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Complementary Equations

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Abstract

Increasing sequences $a()$ and $b()$ that partition the sequence of positive integers are called complementary sequences, and equations that explicitly involve both $a()$ and $b()$ are called complementary equations. This article surveys several families of such equations, including $b(n) = a(jn) \pm r$, $b(n) = a(jn) + kn$, $b(n) = f(a(n))$, and $b(n) = a(b(n-1)) + qn + r$.

1 Introduction

Under the assumption that sequences a and b partition the sequence $N = (1, 2, 3, \ldots)$ of positive integers, the designation *complementary equations* applies to equations such as

$$b(n) = a(a(n)) + 1$$

in much the same way that the designations *functional equations*, *differential equations*, and *Diophantine equations* apply elsewhere. Indeed, complementary equations can be regarded as a class of Diophantine equations.

Various pairs of complementary sequences, such as Beatty sequences ([1, 21]), have been widely studied, as evidenced by many entries in the *Online Encyclopedia of Integer Sequences*[19]. In particular, complementary sequences have been discussed extensively by Fraenkel in connection with Beatty sequences, spectra of numbers, and combinatorial games; see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. See also [16, 18]. However, in the literature the recognition of a single equation involving both sequences tends to occur only parenthetically. The purpose of this paper is to recognize classes of such equations explicitly.

Throughout, the symbols a and b denote strictly increasing complementary sequences; i.e., every number in N is $a(n)$ or $b(n)$ for some n in N, and no term of a is also a term of b. An *ordinary complementary equation* (OCE) is defined by the form

$$b(n) = f(\widehat{a}(n), \widehat{b}(n), n), \tag{1}$$

where

$$\widehat{a}(n) = (a\,(1), a\,(2)\,, \ldots a(p(n))), \quad a(p(n)) < b(n),$$
$$\widehat{b}(n) = (b\,(1), b\,(2)\,, \ldots b(q(n))), \quad b(q(n)) < b(n).$$

That is, the nth term of the complement, b, of a, is determined as a function, f, of n and terms of a and b that are previously defined. (It is common and convenient to use time-suggestive descriptors such as *previously*, but we note that inductive definitions do not, in fact, depend on time.)

In some cases, an OCE (1) forces $a(1) = 1$, but, as in Example 3 below, this need not be the case. By decreeing an initial value, either $a(1) = 1$ or else $b(1) = 1$, the OCE must then have a unique solution b, or equivalently, a.

An equation involving both a sequence and its complement and which cannot be represented in the form (1) is a *partial complementary equation* (PCE). Typically, a PCE determines only a part of a solution; which is to say that there may be many solutions, as in Example 2 just below.

Example 1. The OCE $b(n) = a(a(n)) + 1$ has unique solution $a(n) = \lfloor n\tau \rfloor$, where $\tau = (1 + \sqrt{5})/2$. The complement is given by $b(n) = \lfloor n\tau \rfloor + n$, which is also the unique solution of the OCE $b(n) = a(n) + n$, an equation discussed more generally in section 4.

Example 2. Every solution of the equation

$$b(n) = b(n-1)a(n+1) \tag{2}$$

has $a(1) = 1$. However, the number 2 could be either $a(2)$ or $b(1)$, so that (2) is a PCE.

Example 3. The OCE

$$b(n) = a(n-1) + b(n-1) \tag{3}$$

for $n \geq 2$, with initial condition $b(1) = 1$, has as solution the Hofstadter sequence A005228:

$$b = (1, 3, 7, 12, 18, 26, 35, 45, \ldots)$$

with complement $a =$ A030124.

Example 4. Bode, Harborth, and Kimberling [3] discuss the equation

$$b(n) = a(n-1) + a(n-2)$$

with prescribed initial terms $a(1)$ and $a(2)$.

Example 5. Another PCE is

$$a(b(n)) - b(a(n)) = 1.$$

Solutions include $b(n) = 2n$ and $b(n) = \lfloor n\tau \rfloor$.

In the sequel, we shall solve four types of OCEs: $b(n) = a(jn) \pm r$, $b(n) = a(jn) + kn$, $b(n) = f(a(n))$, and the PCE $b(n) = a(b(n-1)) + qn + r$.

2 The step sequence of an OCE

Suppose an OCE (1) is given. Because a and b are complementary, when jointly ranked they form the sequence N. The joint ranking has the form

$$\begin{aligned}&a(1), \ldots, a(u_1), b\,(1)\,, \ldots, b(v_1),\\ &a(u_1+1), \ldots, a(u_2), b\,(v_1+1)\,, \ldots, b(v_2),\\ &a(u_2+1), \ldots, a(u_3), b\,(v_2+1)\,, \ldots, b(v_3), \ldots,\end{aligned}$$

where the numbers u_i and v_i are nonnegative integers. Note that

$$\begin{aligned}1 = a(1) < b(1), \text{ if } u_1 > 0;\\ 1 = b(1) < a(1), \text{ if } u_1 = 0.\end{aligned}$$

Each $n \geq v_1 + 1$ has a unique representation

$$n = v_{i-1} + r, \quad \text{where } 1 \leq r \leq v_i - v_{i-1},$$

where $i - 1 = \max\{m : n \geq v_m\}$.

Define the *step sequence* $s = (s(2), s(3), \ldots)$ by

$$s(n) = \begin{cases} u_i - u_{i-1} + 1, & \text{if } r = 1; \\ 1, & \text{if } r > 1. \end{cases}$$

Then b is clearly given by

$$b(n) = \begin{cases} u_1 + 1, & \text{if } n = 1; \\ b(n-1) + s(n), & \text{if } n \geq 2. \end{cases} \tag{3}$$

In many cases, we shall see, every $b(n)$ is immediately preceded and followed by a term of a, so that $v_i = i$ for all $i \geq 1$, and

$$s(n) = u_n - u_{n-1} + 1$$

for $n \geq 2$. In the sequel, we shall concentrate on sequences b of this kind.

3 The equations $b(n) = a(jn) \pm r$

Consider the equation

$$b(n) = a(jn) + r, \tag{4}$$

where $1 \le r \le j$. In order to solve this OCE, we find inductively that

$$a(n) = \begin{cases} n, & \text{if } 1 \le n < j + r; \\ n+1, & \text{if } j + r \le n < 2j + r; \\ n+2, & \text{if } 2j + r \le n < 3j + r; \\ \vdots & \vdots \\ n+q, & \text{if } qj + r \le n < (q+1)j + r. \\ \vdots & \vdots \end{cases}$$

For example, we move from $a(n) = n$ to $a(n) = n+1$ when $n = j + r$ in order to make room for $b(1) = j + r$. The inequality for which $a(n) = n + q$ is equivalent to

$$\frac{n - j - r}{j} < q \le \frac{n - r}{j},$$

so that $q = \left\lfloor \frac{n-r}{j} \right\rfloor$ and $a(n) = n + \left\lfloor \frac{n-r}{j} \right\rfloor$. Thus, for $n = 1$, we have $a(jn) = j$. Replacing n by jn gives

$$a(jn) = jn + \left\lfloor \frac{jn - r}{j} \right\rfloor = jn + n - 1.$$

To conclude, we have

$$a(jn) + r = jn + n - 1 + r$$

for all $n \ge 1$, so that the solution of (4) is given by

$$b(n) = (j+1)n + r - 1.$$

The same method applies to the OCE

$$b(n) = a(jn) - r,$$

where $1 \le r \le j - 1$, giving the solution

$$b(n) = (j+1)n - r.$$

4 The equation $b(n) = a(jn) + kn$

Suppose r and s are positive irrational numbers satisfying Beatty's equation [2]:

$$\frac{1}{r} + \frac{1}{s} = 1. \tag{5}$$

Then the sequences a and b given by $a(n) = \lfloor nr \rfloor$ and $b(n) = \lfloor ns \rfloor$ are a pair of complementary sequences known as Beatty sequences ([21], [19], [1]).

The OCE

$$b(n) = a(n) + kn, \tag{6}$$

where k is a positive integer, occurs in Stolarsky [21] where it is solved by means of shift operators, related to morphisms and continued fractions ([1], [21]) and also closely related to the step sequences of section 2. Sequences satisfying (6) were also studied by Fraenkel [10]. The solution of (6) is given by the Beatty sequences

$$a(n) = \lfloor rn \rfloor, \quad b(n) = \lfloor sn \rfloor, \tag{7}$$

where

$$r = 1 + \frac{\sqrt{k^2+4} - k}{2} \quad \text{and} \quad s = 1 + \frac{\sqrt{k^2+4} + k}{2}. \tag{8}$$

We wish to generalize Stolarsky's result to certain equations of the form

$$b(n) = a(jn) + kn; \tag{9}$$

specifically, we seek positive integers j and k for which the solution is a pair of Beatty sequences. Write

$$r = \frac{m + \sqrt{p}}{j}, \tag{10}$$

where m and p are rational numbers and $\sqrt{p}$ is irrational. Equation (5) then leads to

$$s = \frac{j\sqrt{p} + p + jm - m^2}{p - (m-j)^2}. \tag{11}$$

The desired equations

$$\lfloor sn \rfloor = \lfloor jrn \rfloor + kn$$

are equivalent to

$$sn - \delta_n = jrn - \varepsilon_n + kn, \tag{12}$$

where the fractional parts δ_n and ε_n satisfy

$$0 < \delta_n = sn - \lfloor sn \rfloor < 1 \quad \text{and} \quad 0 < \varepsilon_n = jrn - \lfloor jrn \rfloor < 1$$

for all n. Dividing both sides of (12) by n and taking the limit as $n \to \infty$ gives

$$s = jr + k.$$

Thus the coefficient $j/(p-(m-j)^2)$ of $\sqrt{p}$ on the right side of (11) must equal the coefficient of $\sqrt{p}$ in jr, which, by (10) is 1, so that

$$j = p - (m-j)^2,$$

which implies

$$j = \frac{2m - 1 \pm \sqrt{4(p-m)+1}}{2}.$$

In order that j be an integer, $\sqrt{4(p-m)+1}$ must be an odd integer:

$$\sqrt{4(p-m)+1} = 2q-1,$$

so that

$$p = q^2 - q + m.$$

Substituting into (11) and simplifying gives

$$s = q + \sqrt{p}.$$

Thus, for given m and q for which $q^2 - q + m$ is a nonsquare (below, we shall show that it is always a nonsquare), we put

$$\begin{aligned} j &= m + q - 1, \\ k &= q - m, \\ r &= \frac{m + \sqrt{q^2 - q + m}}{j}, \\ s &= q + \sqrt{q^2 - q + m}, \end{aligned}$$

and have the solution (7) of the equation (9).

Instead of starting with m and q, we can start with j and k to produce

$$\begin{aligned} q &= \frac{j + k + 1}{2}, \\ m &= \frac{j - k + 1}{2}, \\ \sqrt{p} &= \frac{\sqrt{(j + k + 1)^2 - 4k}}{2}. \end{aligned}$$

It is this latter representation of p that we now use to show that $\sqrt{p}$ is irrational for all positive integers j and k. It suffices to show that $(j+k+1)^2 - 4k$ is a nonsquare. Let $M = j + k + 1$, and note that for $k = 0$ and $k = M - 1$ we have $M^2 - 4k$ taking the values M^2 and $(M-2)^2$, respectively. There is only one square between those numbers, namely $(M-1)^2$. Therefore, if $M^2 - 4k$ is a square for some k satisfing $1 \le k \le M - 2$, then that value of k must satisfy

$$M^2 - 4k = (M-1)^2. \tag{13}$$

However, (13) implies $4k = 2M - 1$, a number that is both even and odd. As there is no such number, $(j + k + 1)^2 - 4k$ is not a square for any positive integers j and k.

Examples using Beatty-pair solutions of (9) are now easy to write out, as suggested by a table:

j	1	1	2	1	2	3	1	2	3	4
k	1	2	1	3	2	1	4	3	2	1
p	5/4	2	3	13/4	17/4	21/4	5	6	2	8

In connection with heap games, Fraenkel [13] considers the extension of (6) to the OCE

$$b(n) = ja(n) + kn,$$

where j and k are positive integers. For small values of j and k, solutions a and b include the pairs (A045671, A045672), (A045681, A045682), and (A045749, A045750), and (A045774, A045775).

Example 6. Taking $j = 1$ and leaving k arbitrary in (9) gives (8).

Example 7. Taking $j = k$ gives the OCE $b(n) = a(jn) + jn$ with solution (7) using

$$r = 1 + \frac{\sqrt{4j^2+1}}{2j} \quad \text{and} \quad r = \frac{2j+1+\sqrt{4j^2+1}}{2}.$$

5 The OCE of a dispersion, $b(n) = f(a(n))$

In this section, rather than starting with an OCE, we start with a certain kind of array consisting of all the positive integers, and we derive an OCE from it. Suppose f and g are strictly increasing complementary sequences and that $g(1) = 1$. The dispersion, $D(f) = \{d(i,j)\}_{i,j\geq 1}$ of f is defined [17] as the array having first column given by $d(i,1) = g(i)$ and subsequent columns given inductively by

$$d(i,j) = f(d(i,j-1)).$$

We shall see next that the general dispersion $D(f)$ is naturally associated with the OCE

$$b(n) = f(a(n)), \tag{14}$$

and that the dispersion provides a solution to this equation.

Note first that no member of column 1 of $D(f)$ is an image of f, so that the terms of column 1 belong to the sequence a. Next, every member m of column 2 is of the form $f(j)$ where j is in a, so that m is in b. Consequently, each m' in column 3 satisfies $m' = f(m)$ for some m in sequence b. Therefore, every term of column 3 is in a; otherwise, if m' were in b, then m would be in a, a contradiction. This shows that every term of column 3 is in a.

Continuing inductively, we conclude that the terms of the odd numbered columns of $D(f)$ are the terms of a, so that a is the ordered union of all the odd numbered columns. Likewise, b is the ordered union of all the even numbered columns. In $D(f)$, every positive integer occurs exactly once (see [17] for a proof), so that every positive integer is in a or b, which confirms that these are complementary sequences.

Example 8. Let $f(n) = 2n$ and $g(n) = 2n - 1$. The associated OCE is $b(n) = 2a(n)$. The northwest corner of the dispersion $D(f)$ is

$$\begin{array}{ccccccc}
1 & 2 & 4 & 8 & 16 & 32 & \cdots \\
3 & 6 & 12 & 24 & 48 & 96 & \\
5 & 10 & 20 & 40 & 80 & 160 & \\
7 & 14 & 28 & 56 & 112 & 224 & \\
\vdots & & & & & & \ddots
\end{array}$$

so that a is the ordered sequence of numbers

$$(2i+1)2^{2j}, \qquad i \geq 0, \ j \geq 0,$$

this being A003159, with complement $b =$ A036554, described as the numbers whose binary representation ends in an odd number of zeros.

As suggested by Example 8, the OCE

$$b(n) = ka(n),$$

for $k \geq 2$, has solution a the ordered union of the numbers

$$(ki+r)k^{2j}, \quad 1 \leq r \leq k-1, \ i \geq 0, \ j \geq 0,$$

with b the ordered union of the numbers $(ki+r)k^{2j+1}$.

Example 9. Let $f(n) = 2n+1$ for $n \geq 1$, let $g(1) = 1$ and $g(n) = 2n$ for $n \geq 2$. The associated OCE is $b(n) = 2a(n) + 1$. The northwest corner of the dispersion $D(f)$ is

$$\begin{array}{cccccccc} 1 & 3 & 7 & 15 & 31 & 63 & \cdots \\ 2 & 5 & 11 & 23 & 47 & 95 \\ 4 & 9 & 19 & 39 & 79 & 159 \\ 6 & 13 & 27 & 55 & 111 & 223 \\ \vdots & & & & & & \ddots \end{array}$$

so that a is the ordered sequence of numbers

$$2^{2j+1} - 1 \quad \text{and} \quad (2i+1)2^{2j} - 1, \qquad i \geq 1, \ j \geq 0,$$

and b is the ordered sequence of numbers

$$2^{2j+2} - 1 \quad \text{and} \quad (2i+1)2^{2j+1} - 1, \qquad i \geq 1, \ j \geq 0.$$

It is natural to ask what OCEs are associated with well-known dispersions. In the next examples, we answer this question for the Wythoff array, the Wythoff difference array, the Stolarsky array, and the inverse Wythoff array.

Example 10. Column 1 of the Wythoff array $W = \{w(i,j)\}$ is given [20] by

$$w(i,1) = \lfloor \lfloor i\tau \rfloor \tau \rfloor,$$

and the ordered complement of column 1, written as an increasing sequence, is given by

$$f(n) = \lfloor (n+1)\tau \rfloor - 1.$$

The associated OCE is therefore

$$b(n) = \lfloor (a(n)+1)\tau \rfloor - 1.$$

Its solution a is the ordered union of odd numbered columns of W, so that a is simply the lower Wythoff sequence, A000201. The complement, b, is the rest of W, which as an increasing sequence is the upper Wythoff sequence, A001950. Initial terms are given by

$$a = (1, 3, 4, 6, 8, 9, 11, 12, \ldots), \qquad b = (2, 5, 7, 10, 13, 15, 18, 20, \ldots).$$

Fraenkel and Kimberling [12] discuss Example 10 in greater detail.

Example 11. The Wythoff difference array, $D = \{d(i,j)\}$, given by A080164, is the dispersion of the upper Wythoff sequence, which, when written in increasing order, is given by

$$f(n) = \lfloor (\tau + 1)n \rfloor .$$

The associated OCE is therefore

$$b(n) = \lfloor (\tau + 1)a(n) \rfloor .$$

Its solution from columns of D is given by initial terms as follows:

$$a = (1, 3, 4, 5, 6, 8, 9, 11, 12, \ldots), \qquad b = (2, 7, 10, 13, 15, 20, \ldots).$$

Example 12. The inverse Wythoff array, $X = \{x(i,j)\}$, has first column given by

$$x(i,1) = s(n) = \begin{cases} 1, & \text{if } i = 1; \\ \lfloor i\tau \rfloor - 1, & \text{if } i > 1 \end{cases}$$

and ordered complement of column 1 given by

$$f(n) = \lfloor (n+1)\tau \rfloor + n.$$

(The definition of the inverse of a dispersion is given in [17]). The associated OCE is

$$b(n) = \lfloor (a(n)+1)\tau \rfloor + a(n).$$

Its solution from columns of X is given by initial terms as follows:

$$a = (1, 2, 3, 5, 7, 8, 10, 11, 12, 13, \ldots), \qquad b = (4, 6, 9, 14, 19, 22, 27, 30, 33, 35, \ldots).$$

6 The form $b(n) = a(b(n-1)) + qn + r$

We begin with the OCE

$$b(n) = a(b(n-1)) + 1. \tag{15}$$

If $a(1) = 1$, the solution is

$$b(n) = \frac{n^2 + n}{2} + 1,$$

whereas if $b(1) = 1$, the solution is

$$b(n) = \frac{n^2 + n}{2}. \tag{16}$$

We shall prove the latter, starting with a lemma closely related to (3).

Lemma. *Suppose a and b satisfy (15) and the initial condition $b(1) = 1$. Then*

$$a(m+1) = \begin{cases} a(m)+2, & \text{if } m = b(k) \text{ for some } k; \\ a(m)+1, & \text{otherwise.} \end{cases}$$

Proof. Because $b(1) = 1$, we have $a(1) \geq 2$, so that $b(2) \geq 3$, by (15). As a first step in an induction proof, we therefore have $b(2) - b(1) \geq 2$. Assume for arbitrary $k \geq 2$ that $b(k) - b(k-1) \geq 2$. Then, using (15),

$$\begin{aligned} b(k+1) - b(k) &= a(b(k)) - a(b(k-1)) \\ &\geq b(k) - b(k-1) \geq 2. \end{aligned}$$

Thus, for every $k \geq 2$, the numbers $b(k)-1$ and $b(k)+1$ are terms of the sequence a. As every positive integer is in exactly one of the sequences a and b, we have $a\,(m+1) = a\,(m) + 2$ if $a(m)$ is in b and $a\,(m+1) = a\,(m) + 1$ otherwise. ∎

Now, we shall prove that (15), with the intial condition $b(1) = 1$, implies (16). By (15),

$$b(2) = a(b(1)) + 1 = a(1) + 1 \geq 3,$$

so that 2 cannot be $b(2)$ and must therefore be $a(1)$. Then $b(2) = a(b(1))+1 = a(1)+1 = 3$. As a two-part induction hypothesis, assume for arbitrary $k \geq 1$ these two equations:

$$b(k) = \frac{k(k+1)}{2}, \tag{17}$$

$$a(b(k)) = b(k) + k. \tag{18}$$

Then

$$\begin{aligned} b(k+1) &= a(b(k)) + 1 \\ &= b(k) + k + 1 \text{ by (18)} \\ &= \frac{(k+1)(k+2)}{2} \text{ by (17),} \end{aligned}$$

and it remains to be proved that

$$a(b(k+1)) = b(k+1) + k + 1. \tag{19}$$

We have

$$\begin{aligned} a(b(k)+1) &= a(b(k)) + 2 \text{ by the lemma} \\ &= b(k) + k + 2 \text{ by (18).} \end{aligned}$$

Also by the lemma,

$$\begin{aligned} a(b(k)+2) &= a(b(k)+1) + 1 \\ a(b(k)+3) &= a(b(k)+2) + 1 = a(b(k)+1) + 2 \\ &\vdots \\ a(b(k)+k) &= a(b(k)+k-1) + 1 = a(b(k)+1) + k - 1 \\ a(b(k)+k+1) &= a(b(k)+k) + 1 = a(b(k)+1) + k. \end{aligned} \tag{20}$$

Now,

$$\begin{aligned} a(b(k)+1) &= a(b(k))+2 \text{ by the lemma} \\ &= b(k+1)+1 \text{ by (15),} \end{aligned}$$

so that

$$a(b(k)+1)+k = b(k+1)+k+1,$$

which by (20) implies

$$a(b(k)+k+1) = b(k+1)+k+1,$$

and the desired (19) now follows from the fact, already proved, that $b(k)+k+1 = b(k+1)$.

(Note that (18) is a PCE; one solution is given by (17); another, by $b(n) = 3n-2$.)

Similar inductive proofs can be given for various OCEs, including the following:

$$\begin{aligned} \text{Equation:}\quad & b(n) = a(b(n-1))+r, \text{ where } r \geq 1 \\ \text{Initial value:}\quad & b(1) = 1 \\ \text{Solution:}\quad & b(n) = (r-1)(n-1)+\frac{n(n+1)}{2}. \end{aligned}$$

$$\begin{aligned} \text{Equation :}\quad & b(n) = a(b(n-1))+r, \text{ where } r \geq 1 \\ \text{Initial value :}\quad & a(1) = 1 \\ \text{Solution :}\quad & b(n) = rn+\frac{n^2-n+2}{2}. \end{aligned}$$

$$\begin{aligned} \text{Equation :}\quad & b(n) = a(b(n-1))+qn, \text{ where } q \geq 1 \\ \text{Initial value :}\quad & b(1) = 1 \\ \text{Solution :}\quad & b(n) = \frac{q(n^2+n+2)+n^2-n+2}{2}. \end{aligned}$$

$$\begin{aligned} \text{Equation :}\quad & b(n) = a(b(n-1))+qn, \text{ where } q \geq 1 \\ \text{Initial value :}\quad & a(1) = 1 \\ \text{Solution :}\quad & b(n) = \frac{q(n^2+n)+n^2-n+2}{2}. \end{aligned}$$

$$\begin{aligned} \text{Equation :}\quad & b(n) = a(b(n-1))+qn+r, \text{ where } q \geq 1,\ r \geq 1 \\ \text{Initial value :}\quad & b(1) = 1 \\ \text{Solution :}\quad & b(n) = \frac{q(n^2+n+2)+n^2+(r+1)n-2r+2}{2}. \end{aligned}$$

$$\begin{aligned} \text{Equation :}\quad & b(n) = a(b(n-1))+qn+r, \text{ where } q \geq 1,\ r \geq 1 \\ \text{Initial value :}\quad & a(1) = 1 \\ \text{Solution :}\quad & b(n) = \frac{q(n^2+n)+n^2+(2r-1)n+2}{2}. \end{aligned}$$

To summarize, if $q \geq 0$ and $r \geq 0$ and q and r are not both 0, and if an initial value, either $a(1) = 1$ or $b(1) = 1$ is assumed, then the equation

$$b(n) = a(b(n-1)) + qn + r$$

holds for a unique second-degree polynomial in n. Several special cases are tabulated here, along with three examples in which $r < 0$.

$b(n) - a(b(n-1) =$	**Initial**	**Solution**	**Name**
1	$a(1) = 1$	$(n^2 + n + 2)/2$	A000124, Hogben's c. p. nos.
2	$a(1) = 1$	$(n^2 + 3n + 2)/2$	A000217, triangle numbers
n	$b(1) = 1$	n^2	A000290, square numbers
$2n + 2$	$a(1) = 1$	$(3n^2 + 5n + 2)/2$	A000326, pentagonal numbers
$3n + 2$	$a(1) = 1$	$2n^2+3n + 1$	A000384, hexagonal numbers
$2n + 1$	$a(1) = 1$	$2n^2+2n + 1$	A001844, centered square nos.
$n + 1$	$b(1) = 1$	$n^2+n - 1$	A028387
$n - 1$	$b(1) = 1$	$n^2-n + 1$	A002061, central polygonal nos.
$3n - 1$	$a(1) = 1$	$2n^2+1$	A058331
$3n - 2$	$b(1) = 1$	$2n^2-1$	A000384, hexagonal numbers

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