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Positive Integers n Such That $\sigma(\phi(n)) = \sigma(n)$

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Abstract

In this paper, we investigate those positive integers n for which the equality $\sigma(\phi(n)) = \sigma(n)$ holds, where σ is the sum of the divisors function and ϕ is the Euler function.

1 Introduction

For a positive integer n we write $\sigma(n)$ and $\phi(n)$ for the sum of divisors function and for the Euler function of n, respectively. In this note, we study those positive integers n such that

$$\sigma(\phi(n)) = \sigma(n)$$

holds. This is sequence A033631 in Sloane's Online Encylopedia of Integer Sequences. Let $\mathcal{A}$ be the set of all such positive integers n and for a positive real number x we put $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. Our result is the following.

Theorem 1. *The estimate*

$$\#\mathcal{A}(x) = O\left(\frac{x}{(\log x)^2}\right)$$

holds for all real numbers $x > 1$.

The above upper bound might actually be the correct order of magnitude of $\#\mathcal{A}(x)$. Indeed, note that if m is such that

$$\sigma(2\phi(m)) = 2\sigma(m) \tag{1}$$

and if $q > m$ is a Sophie Germain prime, that is a prime number q such that $p = 2q + 1$ is also prime, then $n = mp \in \mathcal{A}$. The numbers $m = 2318,\ 2806,\ 5734,\ 5937,\ 7198,\ 8097,\ \ldots$ all satisfy relation (1), and form Sloane's sequence A137733. More generally, if k and m are positive integers such that

$$\sigma(2^k\phi(m)) = 2^k\sigma(m) \tag{2}$$

and $q_1 < \ldots < q_k$ are primes with $p_i = 2q_i + 1$ also primes for $i = 1, \ldots, k$ and $q_1 > m$, then $n = p_1 \ldots p_k m \in \mathcal{A}$. Now recall that the *Prime K-tuplets Conjecture* of Dickson (see, for instance, [2, 4, 8]) asserts that, except in cases ruled out by obvious congruence conditions, K linear forms $a_i n + b_i$, $i = 1, \ldots, K$, take prime values simultaneously for about $cx/(\log x)^K$ integers $n \le x$, where c is a positive constant which depends only on the given linear forms. Under this conjecture (applied with $K = 2$ and the linear forms n and $2n+1$), we obtain that there should be $\gg x/(\log x)^2$ Sophie Germain primes $q \le x$, which suggests that $\#\mathcal{A}(x) \gg x/(\log x)^2$. We will come back to the Sophie Germain primes later.

Throughout, we use the Vinogradov symbols $\gg$ and $\ll$ and the Landau symbols O and o with their regular meanings. We use log for the natural logarithm and p, q and r with or without subscripts for prime numbers.

2 Preliminary Results

In this section, we point out a subset $\mathcal{B}(x)$ of all the positive integers $n \le x$ of cardinality $O(x/(\log x)^2)$. For the proof of Theorem 1 we will work only with the positive integers $n \in \mathcal{A}(x)\backslash\mathcal{B}(x)$. Further, x_0 is a sufficiently large positive real number, where the meaning of sufficiently large may change from a line to the next.

We put

$$y = \exp\left(\frac{\log x}{\log\log x}\right).$$

For a positive integer n we write $P(n)$ for the largest prime factor of n. It is well known that

$$\Psi(x,y) = \#\{n \le x \mid P(n) \le y\} = x\exp(-(1+o(1))u\log u) \qquad (u \to \infty), \tag{3}$$

where $u = \log x/\log y$, provided that $u \le y^{1/2}$ (see [1], Corollary 1.3 of [6], or Chapter III.5 of [9]). In our case, $u = \log\log x$, so, in particular, the condition $u \le y^{1/2}$ is satisfied for $x > x_0$. We deduce that

$$u\log u = (\log\log x)(\log\log\log x).$$

Thus, if we set $\mathcal{B}_1(x) = \{n \le x \mid P(n) \le y\}$, then

$$\begin{aligned} \#\mathcal{B}_1(x) &= \Psi(x,y) = x \exp\left(-(1+o(1))(\log\log x)(\log\log\log x)\right) \\ &< \frac{x}{(\log x)^{10}} \qquad \text{for } x > x_0. \end{aligned} \tag{4}$$

We now let

$$z = (\log x)^{26}$$

and for a positive integer n we write $\rho(n)$ for its largest *squarefull divisor*. Recall that a positive integer m is squarefull if $p^2 \mid m$ whenever p is a prime factor of m. It is well known that if we write $\mathcal{S}(t) = \{m \le t \mid m \text{ is squarefull}\}$, then

$$\#\mathcal{S}(t) = \frac{\zeta(3/2)}{\zeta(3)} t^{1/2} + O(t^{1/3}),$$

where ζ is the Riemann Zeta-Function (see, for example, Theorem 14.4 in [7]). By partial summation, we easily get that

$$\sum_{\substack{m \ge t \\ m \text{ squarefull}}} \frac{1}{m} \ll \frac{1}{t^{1/2}}. \tag{5}$$

We now let $\mathcal{B}_2(x)$ be the set of positive integers $n \le x$ such that one of the following conditions holds:

(i) $\rho(n) \ge z$,

(ii) $p \mid n$ for some prime p such that $\rho(p \pm 1) \ge z$,

(iii) there exist primes r and p such that $p \mid n$, $p \equiv \pm 1 \pmod{r}$ and $\rho(r \pm 1) \ge z$.

We will find an upper bound for $\#\mathcal{B}_2(x)$. Let $\mathcal{B}_{2,1}(x)$ be the set of those $n \in \mathcal{B}_2(x)$ for which (i) holds. We note that for every $n \in \mathcal{B}_{2,1}(x)$ there exists a squarefull positive integer $d \ge z$ such that $d \mid n$. For a fixed d, the number of such $n \le x$ does not exceed x/d. Hence,

$$\#\mathcal{B}_{2,1}(x) \le \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \frac{x}{d} \ll \frac{x}{(\log x)^{13}}, \tag{6}$$

where we have used estimate (5) with $t = z$. Now let $\mathcal{B}_{2,2}(x)$ be the set of those $n \in \mathcal{B}_2(x)$ for which (ii) holds. We note that each $n \in \mathcal{B}_{2,2}(x)$ has a prime divisor p such that $p \equiv \pm 1 \pmod{d}$, where d is as above. Given d and p, the number of such $n \le x$ does not exceed x/p. Summing up over all choices of p and d we get that

$$\begin{aligned} \#\mathcal{B}_{2,2}(x) &\le \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \sum_{\substack{p \equiv \pm 1 \pmod{d} \\ p \le x}} \frac{x}{p} \ll x \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \frac{\log\log x}{\phi(d)} \\ &\ll x(\log\log x)^2 \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \frac{1}{d} \ll \frac{x(\log\log x)^2}{(\log x)^{13}}, \end{aligned} \tag{7}$$

where in the above estimates we used aside from estimate (5), the fact that the estimate

$$\sum_{\substack{p \equiv a \pmod b \\ p \le x}} \frac{1}{p} \le \frac{1}{p_1(a,b)} + O\left(\frac{\log\log x}{\phi(b)}\right) \tag{8}$$

holds uniformly in a, b and x when $b \le x$, where a and b are coprime and $p_1(a,b)$ is the smallest prime number $p \equiv a \pmod b$ (note that $p_1(1,b) \ge b+1$ and $p_1(-1,b) = p_1(b-1,b) \ge b-1 \ge \phi(b)$ for all $b \ge 2$), together with the well known minimal order $\phi(n)/n \gg 1/\log\log x$, valid for n in the interval $[1,x]$.

Let $\mathcal{B}_{2,3}(x)$ be the set of those $n \in \mathcal{B}_2(x)$ for which (iii) holds. Then there exists r such that $r \equiv \pm 1 \pmod d$ for some d as above, as well as $p \mid n$ such that $r \mid p-1$ or $r \mid p+1$. Given d, r and p, the number of such $n \le x$ does not exceed x/p, and now summing up over all choices of d, r and p, we get that

$$\begin{aligned}
\#\mathcal{B}_{2,3}(x) &\le \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \sum_{\substack{r \equiv \pm 1 \pmod d \\ r \le x}} \sum_{\substack{p \equiv \pm 1 \pmod r \\ p \le x}} \frac{x}{p} \\
&\ll x \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \sum_{\substack{r \equiv \pm 1 \pmod d \\ r \le x}} \frac{\log\log x}{\phi(r)} \\
&\ll x(\log\log x)^2 \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \sum_{\substack{r \equiv \pm 1 \pmod d \\ r \le x}} \frac{1}{r} \\
&\ll x(\log\log x)^3 \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \frac{1}{\phi(d)} \\
&\ll x(\log\log x)^4 \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \frac{1}{d} \ll \frac{x(\log\log x)^4}{(\log x)^{13}},
\end{aligned} \tag{9}$$

where in the above estimates we used again estimate (5), estimate (8) twice as well as the minimal order of the Euler function on the interval $[1,x]$.

Hence, using estimates (6)–(9), we get

$$\#\mathcal{B}_2(x) \le \#\mathcal{B}_{2,1}(x) + \#\mathcal{B}_{2,2}(x) + \#\mathcal{B}_{2,3}(x) < \frac{x}{(\log x)^{10}} \qquad \text{for } x > x_0. \tag{10}$$

We now put

$$w = 10 \log\log x$$

and set

$$S(w,x) = \sum_{\substack{\omega(m) \ge w \\ m \le x}} \frac{1}{m},$$

where $\omega(m)$ denotes the number of distinct prime factors of the positive integer m. Note that, using the fact that $\sum_{p\le t}\frac{1}{p}=\log\log t+O(1)$ and Stirling's formula $k!=(1+o(1))k^ke^{-k}\sqrt{2\pi k}$, we have

$$\begin{aligned}
S(w,x) &= \sum_{k\ge w}\sum_{\substack{\omega(m)=k\\ m\le x}}\frac{1}{m}=\sum_{k\ge w}\frac{1}{k!}\left(\sum_{p\le x}\sum_{\alpha\ge 1}\frac{1}{p^\alpha}\right)^k\\
&= \sum_{k\ge w}\frac{1}{k!}\left(\sum_{p\le x}\frac{1}{p}+O\left(\sum_{p\ge 2}\frac{1}{p^2}\right)\right)^k \ll \sum_{k\ge w}\frac{1}{k!}(\log\log x+O(1))^k\\
&\le \sum_{k\ge w}\left(\frac{e\log\log x+O(1)}{k}\right)^k\le\sum_{k\ge w}\left(\frac{e\log\log x+O(1)}{w}\right)^k\\
&\ll \left(\frac{e\log\log x+O(1)}{w}\right)^w\ll\frac{1}{(\log x)^{10\log(10/e)}}<\frac{1}{(\log x)^{11}} \qquad (11)
\end{aligned}$$

for $x>x_0$ because $10\log(10/e)>11$.

We now let $\mathcal{B}_3(x)$ be the set of positive integers $n\le x$ such that one of the following conditions holds:

(i) $\omega(n)\ge w$,

(ii) $p\mid n$ for some prime p for which $\omega(p\pm 1)\ge w$,

(iii) there exist primes r and p such that $p\mid n$, $p\equiv\pm 1 \pmod r$ and $\omega(r\pm 1)\ge w$.

Let $\mathcal{B}_{3,1}(x)$, $\mathcal{B}_{3,2}(x)$ and $\mathcal{B}_{3,3}(x)$ be the sets of $n\in\mathcal{B}_3(x)$ for which (i), (ii) and (iii) hold, respectively.

To bound the cardinality of $\mathcal{B}_{3,1}(x)$, note that, using (11), we have

$$\#\mathcal{B}_{3,1}(x)=\sum_{\substack{\omega(n)\ge w\\ n\le x}}1\le\sum_{\substack{\omega(n)\ge w\\ n\le x}}\frac{x}{n}=xS(w,x)<\frac{x}{(\log x)^{11}} \qquad (12)$$

for $x>x_0$. To bound the cardinality of $\mathcal{B}_{3,2}(x)$, note that each $n\in\mathcal{B}_{3,2}(x)$ admits a prime divisor p such that $\omega(p\pm 1)\ge w$. Fixing such a p, the number of such $n\le x$ does not exceed x/p. Summing up over all such p we have, again in light of (11),

$$\begin{aligned}
\#\mathcal{B}_{3,2}(x) &\le \sum_{\substack{\omega(p\pm1)\ge w\\ p\le x}}\frac{x}{p}\le x\left(\sum_{\substack{\omega(p+1)\ge w\\ p+1\le x+1}}\frac{2}{p+1}+\sum_{\substack{\omega(p-1)\ge w\\ p-1\le x}}\frac{1}{p-1}\right)\\
&\le x(2S(w,x+1)+S(w,x))<3xS(w,x)+2\\
&\ll \frac{x}{(\log x)^{11}} \qquad (13)
\end{aligned}$$

for $x > x_0$. To bound the cardinality of $\mathcal{B}_{3,3}(x)$, note that for each $n \in \mathcal{B}_{3,3}(x)$ there exist some prime r with $\omega(r \pm 1) \geq w$ and some prime $p \mid n$ such that $p \equiv \pm 1 \pmod{r}$. Given such r and p, the number of such $n \leq x$ does not exceed x/p. Summing up over all choices of r and p given above we get, again using (11),

$$\begin{aligned}
\#\mathcal{B}_{3,3}(x) &\leq \sum_{\substack{\omega(r\pm1)\geq w \\ r\leq x}} \sum_{\substack{p\equiv \pm 1 \pmod{r} \\ p\leq x}} \frac{x}{p} = x(\log\log x) \sum_{\substack{\omega(r\pm1)\geq w \\ r\leq x}} \frac{1}{\phi(r)} \\
&= x(\log\log x) \sum_{\substack{\omega(r\pm1)\geq w \\ r\leq x}} \frac{1}{r-1} \\
&\leq x(\log\log x) \left(\sum_{\substack{\omega(r-1)\geq w \\ r-1\leq x}} \frac{1}{r-1} + \sum_{\substack{\omega(r+1)\geq w \\ r+1\leq x}} \frac{3}{r+1} \right) \\
&\leq x(\log\log x)(S(w,x) + 3S(w,x+1)) \\
&\leq 4x(\log\log x)S(w,x) + O(\log\log x) \ll \frac{x(\log\log x)}{(\log x)^{11}} \qquad (14)
\end{aligned}$$

for $x > x_0$.

Hence, using estimates (12) to (14), we get

$$\#\mathcal{B}_3(x) \leq \#\mathcal{B}_{3,1}(x) + \#\mathcal{B}_{3,2}(x) + \#\mathcal{B}_{3,3}(x) < \frac{x}{(\log x)^{10}} \qquad \text{for } x > x_0. \qquad (15)$$

We now let

$$\mathcal{B}_4(x) = \{n \leq x \mid n \notin (\mathcal{B}_1(x) \cup \mathcal{B}_2(x)) \text{ and } p^2 \mid \phi(n) \text{ for some } p > z\}.$$

Let $n \in \mathcal{B}_4(x)$ and let $p^2 \mid \phi(n)$ for some prime p. Then it is not possible that $p^2 \mid n$ (because $n \notin \mathcal{B}_2(x)$), nor is it possible that $p^2 \mid q-1$ for some prime factor q of n (again because $n \notin \mathcal{B}_2(x)$). Thus, there must exist distinct primes q and r dividing n such that $q \equiv 1 \pmod{p}$ and $r \equiv 1 \pmod{p}$. Fixing such p, q and r, the number of acceptable values of such $n \leq x$ does not exceed $x/(qr)$. Summing up over all the possible values of p, q and r we arrive at

$$\begin{aligned}
\#\mathcal{B}_4(x) &\leq \sum_{z\leq p\leq x} \sum_{\substack{q\equiv 1 \pmod{p} \\ r\equiv 1 \pmod{p} \\ q<r,\ qr\leq x}} \frac{x}{qr} \leq x \sum_{z\leq p\leq x} \frac{1}{2} \left(\sum_{\substack{q\equiv 1 \pmod{p} \\ q\leq x}} \frac{1}{q} \right)^2 \\
&\ll x \sum_{z\leq p\leq x} \frac{(\log\log x)^2}{(p-1)^2} \ll x(\log\log x)^2 \sum_{z\leq p\leq x} \frac{1}{p^2} \\
&\ll \frac{x(\log\log x)^2}{(\log x)^{13}},
\end{aligned}$$

where we used estimates (8) and (5). Hence,

$$\#\mathcal{B}_4(x) < \frac{x}{(\log x)^{10}} \qquad \text{for all } x > x_0. \tag{16}$$

We now let

$$\tau = \exp\left((\log\log x)^2\right)$$

and let $\mathcal{B}_5(x)$ stand for the set of $n \le x$ which are multiples of a prime p for which either $p-1$ or $p+1$ has a divisor $d > \tau$ with $P(d) < z$. Fix such a pair d and p. Then the number of $n \le x$ divisible by p is at most x/p. This shows that

$$\#\mathcal{B}_5(x) \le \sum_{\substack{P(d)<z \\ \tau<d\le x}} \sum_{\substack{p\equiv\pm 1 \pmod d \\ p\le x}} \frac{x}{p} \ll x\log\log x \sum_{\substack{P(d)<z \\ \tau<d\le x}} \frac{1}{d}. \tag{17}$$

It follows easily by partial summation from the estimates (3) for $\Psi(x,v)$, that if we write $v = \log\tau/\log z$, then

$$S = \sum_{\substack{P(d)<z \\ \tau<d\le x}} \frac{1}{d} \le \frac{\log x}{\exp((1+o(1))v\log v)}.$$

Since $v = (\log\log x)/26$, we get that

$$v\log v = (1/26 + o(1))(\log\log x)(\log\log\log x)$$

and therefore that

$$S \le \frac{\log x}{\exp((1+o(1))v\log v)} < \frac{1}{(\log x)^{11}} \tag{18}$$

for all $x > x_0$, which together with estimate (17) gives

$$\#\mathcal{B}_5(x) < \frac{x}{(\log x)^{10}} \qquad \text{for } x > x_0. \tag{19}$$

Thus, setting

$$\mathcal{B}(x) = \bigcup_{i=1}^{5} \mathcal{B}_i(x), \tag{20}$$

we get, from estimates (4), (10), (15), (16) and (19) that

$$\#\mathcal{B}(x) \le \sum_{i=1}^{5} \#\mathcal{B}_i(x) \ll \frac{x}{(\log x)^{10}} \qquad \text{for all } x > x_0. \tag{21}$$

3 The Proof of Theorem 1

We find it convenient to prove a stronger theorem.

Theorem 2. *Let a and b be any fixed positive integers. Setting*

$$\mathcal{A}_{a,b} = \{n \mid \sigma(a\phi(n)) = b\sigma(n)\},$$

then the estimate

$$\#\mathcal{A}_{a,b}(x) \ll_{a,b} \frac{x}{(\log x)^2}$$

holds for all $x \geq 3$.

Proof. Let x be large and let $\mathcal{B}(x)$ be as in (20). We assume that $n \leq x$ is a positive integer not in $\mathcal{B}(x)$. We let $\mathcal{A}_1(x)$ be the set of $n \in \mathcal{A}_{a,b}(x) \backslash \mathcal{B}(x)$ for which $(P(n)-1)/2$ is not prime but such that there exists a prime number $r > z$ and another prime number $q \mid P(n) - 1$ for which $r \mid \gcd(P(n) + 1, q + 1)$. To count the number of such positive integers $n \leq x$, let r and q be fixed primes such that $r \mid q + 1$, and let P be a prime such that $q \mid P - 1$ and $r \mid P + 1$. The number of positive integers $n \leq x$ such that $P(n) = P$ does not exceed x/P. Note that the congruences $P \equiv -1 \pmod r$ and $P \equiv 1 \pmod q$ are equivalent to $P \equiv a_{q,r} \pmod{qr}$, where $a_{q,r}$ is the smallest positive integer m satisfying $m \equiv -1 \pmod r$ and $m \equiv 1 \pmod q$. We distinguish two instances:

Case 1: $qr < P$.

Let $\mathcal{A}_1'(x)$ be the set of such integers $n \in \mathcal{A}_1(x)$. Then

$$\begin{aligned}
\#\mathcal{A}_1'(x) &\leq \sum_{z<r\leq x} \sum_{\substack{q\equiv -1 \pmod r \\ q\leq x}} \sum_{\substack{P\equiv a_{q,r} \pmod{qr} \\ qr<P\leq x}} \frac{x}{P} \\
&\ll x\log\log x \sum_{z<r\leq x} \sum_{\substack{q\equiv -1 \pmod r \\ q\leq x}} \frac{1}{\phi(qr)} \\
&\ll x\log\log x \sum_{z<r\leq x} \frac{1}{r} \sum_{\substack{q\equiv -1 \pmod r \\ q\leq x}} \frac{1}{q} \\
&\ll x(\log\log x)^2 \sum_{z<r\leq x} \frac{1}{r\phi(r)} \\
&\ll x(\log\log x)^2 \sum_{z<r} \frac{1}{r^2} \ll \frac{x(\log\log x)^2}{(\log x)^{11}}, \qquad (22)
\end{aligned}$$

where in the above inequalities we used estimates (8) and (5).

Case 2: $qr \geq P$.

Let $\mathcal{A}_1''(x)$ be the set of such integers $n \in \mathcal{A}_1(x)$. Here we write $n = Pm$. Note that $P > P(m)$ because $y > z$ for large x and $n \notin \mathcal{B}_1(x) \cup \mathcal{B}_2(x)$. Furthermore, since $r \mid q+1$, we may write $q = r\ell - 1$. Since $q \mid P - 1$, we may write $P = sq + 1 = s(r\ell - 1) + 1 = sr\ell + 1 - s$. Since $r \mid P + 1$, we get that $1 - s \equiv -1 \pmod r$, and therefore that $s \equiv 2 \pmod r$. Hence, there exists a nonnegative integer λ such that $s = \lambda r + 2$. If $\lambda = 0$, then $s = 2$ leading to

$P = 2q+1$, which is impossible. Thus, $\lambda > 0$. Let us fix the value of λ as well as that of r. Then

$$r\ell - 1 = q \qquad \text{and} \qquad (\lambda r^2 + 2r)\ell - (\lambda r + 1) = P \tag{23}$$

are two linear forms in the variable ℓ which are simultaneously primes. Note that $P \le x/m$ and since $P \ge \ell r(\lambda r + 2)$, we get that $\ell \le x/(mr(\lambda r+2))$. In particular, $mr(\lambda r + 2) \le x$. Recall that a typical consequence of Brun's sieve (see for example Theorem 2.3 in [3]), is that if

$$L_1(m) = Am + B \qquad \text{and} \qquad L_2(m) = Cm + D$$

are linear forms in m with integer coefficients such that $AD - BC \neq 0$ and if we write E for the product of all primes p dividing $ABCD(AD-BC)$, then the number of positive integers $m \le y$ such that $L_1(m)$ and $L_2(m)$ are simultaneously primes is

$$\le \frac{Ky}{(\log y)^2}\left(\frac{E}{\phi(E)}\right)^2$$

for some absolute constant K. Applying this result for our linear forms in ℓ shown at (23) for which $A = r,\ B = -1,\ C = \lambda r^2 + 2r$ and $D = -(\lambda r + 1)$, we get that the number of acceptable values for ℓ does not exceed

$$\begin{aligned} & K\frac{x}{mr(\lambda r+2)\left(\log(x/(mr(\lambda r+2)))\right)^2}\left(\frac{(\lambda r+2)(\lambda r+1)r}{\phi((\lambda r+2)(\lambda r+1)r)}\right)^2 \\ \ll \ & \frac{x(\log\log x)^2}{mr(\lambda r+2)}, \end{aligned}$$

where for the rightmost inequality we used again the minimal order of the Euler function on the interval $[1, x]$. Here, K is some absolute constant. Summing up over all possible values of λ, r and m, we get

$$\begin{aligned} \#\mathcal{A}_1''(x) & \ll \sum_{z<r\le x}\sum_{1\le\lambda\le x}\sum_{1\le m\le x}\frac{x(\log\log x)^2}{mr(\lambda r+2)} \\ & < x(\log\log x)^2\left(\sum_{z<r\le x}\frac{1}{r^2}\right)\left(\sum_{1\le\lambda\le x}\frac{1}{\lambda}\right)\left(\sum_{1\le m\le x}\frac{1}{m}\right) \\ & \ll \frac{x(\log\log x)^2(\log x)^2}{(\log x)^{13}} = \frac{x(\log\log x)^2}{(\log x)^{11}}, \end{aligned} \tag{24}$$

where we used again estimate (5).

From estimates (22) and (24), we get

$$\#\mathcal{A}_1(x) \le \#\mathcal{A}_1'(x) + \mathcal{A}_1''(x) < \frac{x}{(\log x)^{10}} \qquad \text{for all } x > x_0. \tag{25}$$

Now let $\mathcal{A}_2(x)$ be the set of those $n \in \mathcal{A}_{a,b}(x)\backslash(\mathcal{A}_1(x)\cup\mathcal{B}(x))$ and such that $(P(n)-1)/2$ is not prime. With $n \in \mathcal{A}_2(x)$, we get that $n = Pm$, where $P > P(m)$ for $x > x_0$ because

$y > z$ for large x and $n \notin \mathcal{B}_1(x) \cup \mathcal{B}_2(x)$. Then $b\sigma(n) = b\sigma(m)(P+1)$. Let d_1 be the largest divisor of $P+1$ such that $P(d_1) \le z$. Then $P+1 = d_1\ell_1$, and $b\sigma(n) = b\sigma(m)d_1\ell_1$. Furthermore, $\phi(n) = \phi(m)(P-1)$, so that $a\phi(n) = a\phi(m)(P-1)$. Let d be the largest divisor of $P-1$ which is z-smooth; that is, with $P(d) \le z$. Then $P-1 = d\ell$. Note that ℓ is squarefree (because $n \notin \mathcal{B}_2(x)$) and ℓ and $\phi(m)$ are coprime (for if not, there would exist a prime $r > z$ such that $r \mid \ell$ and $r \mid \phi(m)$, so that $r^2 \mid \phi(n)$, which is impossible because $n \notin \mathcal{B}_4(x)$). Since $z > a$ for $x > x_0$, we get that $a\phi(m)d$ and ℓ are coprime, and therefore that $\sigma(\phi(n)) = \sigma(a\phi(m)d)\sigma(\ell)$. We thus get the equation

$$\sigma(a\phi(m)d)\sigma(\ell) = b\sigma(m)d_1\ell_1.$$

Now note that ℓ_1 and $\sigma(\ell)$ are coprime. Indeed, if not, since ℓ is squarefree, there would exist a prime factor r of ℓ_1 (necessarily exceeding z) dividing $q+1$ for some prime factor q of ℓ, that is of $P-1$. But this is impossible because $n \notin \mathcal{A}_1(x)$.

Thus, $\ell_1 \mid \sigma(a\phi(m)d)$. Note now that $Pm \le x$, so that $m \le x/P \le x/y$. Furthermore, $\max\{d, d_1\} \le \tau$ because $n \notin \mathcal{B}_5(x)$. Let us now fix m, d and d_1. Then $\ell_1 \mid \sigma(a\phi(m)d)$, and therefore the number of choices for ℓ_1 does not exceed $\tau(\sigma(a\phi(m)d))$, where $\tau(k)$ is the number of divisors of the positive integer k. The above argument shows that if we write $\mathcal{M}$ for the set of such acceptable values for m, then

$$\#\mathcal{A}_2(x) \le \frac{x\tau^2}{y} \max\{\tau(\sigma(a\phi(m)d)) \mid m \in \mathcal{M},\ d \le \tau,\ d_1 \le \tau\}. \tag{26}$$

To get an upper bound on $\tau(\sigma(a\phi(m)d))$, we write $a\phi(m)d$ as

$$a\phi(m)d = AB,$$

where A is squarefull, B is squarefree, and A and B are coprime. Clearly,

$$\sigma(a\phi(m)d) = \sigma(AB) = \sigma(A)\sigma(B),$$

so that

$$\tau(\sigma(a\phi(m)d)) \le \tau(\sigma(A))\tau(\sigma(B)).$$

Since B is squarefree, it is clear that

$$\sigma(B) \Big| \prod_{q \mid a\phi(m)d} (q+1).$$

Furthermore,

$$\omega(a\phi(m)d) \le \omega(a\phi(n)) \le \omega(a) + \omega(n) + \sum_{p \mid n} \omega(p-1) \le w^2 + w + O(1), \tag{27}$$

so that we have $\omega(a\phi(m)d) \le 2w^2$ if $x > x_0$. The above inequalities follow from the fact that $n \notin \mathcal{B}_3(x)$. Also, for each one of the at most $2w^2$ prime factors q of $a\phi(m)d$, the number $q+1$ has at most w prime factors itself and its squarefull part does not exceed z, again because

$n \notin \mathcal{B}_3(x)$. In conclusion, $\sigma(B) \mid C$, where C is a number with at most $2w^3$ prime factors whose squarefull part does not exceed z^{2w^2}. Thus,

$$\tau(\sigma(B)) \le \tau(C) \le 2^{\omega(C)}\tau(\rho(C)),$$

where $\omega(C) \le 2w^3$ and $\rho(C) \le z^{2w^2}$. Clearly,

$$\tau(\rho(C)) \le \rho(C) \le z^{w^2} = \exp(w^2 \log z) = \exp(O(\log\log x)^3),$$

and since also

$$2^{\omega(C)} \le 2^{2w^3} = \exp(O(\log\log x)^3),$$

we finally get that

$$\tau(\sigma(B)) = \exp(O(\log\log x)^3). \tag{28}$$

We now deal with $\sigma(A)$. We first note that $P(A) \le z$ for $x > x_0$. Indeed, if $q > z$ divides A, then $q^2 \mid a\phi(n)$. If $x > x_0$, then $z > a$, in which case the above divisibility relation forces $q^2 \mid \phi(n)$ which is not possible because $n \notin \mathcal{B}_3(x)$. By looking at the multiplicities of the prime factors appearing in A, we easily see that

$$A \Big| \rho(a)\rho(d)\prod_{p|m}\rho(p-1)\left(\prod_{\substack{q \mid a\phi(m)d\\ q\le z}} q\right)^{\omega(a\phi(m)d)}.$$

As we have seen at estimate (27), $\omega(a\phi(m)d) \le 2w^2$, in which case the above relation shows that

$$A \le \rho(a) z^{\omega(a\phi(m)d)+\omega(a\phi(m)d)^2} \ll z^{4w^4+2w^2} = \exp(O(\log\log x)^5).$$

But since $\sigma(A) < A^2$ and $\tau(\sigma(A)) \ll \sigma(A) \le A^2$, we get that

$$\tau(\sigma(A)) = \exp(O(\log\log x)^5), \tag{29}$$

which together with estimate (28) gives

$$\tau(\sigma(a\phi(m)d)) \le \tau(\sigma(A))\tau(\sigma(B)) = \exp(O(\log\log x)^5).$$

Returning to estimate (26), we get that

$$\begin{aligned}
\#\mathcal{A}_2(x) &\le \frac{x\tau^2}{y}\exp(O(\log\log x)^5) = x\exp\left(-\frac{\log x}{\log\log x} + O((\log\log x)^5)\right)\\
&< \frac{x}{(\log x)^{10}} \qquad \text{for all } x > x_0.
\end{aligned} \tag{30}$$

Thus, writing $\mathcal{C}(x) = \mathcal{A}_{a,b}(x)\backslash(\mathcal{A}_1(x) \cup \mathcal{A}_2(x) \cup \mathcal{B}(x))$, we get that

$$\mathcal{A}_{a,b}(x) = \#\mathcal{C}(x) + O\left(\frac{x}{(\log x)^{10}}\right). \tag{31}$$

Moreover, if $n \in \mathcal{C}(x)$, then $n = Pm$, with $P > P(m)$ for $x > x_0$, and $(P-1)/2$ is a prime. Hence, $P - 1 = 2q$, where q is a Sophie Germain prime. Since also $P \le x/m$, it follows by Brun's method that the number of such values for P is

$$\ll \frac{x}{m(\log x/m)^2} \le \frac{x}{m(\log y)^2} = \frac{x(\log\log x)^2}{m(\log x)^2}.$$

In the above inequalities we used the fact that $x/m \ge P \ge y$. Summing up over all $m \le x$, we get

$$\#\mathcal{C}(x) \ll \sum_{m \le x} \frac{x(\log\log x)^2}{m(\log x)^2} \ll \frac{x(\log\log x)^2}{\log x}. \tag{32}$$

In particular,

$$\#\mathcal{A}_{a,b}(x) \ll \frac{x(\log\log x)^2}{\log x}. \tag{33}$$

This is weaker than the bound claimed by our Theorem 2. However, it implies, by partial summation, that

$$\begin{aligned} \sum_{n \in \mathcal{A}_{a,b}(x)} \frac{1}{n} &\le 1 + \int_{2_-}^{x} \frac{1}{t} d(\#\mathcal{A}_{a,b}(t)) \\ &= 1 + \frac{\#\mathcal{A}_{a,b}(t)}{t}\Big|_{t=2_-}^{t=x} + O\left(\int_{2_-}^{x} \frac{t(\log\log t)^2}{t^2 \log t} dt\right) \\ &= O((\log\log x)^3). \end{aligned} \tag{34}$$

To get some improvement, we return to $\mathcal{C}(x)$. Let $n \in \mathcal{C}(x)$ and write it as $n = Pm$, where $P > P(m)$. Write also $P = 2q + 1$. Then $\phi(n) = 2\phi(m)q$. Moreover, $q > (y-1)/2 > z$ for $x > x_0$ so that q does not divide $a\phi(m)$ (because $q > a$ and $n \notin \mathcal{B}_3(x)$). Hence, $\sigma(a\phi(n)) = \sigma(2a\phi(m))(q+1)$. On the other hand, $b\sigma(n) = b\sigma(m)(P+1) = 2b\sigma(m)(q+1)$. Thus, the equation $\sigma(a\phi(n)) = \sigma(bn)$ forces $\sigma(2a\phi(m)) = 2b\sigma(m)$, implying that $m \in \mathcal{A}_{2a,2b}(x/P) \subset \mathcal{A}_{2a,2b}(x/y)$. The argument which leads to estimate (32) now provides the better estimate

$$\#\mathcal{C}(x) \ll \sum_{m \in \mathcal{A}_{2a,2b}(x)} \frac{x(\log\log x)^2}{m(\log x)^2} \ll \frac{x(\log\log x)^5}{(\log x)^2}.$$

In particular, we get

$$\#\mathcal{A}_{a,b}(x) \ll \frac{x(\log\log x)^5}{(\log x)^2}. \tag{35}$$

This is still somewhat weaker than what Theorem 2 claims. However, it implies that the sum of the reciprocals of the numbers in $\mathcal{A}_{a,b}$ is convergent. In fact, by partial summation, it follows, as in estimates (34), that

$$\begin{aligned}
\sum_{\substack{n\in\mathcal{A}_{a,b}\\ n\geq y}}\frac{1}{n} &\leq \int_{y-}^{\infty}\frac{1}{t}d(\#\mathcal{A}_{a,b}(t))\\
&= \frac{\#\mathcal{A}_{a,b}(t)}{t}\Big|_{t=y-}^{t=\infty}+O\left(\int_{y-}^{x}\frac{t(\log\log t)^5}{t^2(\log t)^2}dt\right)\\
&\ll \frac{(\log\log y)^5}{(\log y)^2}+\int_{y-}^{\infty}\frac{1}{t(\log t)^{3/2}}dt\\
&\ll \frac{1}{(\log y)^{1/2}}. \qquad (36)
\end{aligned}$$

We now take another look at $\mathcal{C}(x)$. Let again $n\in\mathcal{C}(x)$ and write $n=Pm$. Fixing m and using the fact that $P\leq x/m$ is a prime such that $(P-1)/2$ is also a prime, we get that the number of choices for P is

$$\ll \frac{x}{m(\log(x/m))^2}.$$

Hence,

$$\#\mathcal{C}(x)\ll \sum_{m\in\mathcal{A}_{2a,2b}(x/y)}\frac{x}{m(\log(x/m))^2}.$$

We now split the above sum at $m=x^{1/2}$ and use estimate (36) to get

$$\begin{aligned}
\#\mathcal{C}(x) &\leq \sum_{m\in\mathcal{A}_{2a,2b}(x^{1/2})}\frac{x}{m(\log(x/m))^2}+\sum_{\substack{m\in\mathcal{A}_{2a,2b}\\ x^{1/2}\leq m\leq x/y}}\frac{x}{m(\log(x/m))^2}\\
&\leq \frac{x}{(\log x^{1/2})^2}\sum_{m\in\mathcal{A}_{2a,2b}}\frac{1}{m}+\frac{x}{(\log y)^2}\sum_{\substack{m\in\mathcal{A}_{2a,2b}\\ m\geq x^{1/2}}}\frac{1}{m}\\
&\ll \frac{x}{(\log x)^2}+\frac{x}{(\log y)^{5/2}}=\frac{x}{(\log x)^2}+\frac{x(\log\log x)^{5/2}}{(\log x)^{5/2}}\\
&\ll \frac{x}{(\log x)^2}, \qquad (37)
\end{aligned}$$

which together with estimate (31) leads to the desired conclusion of Theorem 2. ☐

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