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Complementary Equations and Wythoff Sequences

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Abstract

The lower Wythoff sequence $a = (a(n))$ and upper Wythoff sequence $b = (b(n))$ are solutions of many complementary equations $f(a, b) = 0$. Typically, $f(a, b)$ involves composites such as $a(a(n))$ and $a(b(n))$, and each such sequence is treated as a binary word (e.g., aa and ab). Conversely, each word represents a sequence and, as such, is a linear combination of a, b, and 1, in which the coefficients of a and b are consecutive Fibonacci numbers. For example, $baba = 3a + 5b - 6$.

1 Introduction

An example of a complementary equation is

$$a(a(n)) = b(n) - 1, \tag{1}$$

where it is given that the sequences a and b are complementary—that is, they are strictly increasing sequences of positive integers, and every positive integer is in exactly one of the sequences. Given the initial value $a(1) = 1$, equation (1) has as a unique solution the lower Wythoff sequence, $a(n) = \lfloor \tau n \rfloor$, or equivalently, the upper Wythoff sequence, $b(n) = \lfloor \tau^2 n \rfloor$, where $\tau = (1 + \sqrt{5})/2$.

Equation (1) can be abbreviated as $aa = b - 1$. Here, the concatenation aa (which will also be written as a^2) represents composition, and the reduced notation provides a convenient way to express complementary equations, such as $ab = a + b$, $ba = a + b - 1$, and $b^2 = a + 2b$.

The Wythoff sequences are solutions of these equations. The main purpose of this article is to show that *every* word of the sort suggested by a^2, ab, ba, and b^2 lends itself to a simple complementary equation having Wythoff solutions.

Suppose

$$w = l_1 l_2 \cdots l_k$$

is a word on the two-symbol alphabet $\{a, b\}$. We shall use w to denote not only the word but also the sequence $(w(n))$ defined as $w = w_k$, where

$$w_1(n) = l_1(n), \qquad w_2(n) = w_1(l_2(n)), \qquad \ldots, \qquad w_k(n) = w_{k-1}(l_k(n)).$$

Call $k = k(w)$ the *length* of w. Let $m = m(w)$ be the number of occurrences of the letter b in w, and call m the *weight* of w.

Regarding notation, set braces $\{\ \}$ denote fractional parts as well as sets; N denotes the set of positive integers, and the Fibonacci sequence (F_n) is defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

In the following four lemmas, a and b denote the Wythoff sequences. These symbols may also be regarded as representing a general pair of complementary sequences defining a complementary equation; in this case, given the initial value $a(1) = 1$, a solution of the equation is the Wythoff solution, but there are, for some of the equations, also other solutions.

Possibly these lemmas all appear in other settings, but proofs are included here for the sake of completeness.

Lemma 1. $a^2 = b - 1$.

Proof. For every n in N,

$$0 < (\tau - 1)\{\tau n\} < 1,$$

so that

$$\begin{aligned} -1 &= \lfloor \{\tau n\} - \tau\{\tau n\} \rfloor \\ &= \lfloor \tau n - \lfloor \tau n \rfloor - \tau\{\tau n\} \rfloor \\ &= \lfloor \tau n - \tau\{\tau n\} \rfloor - \lfloor \tau n \rfloor, \end{aligned}$$

so that $\lfloor \tau n \rfloor = \lfloor \tau n - \tau\{\tau n\} \rfloor + 1$. Then using $b(n) = \lfloor \tau n \rfloor + n$,

$$\begin{aligned} b(n) &= \lfloor \tau n + n - \tau\{\tau n\} \rfloor + 1 \\ &= \left\lfloor \tau^2 n - \tau\{\tau n\} \right\rfloor + 1 \\ &= \lfloor \tau(\tau n - \{\tau n\} \rfloor + 1 \\ &= \lfloor \tau \lfloor \tau n \rfloor \rfloor + 1 \\ &= a(a(n)) + 1. \end{aligned}$$

$\square$

Lemma 2. $ab = a + b$.

Proof. Adapting a proof in [5], for every n in N we have

$$\begin{aligned}
a(b(n)) &= \left\lfloor \tau \left\lfloor \tau^2 n \right\rfloor \right\rfloor \\
&= \lfloor \tau n + \tau \lfloor \tau n \rfloor \rfloor \\
&= n + \lfloor 2\tau n - \tau n - n + \tau \lfloor \tau n \rfloor \rfloor \\
&= 2 \lfloor \tau n \rfloor + n + \lfloor 2\tau n - 2 \lfloor \tau n \rfloor - \tau n - n + \tau \lfloor \tau n \rfloor \rfloor \\
&= \lfloor \tau n \rfloor + \left\lfloor \tau^2 n \right\rfloor + \left\lfloor 2\tau n - 2 \lfloor \tau n \rfloor - \tau^2 n + \tau \lfloor \tau n \rfloor \right\rfloor \\
&= a(n) + b(n),
\end{aligned}$$

because

$$\left\lfloor 2\tau n - 2 \lfloor \tau n \rfloor - \tau^2 n + \tau \lfloor \tau n \rfloor \right\rfloor = \lfloor (2-\tau)\{\tau n\} \rfloor = 0.$$

□

Lemma 3. $ba = a + b - 1$.

Proof. For every n in N,

$$\begin{aligned}
b(a(n)) &= b(\lfloor \tau n \rfloor) = \lfloor \tau \lfloor \tau n \rfloor \rfloor + \lfloor \tau n \rfloor \\
&= a(a(n)) + a(n),
\end{aligned}$$

and now Lemma 1 applies. □

Lemma 4. $b^2 = a + 2b$.

Proof. Using Lemma 2,

$$b(b(n)) = a(b(n)) + b(n) = a(n) + b(n) + b(n).$$

Lemmas 1-4 show the four words w of length 2 as sequences which, as solutions of complementary equations, are simple linear combinations of a, b, and 1. This result will be extended to longer words in the next section. □

2 Main Theorem on Wythoff sequences

Lemmas 1-4 enable a proof that *every* word, as a sequence, is a linear combination of a, b, and 1 in which the coefficients of a and b are consecutive Fibonacci numbers.

Theorem 5. *Let $w = l_1 l_2 \cdots l_k$ with length $k \geq 2$ and weight m. Then, as a sequence,*

$$w = F_{k+m-2}a + F_{k+m-1}b - c, \tag{2}$$

where $c \geq 0$, invariant of n, is given by

$$c = F_{k+m+1} - w(1).$$

Proof. For $k = 2$, there are four words w, namely a^2, ab, ba, b^2, and by Lemmas 1-4, the complementary equation (2) holds for each of these. As an induction hypothesis, assume for arbitrary $k \geq 2$ and every word

$$w = l_1 l_2 \cdots l_k \tag{3}$$

that (2) holds. Let

$$w' = l_1 l_2 \cdots l_k l_{k+1}$$

be an arbitrary word of length $k+1$, and let k' and m' denote the length and weight of w'.

Case 1. $l_{k+1} = a$, so that $k' = k+1$ and $m' = m$. We have $w' = wa$ where w is as in (3). The induction hypothesis is that

$$w(\upsilon) = F_{k+m-2}a(\upsilon) + F_{k+m-1}b(\upsilon) - c$$

for every υ in N. Thus for any n in N, we put $\upsilon = a(n)$:

$$w'(n) = w(a(n)) = F_{k+m-2}a(a(n)) + F_{k+m-1}b(a(n)) - c,$$

so that

$$\begin{aligned}
w' &= F_{k+m-2}a^2 + F_{k+m-1}ba - c \\
&= F_{k+m-2}(b-1) + F_{k+m-1}(a+b-1) - c \\
&= F_{k+m-1}a + F_{k+m}b - c - F_{k+m} \\
&= F_{k'+m'-2}a + F_{k'+m'-1}b - c - F_{k+m}.
\end{aligned}$$

Case 2. $l_{k+1} = b$, so that $k' = k+1$, $m' = m+1$ and $w' = wb$, and

$$w'(n) = w(b(n)) = F_{k+m-2}a(b(n)) + F_{k+m-1}b(b(n)) - c,$$

which is

$$\begin{aligned}
w' &= F_{k+m-2}ab + F_{k+m-1}b^2 - c \\
&= F_{k+m-2}(a+b) + F_{k+m-1}(a+2b) - c \\
&= F_{k+m}a + F_{k+m+1}b - c \\
&= F_{k'+m'-2}a + F_{k'+m'-1}b - c.
\end{aligned}$$

Thus, (2) is proved, and clearly c is invariant of n. Put $n = 1$ to find from (2) that

$$\begin{aligned}
c &= -w(1) + a(1)F_{k+m-2} + b(1)F_{k+m-1} \\
&= F_{k+m+1} - w(1).
\end{aligned}$$

That $c \geq 0$ follows inductively from the observations that $c \geq 0$ in each of Lemmas 1-4 and that in both cases above, as we pass from w to w', the constant c passes to a new constant $c' \geq c$. $\square$

3 Tables and examples

Table 1. Words of length 2		
$aa = b - 1$		$ba = a + b - 1$
$ab = a + b$		$bb = a + 2b$

Sequences in [7] corresponding to the words in Table 1 are A003622, A003623, A035336, and A101864, respectively.

Table 2. Words of length 3		
$aaa = a + b - 2$		$abb = 2a + 3b$
$aab = a + 2b - 1$		$bab = 2a + 3b - 1$
$aba = a + 2b - 2$		$bba = 2a + 3b - 3$
$baa = a + 2b - 3$		$bbb = 3a + 5b$

Sequences in [7] corresponding to the words in Table 2 are A134859, A134864, A035337, A134861, A134862, A134863, A035338, and A134864, respectively.

Table 3. Words of length 4		
$aaaa = a + 2b - 4$		$abba = 3a + 5b - 5$
$aaab = 2a + 3b - 2$		$baba = 3a + 5b - 6$
$aaba = 2a + 3b - 4$		$bbaa = 3a + 5b - 8$
$abaa = 2a + 3b - 5$		$abbb = 5a + 8b$
$baaa = 2a + 3b$		$babb = 5a + 8b - 1$
$aabb = 3a + 5b - 1$		$bbab = 5a + 8b - 3$
$abab = 3a + 5b - 2$		$bbba = 5a + 8b - 8$
$baab = 3a + 5b - 3$		$bbbb = 8a + 13b$

Example 6. $a^k = F_{k-2}a + F_{k-1}b - F_{k+1} + 1$ for all k in N.

Example 7. $b^k = F_{2k-2}a + F_{2k-1}b$ for all k in N.

4 The constant c

The constant c in Theorem 5 is not easily written out in general. However, we can gain some insights using the following theorem.

Theorem 8. *Suppose, as in Theorem 5, that*

$$w = F_{k+m-2}a + F_{k+m-1}b - c.$$

Then for $i \geq 0$,

$$\begin{aligned}
wa^{2i} &= F_{k+m+2i-2}a + F_{k+m+2i-1}b - c - \sum_{h=1}^{i} F_{k+m+2h} && (4)\\
wa^{2i+1} &= F_{k+m+2i-1}a + F_{k+m+2i}b - c - F_{k+m+2i} - \sum_{h=1}^{i} F_{k+m+2h} && (5)\\
wb^{i} &= F_{k+m+2i-2}a + F_{k+m+2i-1}b - c. && (6)
\end{aligned}$$

Proof. Each equation is easily proved by induction on i. □

As every word is a concatenation of subwords covered by (4)-(6), it follows that the constant c in (2) is a sum of Fibonacci numbers, each appearing at most twice, as in (5), where F_{k+m+2i} occurs twice.

Corollary 9. *Suppose w is a word. In the representation (2), $c = 0$ if and only if $w = b^{i+1}$ or $w = ab^i$ for some $i \geq 1$.*

Proof. The assertion is clearly true for words of length 1. By Lemmas 1-4, among words of length 2, we have $c = 0$ only for $w = b^2$ and $w = ab$. The remaining cases now follow from (6). □

5 Columns of the Wythoff array

Let W denote the Wythoff array [1, 8, 9], for which the entry in row n, column h, is

$$W(n, h) = (n - 1)F_h + \lfloor n\tau \rfloor F_{h+1}.$$

Words w, taken as sequences, have close connections to columns of W.

Theorem 10. *Column h of W is given by*

$$w_h := \begin{cases} ab^{(h-1)/2}a & \text{if } h \text{ is odd} \\ b^{h/2}a & \text{if } h \text{ is even.} \end{cases}$$

Proof. By Theorem 5, for all h and n in N,

$$\begin{aligned}
w_h(n) &= F_{h-1}a(n) + F_hb(n) - F_h \\
&= F_{h-1} \lfloor n\tau \rfloor + F_h(\lfloor n\tau \rfloor + n - 1) \\
&= (F_{h-1} + F_h) \lfloor n\tau \rfloor + (n - 1)F_h \\
&= (n - 1)F_h + \lfloor n\tau \rfloor F_{h+1} \\
&= W(n, h).
\end{aligned}$$

The next theorem tells that the sequences in Corollary 9 are ordered unions of columns of W. □

Theorem 11. *For all h in N,*

$$ab^{h-1} = \textit{ordered} \cup \{w_{2j-1} : \quad j = h, h+1, h+2, ...\}$$

and

$$b^h = \textit{ordered} \cup \{w_{2j} : \quad j = h, h+1, h+2, ...\}.$$

Proof of Theorem 11 is left to the reader, who may wish to prove also, regarding all words not covered by Theorems 10 and 11, that every word a^2w', as a sequence, is a subsequence of column 1 of W, that every baw' is a subsequence of column 2, that every $abaw'$ is a subsequence of column 3, that every $bbaw'$ is a subsequence of column 4, and so on.

6 Concluding remarks

Some complementary equations having Wythoff solutions have been presented. Others, such as $ab - ba = 1$, can be written easily by inspecting Tables 1-3. Most of these equations have not only the Wythoff solutions, but also others. For example, the equation $ab = a + b$ has the sequence $(2n - 1)$ as a solution. One wonders if *general* solutions can be obtained for this and other complementary equations, and not only for the initial value $a(1) = 1$. The interested reader may wish to consult [1] (especially Chapters 7 and 9) and [2, 3, 4, 5, 6].

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