case when 2^2 is the highest power of 2 that divides n. For $n = p_1^{r_1} \cdot \ldots \cdot p_s^{r_s}$, this gives

$$|a_n| \le \frac{c}{n} \prod_{\{i | r_i = 1\}} \frac{p_i}{p_i - 1}.$$

This product can be related to Euler's ϕ-function,

$$\phi(n) = n \prod_{p|n}\left(1 - \frac{1}{p}\right) = n \prod_{p|n}\left(\frac{p-1}{p}\right) \le n \prod_{\substack{p|n \\ p^2 \nmid n}}^{r} \left(\frac{p-1}{p}\right),$$

giving $\prod_{\{i|r_i=1\}} \frac{p_i}{p_i-1} \le \frac{n}{\phi(n)}$. Using

$$\limsup_{n\to\infty} \frac{n}{\phi(n)\log(\log n)} = e^\gamma,$$

where $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)$ denotes Euler's constant, compare [6, Thm. 328], we obtain

$$|a_n| \le \frac{C \cdot \log(\log(n))}{n}$$

for some constant C, which implies $\sum_{n=1}^{\infty} |a_n|^{1+\varepsilon} < \infty$. $\square$

In particular, $a \in \ell^2$, and (numerically) one finds $\sum_{n=1}^{\infty}|a_n|^2 \approx 2.99635 < 3$. Since $\sum_p \frac{1}{p}$ diverges and $\left|a_p\right| = \frac{1}{p-1} > \frac{1}{p}$, it is clear that the statement of Proposition 4 is not true for $\varepsilon = 0$.

Lemma 5. *Let $D_a(s)$ be the Dirichlet series of the eigenvector a, with $s = \sigma + it$.*

(i) *$D_a(s)$ converges absolutely in $\mathcal{S} = \{s \in \mathbb{C} \mid \sigma > 1\}$.*

(ii) $\lim_{\sigma\to\infty} D_a(\sigma + it) = 1$, *uniformly in $t \in \mathbb{R}$.*

Proof. We employ the methods of [1, Chapter 11]. The first claim follows from the absolute convergence of $\zeta(\sigma)$ for $\sigma > 1$, because $\sum_m \left|\frac{a_m}{m^s}\right| \leq \sum_m \frac{|a_m|}{m^\sigma} \leq \zeta(\sigma)$.

For (ii), observe that $a_1 = 1$, wherefore one has

$$|D_a(s) - 1| \leq \sum_{m=2}^{\infty} \frac{1}{m^\sigma} = \zeta(\sigma) - 1,$$

which tends to 0 (independently of t) as $\sigma \to \infty$. □

Using the fixed point relations $\mathrm{fix}(a) = a$ and $\mathrm{orb}(a) = a$, the first equation of Lemma 1 becomes a 'recursion' that can be solved for an explicit expression in terms of the Riemann zeta function.

Theorem 6. *The Dirichlet series of the fixed point a satisfies*

$$D_a(s) = \prod_{\ell=1}^{\infty} \frac{1}{\zeta(s+\ell)} = \prod_p \prod_{\ell\geq 1} \left(1 - \frac{1}{p^{s+\ell}}\right),$$

where the infinite product converges on the set $\mathcal{S}$ from Lemma 5.

Proof. Using Lemma 1, one inductively obtains $D_a(s) = \frac{D_a(s+m)}{\prod_{\ell=1}^m \zeta(s+\ell)}$ for arbitrary $m \in \mathbb{N}$. For $m \to \infty$, the numerator converges to 1, since

$$D_a(s+m) = a_1 + \sum_{n\geq 2} \frac{a_n}{n^s} \frac{1}{n^m} \leq a_1 + \frac{1}{2^m} D_a(s).$$

Due to the asymptotic behaviour $\zeta(\sigma + it) = 1 + \mathcal{O}(1/2^\sigma)$ for $\sigma \to \infty$, the product in the denominator of the resulting expression $D_a(s) = \prod_{\ell=1}^\infty \frac{1}{\zeta(s+\ell)}$ converges as well, giving the first part of the claim. The second equality follows from the Euler product representation $\zeta(s+\ell)^{-1} = \prod_p (1 - p^{-(s+\ell)})$ for $\sigma > 1$, in which the order of taking products over p and ℓ may be exchanged due to absolute convergence. □

Remark. Since $a_1 \neq 0$, a admits a Dirichlet inverse. Its terms can be calculated via to the general recursion formula, compare [1, Thm. 2.8]. The Dirichlet inverse satisfies $D_{f^{-1}} = 1/D_f$, wherefore $b := a^{-1}$ has the Dirichlet series $D_b = \prod_{\ell\geq 1} \zeta(s+\ell)$. Determining the coefficients of this series leads to an analogue of Proposition 3; the n-th entry of b is

$$b_n = \prod_{j=1}^{s} \prod_{k=1}^{r_j} \frac{p_j^{k-1}}{p_j^k - 1} = \prod_{j=1}^{s} \frac{p_j^{\frac{1}{2} r_j (r_j - 1)}}{\prod_{\ell=1}^{r_j} (p_j^\ell - 1)},$$

with $n = p_1^{r_1} \cdot \ldots \cdot p_s^{r_s}$ as before; see [8] for details. The first few terms of a and b read

$$a = \Bigl(1, -1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{6}, -\frac{1}{21}, \ldots\Bigr) \quad \text{and} \quad b = \Bigl(1, 1, \frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{2}, \frac{1}{6}, \frac{8}{21}, \ldots\Bigr).$$

Although it appears that a and b share the same denominator sequence, this is not true when the terms are represented as reduced fractions, because cancellation can occur for the b_n. The first instance of this phenomenon is $b_{12} = \frac{1}{3}$, whereas $a_{12} = -\frac{1}{6}$.

Returning to the task of identifying all eigenvalues and their corresponding eigenspaces, we consider the equation $x = \lambda \cdot \mathrm{fix}(x)$ for $\lambda \neq 1$ or, equivalently,

$$(\lambda - n)\, x_n = \lambda \sum_{n>d|n} d\, x_d$$

for all $n \in \mathbb{N}$. Since $\mathrm{fix}(x)_1 = x_1$ by definition, the first term of such an eigenvector has to be 0. In general, if $x_d = 0$ for all $d|n$ with $d < n$, the n-th term x_n will vanish as well, unless $\lambda = n$. Thus, the first non-zero term is x_m where $m = \lambda$, and all eigenvalues are thus natural numbers. For fixed $1 < m \in \mathbb{N}$, the recursion analogous to the case $m = 1$ given in (4) is

$$x_n = \frac{1}{m-n} \sum_{n>d|n} d\, x_d \qquad (\text{for} \quad n \neq m), \tag{5}$$

which shows that $x_n \neq 0$ if and only if $n = km$ for some $k \geq 1$. The recursion for the non-vanishing terms simplifies to

$$x_{km} = \frac{1}{m(1-k)} \sum_{k>d|k} dm\, x_{dm} = \frac{1}{1-k} \sum_{k>d|k} d\, x_{dm}$$

for $k > 1$,where the only free parameter is now the first non-zero term x_m, indicating a one-dimensional eigenspace. Choosing $x_m = a_1 = 1$, the resulting eigenvector $a^{(m)}$ with eigenvalue m can be expressed in terms of $a = a^{(1)}$:

$$a^{(m)}(n) = \begin{cases} a\left(\frac{n}{m}\right), & \text{if } m|n; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

In particular, $a^{(m)} \in \ell^{1+\varepsilon}$ for all $\varepsilon > 0$ and all $m \geq 1$, where $a^{(m)}$ and a clearly have the same norm. Being eigenvectors for different eigenvalues, the $a^{(m)}$ are linearly independent. The next proposition states that they even generate the space of arithmetic functions in the sense of formal linear combinations.

Proposition 7. *The eigenvectors of* fix *form a basis of $\mathcal{A}$, so that $\bigl\langle a^{(m)} \mid m \in \mathbb{N} \bigr\rangle_{\mathbb{C}} = \mathcal{A}$, and the representation of $f \in \mathcal{A}$ as a formal linear combination of the $a^{(m)}$ is unique.*

Proof. Let $f \in \mathcal{A}$, $a = a^{(1)}$ and $b := (a)^{-1}$, and set $\alpha = f * b$. Then, $\alpha * a = (f * b) * a = f$. On the other hand, one has

$$\bigl(\alpha * a\bigr)(k) = \sum_{d|k} \alpha_d\, a^{(d)}(k) = \sum_{m \geq 1} \alpha_m\, a^{(m)}(k)$$

for all $k \in \mathbb{N}$, so $f \in \bigl\langle a^{(m)} \mid m \in \mathbb{N} \bigr\rangle_{\mathbb{C}}$. Uniqueness of the linear combination follows from the uniqueness of the Dirichlet inverse. □

The spectrum of an operator $T\colon \mathcal{A} \to \mathcal{A}$ is defined by

$$\operatorname{spec}(T) := \{\lambda \in \mathbb{C} \mid T - \lambda\,\mathrm{id} \text{ is not invertible}\}.$$

Note that fix and orb could be investigated as operators on ℓ^2, restricting their domains to appropriate invariant subspaces. But since we need a more general setting later when considering their behaviour under iteration, we work with the spectrum as a purely algebraic notion.

Theorem 8. *The spectra of* fix *and* orb *are*

$$\operatorname{spec}(\mathrm{fix}) = \mathbb{N} \quad \textit{and} \quad \operatorname{spec}(\mathrm{orb}) = \left\{ \frac{1}{N} \mid N \in \mathbb{N} \right\}.$$

Proof. Obviously, $\mathrm{fix}(f) = \lambda f$ implies $f = 0$ if and only if $\lambda \in \mathbb{C} \setminus \mathbb{N}$, so $\mathrm{fix} - \lambda\,\mathrm{id}$ is injective for these λ. Surjectivity can be seen by constructing a preimage $f \in \mathcal{A}$ of an arbitrary $g \in \mathcal{A}$ under $\mathrm{fix} - \lambda\,\mathrm{id}$. To this end, let $f_1 = \frac{g_1}{1-\lambda}$ and, having already defined $f_1, \ldots, f_{n-1}$ for some $n > 1$, set

$$f_n = \frac{1}{n - \lambda}\Bigl(g_n - \sum_{n > d \mid n} d\, f_d\Bigr).$$

When $\lambda \notin \mathbb{N}$, the resulting arithmetic function f is a term-wise well-defined element of $\mathcal{A}$ with $\mathrm{fix}(f) - \lambda f = g$. In summary, $\mathrm{fix} - \lambda\,\mathrm{id}$ is invertible if and only if $\lambda \in \mathbb{C} \setminus \mathbb{N}$. The spectrum of orb can be derived from that of fix by the equivalence

$$\mathrm{fix}(a^{(N)}) = N a^{(N)} \iff \mathrm{orb}(a^{(N)}) = \frac{1}{N}\, a^{(N)},$$

which follows from $\mathrm{fix} \circ \mathrm{orb} = \mathrm{orb} \circ \mathrm{fix} = \mathrm{id}$. □

3 Iterations of the operator fix

Puri and Ward [10] raised the question of the nature of the orbits under fix. In particular, the asymptotic properties of such orbits are sought. In our terminology, this is to ask how elements of $\mathcal{A}$ behave under iteration of the operator fix. Starting from $\delta_1 = (1, 0, 0, \ldots)$, the first iterates are the sequences A000007, A000012, A000203, A001001 and A038991 – A038999 of the OEIS, see [10, 11]. Here, we start from the sequences δ_m, $m \geq 1$, defined via $\delta_m(n) = \delta_{m,n}$ for all n, and give term-wise exact representations of $\mathrm{fix}^n(\delta_m)$, from which the corresponding quantities of general starting sequences can later be constructed.

Let $y^{(0)} = \delta_1$ and $y^{(n)} := \mathrm{fix}(y^{(n-1)})$ for $n \geq 1$, so that

$$y^{(n)}(m) = \bigl(\mathrm{fix}(y^{(n-1)})\bigr)_m = \bigl(\mathrm{fix}^n(y^{(0)})\bigr)_m$$

holds for all $m \geq 1$.

Proposition 9. *Let p be a prime and $r \in \mathbb{N}$. Then*

$$y^{(n)}(p^r) = \begin{bmatrix} n + r - 1 \\ r \end{bmatrix}_p = \prod_{i=0}^{r-1} \frac{1 - p^{n+r-1-i}}{1 - p^{i+1}},$$

where $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_q$ denotes the Gaussian or q-binomial coefficient.

Proof. The claim can be verified by induction in n. By definition of fix, we have

$$\begin{aligned}
y^{(n+1)}(p^r) = \mathrm{fix}(y^{(n)})(p^r) &= \sum_{d|p^r} d \cdot y^{(n)}(d) = \sum_{k=0}^{r} p^k \cdot y^{(n)}(p^k) \\
&= 1 + p \begin{bmatrix} n \\ 1 \end{bmatrix}_p + \sum_{k=2}^{r} p^k \cdot y^{(n)}(p^k).
\end{aligned} \tag{7}$$

Applying *Pascal's identity* for Gaussian binomials,

$$\begin{bmatrix} m \\ \ell \end{bmatrix}_q = q^r \begin{bmatrix} m-1 \\ \ell \end{bmatrix}_q + \begin{bmatrix} m-1 \\ \ell-1 \end{bmatrix}_q,$$

to the k-th summand in (7) yields

$$p^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_p = \begin{bmatrix} n+k \\ k \end{bmatrix}_p - \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_p.$$

Plugging this and the relation $\left[{n \atop 1} \right]_p = \sum_{k=0}^{n-1} p^k$ into (7), we obtain

$$y^{(n+1)}(p^r) = \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_p + \sum_{k=2}^{r} \begin{bmatrix} n+k \\ k \end{bmatrix}_p - \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_p = \begin{bmatrix} n+r \\ r \end{bmatrix}_p$$

which completes the proof. □

Corollary 10. *For $1 \neq m \in \mathbb{N}$ with prime decomposition $m = p_1^{r_1} p_2^{r_2} \cdot \ldots \cdot p_k^{r_k}$, one has*

$$y^{(n)}(m) = y^{(n)}(p_1^{r_1})\, y^{(n)}(p_2^{r_2}) \cdot \ldots \cdot y^{(n)}(p_k^{r_k}) = \prod_{j=1}^{k} \prod_{i=0}^{r_j-1} \frac{1 - p_j^{n+r_j-1-i}}{1 - p_j^{i+1}}.$$

Proof. It is easy to check that fix, and hence all iterates fix^n with $n \geq 1$, preserve multiplicativity. Thus, in view of Proposition 9, the claim follows from δ_1 being multiplicative. □

Remark. A different approach to calculating $y^{(n)}$ exploits the fact that its Dirichlet series $D_{y^{(n)}}(s)$ is known from the context of counting sublattices of $\mathbb{Z}^n$. Let

$$h_n(m) = \left| \{ \mathit{\Lambda} \mid \mathit{\Lambda} \text{ is a sublattice of } \mathbb{Z}^n \text{ of index } m \} \right|$$

and $D_{h_n}(s) = \sum_{m \geq 1} \frac{h_n(m)}{m^s}$. Proposition A.1 of [2] states that

$$h_n(m) = \sum_{d_1 \cdot \ldots \cdot d_n = m} d_1^0 \cdot d_2^1 \cdot \ldots \cdot d_n^{n-1} \quad \text{and} \quad D_{h_n}(s) = \zeta(s) \cdot D_{h_{n-1}}(s-1),$$

where the first sum runs over all n-tuples $(d_1, \ldots, d_n)$ of positive integers with $d_1 \cdot \ldots \cdot d_n = m$. In particular, $D_{h_1}(s) = \zeta(s) = D_{y^{(1)}}(s)$. On the other hand, Lemma 1 gives $D_{y^{(n+1)}}(s) =$

$\zeta(s)D_{y^{(n)}}(s-1)$, so $D_{y^{(n)}}(s) = D_{h_n}(s)$ for all $n \geq 1$ and thus $h_n(m) = y^{(n)}(m)$ for all $n, m \geq 1$. For $m = p_1^{r_1} \cdot \ldots \cdot p_k^{r_k}$, the coefficient $h_n(m)$ can be written as

$$h_n(m) = \prod_{j=1}^{k} \prod_{i=1}^{r_j} \frac{p_j^{n+i-1} - 1}{p_j^i - 1} = \prod_{j=1}^{k} \begin{bmatrix} n + r_j - 1 \\ r_j \end{bmatrix}_{p_j},$$

as shown in [5] and [12], leading to the result of Corollary 10. This also provides a concrete interpretation of the iteration sequences mentioned earlier.

Similarly to calculating fix-eigenvectors $a^{(m)}$ for $m > 1$, the sequences arising from fix-iterations on starting sequences δ_m with $m > 1$ can be related to the case $m = 1$. From $\big(\mathrm{fix}(\delta_m)\big)_k = m \cdot \delta_{k,jm}$ for some $j \in \mathbb{N}$, one can conclude inductively

$$\big(\mathrm{fix}^n(\delta_m)\big)_k = \begin{cases} m^n \cdot y^{(n)}\big(\frac{k}{m}\big), & \text{if } m|k; \\ 0, & \text{otherwise.} \end{cases}$$

Since arbitrary elements from $\mathcal{A}$ can be written as complex linear combinations of the δ_m, we find the following behaviour of general arithmetic sequences under fix-iterations.

Lemma 11. *Let $f = (f_1, f_2, \ldots) \in \mathcal{A}$. Then, for all $M \geq 1$,*

$$\big(\mathrm{fix}^n(f)\big)_M = \sum_{k=1}^{M} f_k \left(\mathrm{fix}^n(\delta_k)\right)_M = \sum_{k|M} f_k\, k^n y^{(n)}\big(\tfrac{M}{k}\big).$$

As a convolution product, this is

$$\mathrm{fix}^n(f) = \big(f \cdot N^n\big) * y^{(n)}, \quad \textit{for all } n \in \mathbb{N},$$

where N^n denotes the arithmetic function defined by $m \mapsto m^n$.

Proof. Note first that, for all $M \geq 1$, the value $\mathrm{fix}(f)_M$ depends on the f_n with $n|M$ only. Each single term of $\mathrm{fix}(f)_M$ can thus be obtained by applying fix to the finite linear combination $\sum_{k=1}^{M} f_k \delta_k$, resulting in a component-wise well-defined arithmetic function. The last identity follows from the definition of the convolution product. ☐

Given the unbounded spectrum of fix, it is not surprising that the image sequences $\mathrm{fix}^n(f)$ of a non-negative f component-wise tend to infinity when $n \to \infty$. Nevertheless, the 'quotient sequences' $\left(\frac{\mathrm{fix}^{n+1}(f)_M}{\mathrm{fix}^n(f)_M}\right)_n$ have a well-defined convergence behaviour for each M as follows.

Theorem 12. *Let f be a non-negative arithmetic function and let $M \in \mathbb{N}$ be such that $\big(\mathrm{fix}^r(f)\big)_M \neq 0$ for some $r \in \mathbb{N}$. Then,*

$$\lim_{n \to \infty} \frac{\big(\mathrm{fix}^{n+1}(f)\big)_M}{\big(\mathrm{fix}^n(f)\big)_M} = M.$$

Proof. Expanding the quotient in question yields

$$\frac{\big(\mathrm{fix}^{n+1}(f)\big)_M}{\big(\mathrm{fix}^{n}(f)\big)_M} = \sum_{d|M} d\, \frac{\big(\mathrm{fix}^{n}(f)\big)_d}{\big(\mathrm{fix}^{n}(f)\big)_M} = M + \sum_{M>d|M} d\, \frac{\big(\mathrm{fix}^{n}(f)\big)_d}{\big(\mathrm{fix}^{n}(f)\big)_M}.$$

The number of terms in the last sum being independent of n, it suffices to show that, for each divisor d of M, the corresponding summand approaches 0 as $n \to \infty$. Let $d_1 = 1, \ldots, d_R = d$ denote the divisors of d. Then, using Lemma 11, one gets

$$\frac{\big(\mathrm{fix}^{n}(f)\big)_d}{\big(\mathrm{fix}^{n}(f)\big)_M} = \sum_{i=1}^{R} \frac{f_{d_i}\; d_i^n\; y^{(n)}\big(\frac{d}{d_i}\big)}{\sum_{k|M} f_k\, k^n y^{(n)}\big(\frac{M}{k}\big)} \le \sum_{i=1}^{R} \frac{d_i^n\; f_{d_i} y^{(n)}(d/d_i)}{d_i^n\; f_{d_i} y^{(n)}(M/d_i)} = \sum_{i=1}^{R} \frac{y^{(n)}(d/d_i)}{y^{(n)}(M/d_i)}.$$

This upper bound can be seen to converge to 0 by splitting numerator and denominator of each summand in the last sum into the factors that arise from the prime decomposition of d/d_i and M/d_i. For $r > s$, the quotient

$$\frac{y^{(n)}(p^s)}{y^{(n)}(p^r)} = \prod_{i=0}^{s-1} \frac{1-p^{n+s-1-i}}{1-p^{n+r-1-i}} \prod_{i=s}^{r-1} \frac{1-p^{i+1}}{1-p^{n+r-1-i}} \tag{8}$$

converges to 0 as $n \to \infty$, since the first product tends to $p^{s-r} > 0$, while the second tends to 0. Being composed of factors such as (8), the quotient $\big(\mathrm{fix}^{n}(f)\big)_d / \big(\mathrm{fix}^{n}(f)\big)_M$ tends to 0 for all $d|M$ as $n \to \infty$, which proves the claim. □

Remark. Unlike fix, the operator orb's spectrum is bounded by 0 from below and by 1 from above. One would expect $\mathrm{orb}^n(f)$ to converge to an element of the eigenspace for the largest eigenvalue ($\lambda = 1$), hence to a scalar multiple of $a^{(1)} = a$. Indeed, employing the eigenvector basis representation $f = \sum_{m\ge1} \alpha_m a^{(m)}$, one obtains

$$\lim_{n\to\infty} (\mathrm{orb}^n f)_M = \lim_{n\to\infty} \sum_{m=1}^{M} \alpha_m \mathrm{orb}^n(a^{(m)})_M = \lim_{n\to\infty} \sum_{m=1}^{M} \alpha_m \frac{1}{m^n} a_M^{(m)} = \alpha_1 a_M^{(1)}$$

for all $M \in \mathbb{N}$. In other words, $\mathrm{orb}^n(f) \xrightarrow{n\to\infty} f_1\, a$, for all $f \in \mathcal{A}$.

4 Concluding remarks

The eigenvector a has an interesting consequence in the setting of dynamical (or Artin-Mazur) zeta functions, where one considers the generating function

$$\exp\left(\sum_{m=1}^{\infty} \frac{f_m}{m}\, z^m\right) = \prod_{m=1}^{\infty} (1-z^m)^{-\mathrm{orb}(f)_m},$$

usually for arithmetic functions f that count (isolated) fixed points of a discrete dynamical system, see [4] for background material. Using a together with $a = \mathrm{orb}(a)$, one finds the product identity

$$\exp\left(\sum_{m=1}^{\infty} \frac{a_m}{m}\, z^m\right) = \prod_{m=1}^{\infty} (1-z^m)^{-a_m}$$

or, equivalently, the series identity

$$\sum_{m=1}^{\infty} \frac{a_m}{m} z^m = - \sum_{m=1}^{\infty} a_m \log(1 - z^m),$$

with absolute convergence for $|z| < 1$. It would be interesting to know whether this gives rise to another interpretation of the meaning of the eigenvector a in this setting.

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