Journal of Integer Sequences, Vol. 10 (2007),
Article 07.4.2

The Average Order of the Dirichlet Series of the gcd-sum Function

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Abstract

Using a result of Bordellès, we derive the second term and improved error expressions for the partial sums of the Dirichlet series of the gcd-sum function, for all real values of the parameter.

1 Introduction

The second term and improved error expressions for the partial sums of the Dirichlet series of the gcd-sum function, first given in [2], are derived, for all real values of the parameter, using a result of Bordellès [1].

Let (m, n) represent the positive greatest common divisor of the positive integers m and n, let $g(n) := \sum_{i=1}^{n}(i, n)$ be the gcd-sum, let $\tau(n)$ be the number of divisors of n, let $Id(n) := n$ be the identity function, for $\alpha \in \mathbb{R}$ let $m_\alpha(n) = 1/n^\alpha$ be the monomial power function, and let $\phi(n)$ be Euler's phi function. For $x \geq 1$ let

$$G_\alpha(x) := \sum_{n \leq x} \frac{g(n)}{n^\alpha}$$

be the average order.

Let θ be the smallest positive real number such that

$$\sum_{n \leq x} \tau(n) = x \log x + x(2\gamma - 1) + O_\epsilon(x^{\theta+\epsilon}) \qquad (1)$$

for all $\epsilon > 0$ and $x \geq 1$. For a discussion about the value of θ and references see [1]. The current best value is slightly more than $1/4$.

The lemma of Bordellès [1] enables us to write $g = \mu * (\tau \cdot Id)$ and this, combined with a best available error for the average order of $\tau(n)$, enables a further term and greater degree of precision to be obtained for the $G_\alpha(x)$ than the expression $g = \phi * Id$ used in [2]. This is true for every value of α except for $\alpha = 2$ in which case both expressions for $g(n)$ give the same terms and error form.

In Section 3 we correct some errors in [2].

2 Average values of the Dirichlet series

First we give two summations for the sum of divisor function $\tau(n)$.

Lemma 1. *For all β satisfying $\beta \geq -\theta$ and $\epsilon > 0$:*

$$\sum_{n \leq x} n^\beta \tau(n) = \frac{x^{\beta+1}}{\beta+1} \log x + \frac{x^{\beta+1}}{\beta+1}\big(2\gamma - \frac{1}{\beta+1}\big) + O_\epsilon(x^{\theta+\beta+\epsilon}).$$

Proof. This follows from (1) and Abel summation as in Lemma 1 of [1]. □

Lemma 2. *For all β satisfying $-1 < \beta < -\theta$:*

$$\sum_{n \leq x} n^\beta \tau(n) = \frac{x^{\beta+1}}{\beta+1} \log x + \frac{x^{\beta+1}}{\beta+1}\big(2\gamma - \frac{1}{\beta+1}\big) + O(1)$$

where the implied constant is absolute.

Let $c_1 := 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}$. The theorem of Bordellès [1] can be written:

Lemma 3. *For all $\epsilon > 0$, as $x \to \infty$*

$$G_0(x) = \frac{x^2 \log x}{2\zeta(2)} + c_1 \frac{x^2}{2\zeta(2)} + O_\epsilon(x^{1+\theta+\epsilon}).$$

It is not needed in what follows, but for completeness the corresponding sum for the power of the multiplier -1 is [3]:

$$\sum_{n \leq x} \frac{\tau(n)}{n} = \frac{\log^2 x}{2} + 2\gamma \log x + \gamma^2 - 2\gamma_1 + O(\frac{1}{\sqrt{x}}),$$

where γ_1 is the first Stieltjes constant.

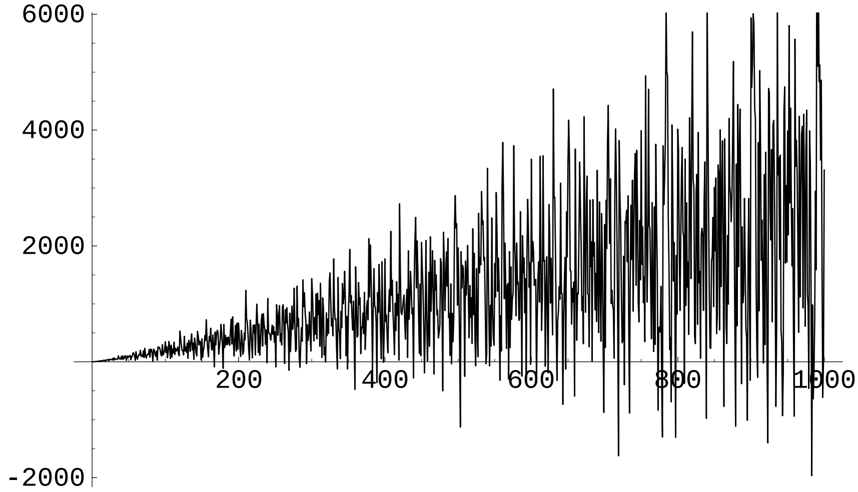

Figure 1: Error value for $G_0(x)$.

Figure 1 is a plot of $G_0(x) - \frac{x^2 \log x}{2\zeta(2)} - c_1 \frac{x^2}{2\zeta(2)}$ for $1 \le x \le 1000$. This should be compared with the corresponding plots for G_α given below. It shows that, maybe, another continuous term can be extracted from the error. The existing computed error is large, but does, on the face of it, go to zero when divided by x^2, consistent with the given value of the constant c_1.

We can now improve each part of [2, Theorem 4.4]:

Theorem 4. *For all $\epsilon > 0$ as $x \to \infty$*

(1) If $\alpha \le 1+\theta$:

$$G_\alpha(x) = \frac{x^{2-\alpha}\log x}{(2-\alpha)\zeta(2)} + \frac{x^{2-\alpha}}{(2-\alpha)\zeta(2)}\big(2\gamma - \frac{1}{2-\alpha} - \frac{\zeta'(2)}{\zeta(2)}\big) + O_\epsilon(x^{\theta+1-\alpha+\epsilon}).$$

(2) If $1+\theta < \alpha < 2$:

$$G_\alpha(x) = \frac{x^{2-\alpha}\log x}{(2-\alpha)\zeta(2)} + \frac{x^{2-\alpha}}{(2-\alpha)\zeta(2)}\big(2\gamma - \frac{1}{2-\alpha} - \frac{\zeta'(2)}{\zeta(2)}\big) + O(1).$$

(3) $G_2(x) = \frac{\log^2 x}{2\zeta(2)} + \frac{\log x}{\zeta(2)}\big(2\gamma - \frac{\zeta'(2)}{\zeta(2)}\big) + O(1)$.

(4) If $\alpha > 2$:

$$\begin{aligned} G_\alpha(x) &= \frac{\zeta(\alpha-1)^2}{\zeta(\alpha)} - \frac{\log x}{(\alpha-2)\zeta(2)x^{\alpha-2}} \\ &\quad - \frac{1}{(\alpha-2)\zeta(2)x^{\alpha-2}}\big(2\gamma + \frac{1}{\alpha-2} - \frac{\zeta'(2)}{\zeta(2)}\big) + O_\epsilon\big(\frac{1}{x^{\alpha-1-\theta-\epsilon}}\big). \end{aligned}$$

Proof. (1) In this case $\alpha \le 1+\theta$. Let $\beta := 1-\alpha$, $c_2 := 2\gamma - 1/(1+\beta)$. Then, using Lemma

2.1 and the complete multiplicativity of m_α:

$$\begin{aligned}
G_\alpha(x) &= \sum_{n\le x}\frac{g(n)}{n^\alpha} = \sum_{n\le x}(m_\alpha\cdot g)(n)\\
&= \sum_{n\le x}((m_\alpha\cdot\mu)*(m_\alpha\cdot Id\cdot\tau))(n)\\
&= \sum_{d\le x}\frac{\mu(d)}{d^\alpha}\sum_{e\le x/d}e^{1-\alpha}\tau(e)\\
&= \sum_{d\le x}\frac{\mu(d)}{d^\alpha}\Big[\frac{1}{\beta+1}\frac{x^{\beta+1}}{d^{\beta+1}}(\log x-\log d)+c_2\frac{x^{\beta+1}}{d^{\beta+1}}+O_\epsilon((\frac{x}{d})^{\theta+\beta+\epsilon})\Big]\\
&= \frac{x^{\beta+1}\log x}{\beta+1}\sum_{d\le x}\frac{\mu(d)}{d^{\alpha+\beta+1}}-\frac{x^{\beta+1}}{\beta+1}\sum_{d\le x}\frac{\mu(d)\log d}{d^{\alpha+\beta+1}}\\
&\quad +c_2x^{\beta+1}\sum_{d\le x}\frac{\mu(d)}{d^{\alpha+\beta+1}}+O_\epsilon(x^{\theta+\beta+\epsilon}\sum_{d\le x}\frac{1}{d^{\theta+\alpha+\beta+1}})
\end{aligned}$$

Therefore

$$\begin{aligned}
G_\alpha(x) &= \frac{x^{2-\alpha}\log x}{2-\alpha}\Big[\frac{1}{\zeta(2)}-\sum_{d>x}\frac{\mu(d)}{d^2}\Big]-\frac{x^{2-\alpha}}{2-\alpha}\Big[\frac{\zeta'(2)}{\zeta(2)^2}-\sum_{d>x}\frac{\mu(d)\log d}{d^2}\Big]\\
&\quad +c_2x^{2-\alpha}\Big[\frac{1}{\zeta(2)}-\sum_{d>x}\frac{\mu(d)}{d^2}\Big]+O_\epsilon(x^{\theta+\epsilon+1-\alpha})\\
&= \frac{x^{2-\alpha}\log x}{(2-\alpha)\zeta(2)}+\frac{x^{2-\alpha}}{(2-\alpha)\zeta(2)}\Big[2\gamma-\frac{1}{2-\alpha}-\frac{\zeta'(2)}{\zeta(2)}\Big]\\
&\quad +O_\epsilon(x^{\theta+\epsilon+1-\alpha})
\end{aligned}$$

(2) The derivation is the same as in (1), except we use Lemma 2.2 instead of Lemma 2.1.
(3) This is given in the proof of [2, Theorem 4.4, Case 2].
(4) Let $\alpha>2$. Then, using [2, Theorem 4.1] and Abel summation:

$$\begin{aligned}
G_\alpha(x) &= \frac{\zeta(\alpha-1)^2}{\zeta(\alpha)}-\sum_{n>x}\frac{g(n)}{n^\alpha}\\
&= \frac{\zeta(\alpha-1)^2}{\zeta(\alpha)}-\lim_{y\to\infty}\frac{G_0(y)}{y^\alpha}+\frac{G_0(x)}{x^\alpha}-\alpha\int_x^\infty\frac{G_0(t)}{t^{\alpha+1}}dt\\
&= \frac{\zeta(\alpha-1)^2}{\zeta(\alpha)}+\frac{G_0(x)}{x^\alpha}-\alpha\int_x^\infty\frac{G_0(t)}{t^{\alpha+1}}dt\\
&= \frac{\zeta(\alpha-1)^2}{\zeta(\alpha)}-\frac{\log x}{(\alpha-2)\zeta(2)x^{\alpha-2}}\\
&\quad -\frac{1}{2(\alpha-2)\zeta(2)x^{\alpha-2}}\Big[2c_1+\frac{\alpha}{\alpha-2}\Big]+O_\epsilon(\frac{1}{x^{\alpha-1-\theta-\epsilon}}),
\end{aligned}$$

and the result follows after substituting for c_1. □

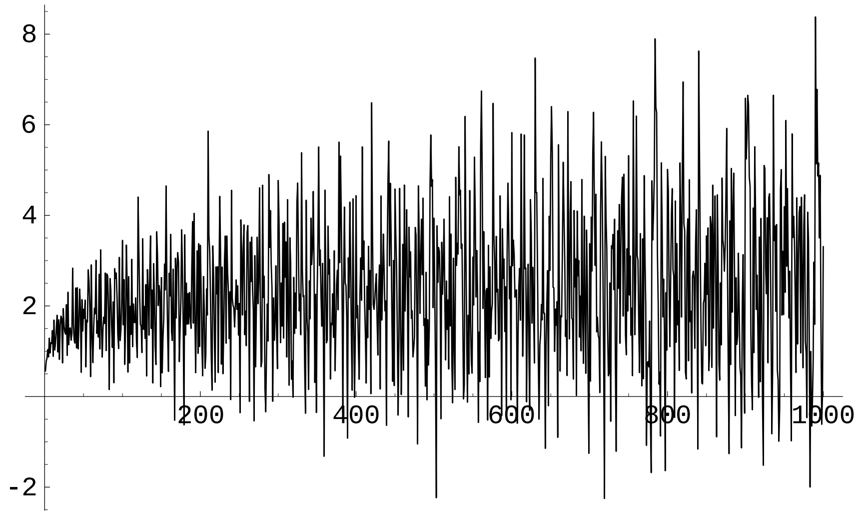

Figure 2: Error value for $G_\alpha(x)$ with $\alpha = 1$.

Figure 2 is a plot of $G_1(x) - \frac{x \log x}{\zeta(2)} - \frac{x}{\zeta(2)}\big(2\gamma - 1 - \frac{\zeta'(2)}{\zeta(2)}\big)$ for $1 \le x \le 1000$.

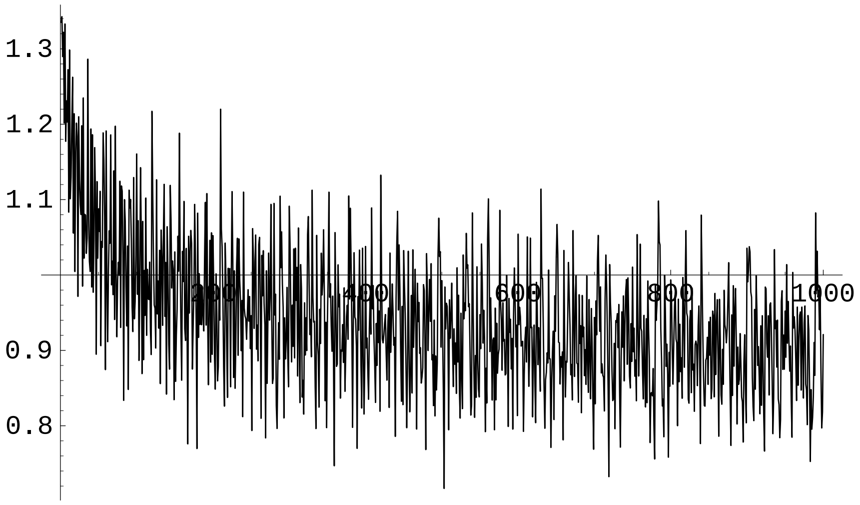

Figure 3: Error value for $G_\alpha(x)$ with $\alpha = 3/2$.

Figure 3 is a plot of

$$G_\alpha(x) - \frac{x^{2-\alpha}\log x}{(2-\alpha)\zeta(2)} - \frac{x^{2-\alpha}}{(2-\alpha)\zeta(2)}\big(2\gamma - \frac{1}{2-\alpha} - \frac{\zeta'(2)}{\zeta(2)}\big)$$

for $\alpha = 3/2$ and $1 \le x \le 1000$.

3 Errors in "The gcd-sum function"

Corrections to some errors in [2]. Page 7 line 8: the second g should be Id, page 9 line 3: 5.1 should be 4.1, page 12 line 2: 5.4 should be 4.3, page 12 line 3: 5.3 should be 4.2, page 13 line 13 and line -1: e, d should be $e.d$, page 14 line -1: 5.4 should be 4.4.

4 Acknowledgments

The support of the University of Waikato is gratefully acknowledged. The plots were produced using the Mathematica functions ListPlot, and DivisorSigma. Helpful suggestions made by a referee are also gratefully acknowledged.

References

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2000 *Mathematics Subject Classification*: Primary 11A05; Secondary 11A25, 11M06, 11N37, 11N56.
Keywords: greatest common divisor, Dirichlet series, gcd-sum.

(Concerned with sequence A018804.)

Received February 28, 2007; revised version received April 13 2007. Published in *Journal of Integer Sequences*, April 13 2007.

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