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On Sums Involving Binomial Coefficients

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Abstract

We give closed forms for the series $\sum_{m=1}^{\infty} \frac{(2x)^{2m+2k}}{m^2(m+k)\binom{2m}{m}}$ and $\sum_{m=1}^{\infty} \frac{(2x)^{2m}(-1)^{m+k}}{m^2(m+k)\binom{2m}{m}}$ for integers $k \geq 0$.

1 Introduction

D. H. Lehmer [1] studied various series with binomial coefficients in the denominator, for example,

$$\sum_{m\geq 1} \frac{(2x)^{2m}}{m\binom{2m}{m}} = \frac{2x}{\sqrt{1-x^2}} \arcsin x, \tag{1}$$

valid for $|x| < 1$. In this note we consider some related results.

2 Main Results

The main results of the paper can be stated as follows:

Theorem 2.1. *(a) For $t \in (-1, 1)$ we have*

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{(2t)^{2m+2k}}{m(2m+2k)\binom{2m}{m}} &= \arcsin(t)^2 \binom{2k}{k} + \sum_{j=1}^{k} \binom{2k}{k-j} \frac{(-1)^{j+1}}{j^2} \\ &\quad + \sum_{j=1}^{k} (-1)^j \binom{2k}{k-j} \left(\frac{2 \arcsin t \sin(2j \arcsin t)}{j} + \frac{\cos(2j \arcsin t)}{j^2} \right).\end{aligned}$$

(b) If we replace t by $\sqrt{-1}t$, we get another form

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{(-1)^{m+k}(2t)^{2m+2k}}{m(2m+2k)\binom{2m}{m}} &= -\log(t+\sqrt{1+t^2})^2\binom{2k}{k}+\sum_{j=1}^{k}\binom{2k}{k-j}\frac{(-1)^{j+1}}{j^2}\\ &+\sum_{j=1}^{k}(-1)^j\binom{2k}{k-j}\left(\frac{(-t+\sqrt{t^2+1})^{2j}-(t+\sqrt{t^2+1})^{2j}}{j}\right)\log(t+\sqrt{t^2+1})\\ &+\sum_{j=1}^{k}(-1)^j\binom{2k}{k-j}\left(\frac{(-t+\sqrt{t^2+1})^{2j}+(t+\sqrt{t^2+1})^{2j})}{2j^2}\right).\end{aligned}$$

Theorem 2.2. *(a) For $t\in(-1,1)$ we have*

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{(2t)^{2m+2k}}{m^2(2m+2k)\binom{2m}{m}} &= \frac{-(\arcsin t)^2}{k}\binom{2k}{k}+\frac{(2t)^{2k}}{k}(\arcsin t)^2-\sum_{j=1}^{k}\binom{2k}{k-j}\frac{(-1)^{j+1}}{kj^2}\\ &- \sum_{j=1}^{k}(-1)^j\binom{2k}{k-j}\left(\frac{2\arcsin t\sin(2j\arcsin t)}{kj}+\frac{\cos(2j\arcsin t)}{kj^2}\right).\end{aligned}$$

(b) The identity in (a) is valid if we replace t by it and $\arcsin(it)$ by $i\log(1+\sqrt{1+t^2})$:

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{(2t)^{2m+2k}(-1)^{m+k}}{m^2(2m+2k)\binom{2m}{m}} &= \left(\frac{1}{k}\binom{2k}{k}-\frac{(2t)^{2k}}{k}\right)(\log(1+\sqrt{1+t^2}))^2-\sum_{j=1}^{k}\binom{2k}{k-j}\frac{(-1)^{j+1}}{kj^2}\\ &-\sum_{j=1}^{k}(-1)^j\binom{2k}{k-j}\left(\frac{(-t+\sqrt{t^2+1})^{2j}-(t+\sqrt{t^2+1})^{2j}}{jk}\right)\log(t+\sqrt{t^2+1})\\ &-\sum_{j=1}^{k}(-1)^j\binom{2k}{k-j}\left(\frac{(-t+\sqrt{t^2+1})^{2j}+(t+\sqrt{t^2+1})^{2j})}{2kj^2}\right).\end{aligned}$$

We use the basic method to prove various combinatorial identities where binomial coefficients occur in the denominator. As a particular consequence of our results we get the sums $S_1(k), S_2(k), T_1(k), T_2(k)$ computed in [2] in a compact form.

3 Proofs of the Theorems

Proof of Theorem 2.1.

We start with the following identity, which was discovered by D. H. Lehmer [1]. If $|x|<1$, then

$$\sum_{m\ge 1}\frac{(2x)^{2m}}{m\binom{2m}{m}} = \frac{2x}{\sqrt{1-x^2}}\arcsin x. \qquad (2)$$

If we replace x by $\sqrt{-1}x$, we get another form of the identity (2):

$$\sum_{m\geq 1}\frac{(-1)^m(2x)^{2m}}{m\binom{2m}{m}} = \frac{-2x}{\sqrt{1+x^2}}\log(x+\sqrt{1+x^2})$$

If we multiply both members of the equation (2) by x^{2k-1} and then integrate, we obtain

$$\sum_{m\geq 1}\frac{2^{2m}t^{2m+2k}}{m(2m+2k)\binom{2m}{m}} = \int_0^t \frac{2x^{2k}}{\sqrt{1-x^2}}\arcsin x dx.$$

The left-hand side of this equation can be written

$$\int_0^{\arcsin t} 2x(\sin(x))^{2k}dx = \frac{(-1)^k}{2^{2k-1}}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j\int_0^{\arcsin t} x\exp(2ix(k-j))dx$$

from which we get

$$\begin{aligned}\int_0^{\arcsin t} 2x(\sin(x))^{2k}dx &= \frac{(\arcsin t)^2}{2^{2k}}\binom{2k}{k} \\ &+ \frac{1}{2^{2k-1}}\sum_{j=1}^{k}(-1)^j\binom{2k}{k-j}\left(\frac{\tau\sin(2j\tau)}{j}+\frac{\cos(2j\tau)}{2j^2}-\frac{1}{2j^2}\right)\end{aligned}$$

where $\tau = \arcsin(t)$. Finally we get

$$\begin{aligned}\sum_{m\geq 1}\frac{(2t)^{2m+2k}}{m(2m+2k)\binom{2m}{m}} &= (\arcsin t)^2\binom{2k}{k}+\sum_{j=1}^{k}\binom{2k}{k-j}\frac{(-1)^{j+1}}{j^2} \\ &+ \sum_{j=1}^{k}(-1)^j\binom{2k}{k-j}\left(\frac{2(\arcsin t)\sin(2j\arcsin t)}{j}+\frac{\cos(2j\arcsin t)}{j^2}\right),\end{aligned}$$

which gives Theorem 2.1.

Proof of Theorem 2.1.

If we integrate (2) from 0 to x we obtain

$$\sum_{m\geq 1}\frac{(2x)^{2m}}{m^2\binom{2m}{m}} = 2(\arcsin x)^2$$

If we multiply both members of this equation by x^{2k-1} and then integrate, we obtain

$$\sum_{m\geq 1}\frac{2^{2m}t^{2m+2k}}{m^2(2m+2k)\binom{2m}{m}} = \int_0^t 2x^{2k-1}(\arcsin x)^2dx.$$

The left-hand side of can be written

$$\int_0^t 2x^{2k-1}(\arcsin x)^2dx = \frac{t^{2k}}{k}(\arcsin t)^2 - \frac{1}{k}\int_0^t\frac{2x^{2k}}{\sqrt{1-x^2}}\arcsin x dx.$$

By using the proof of Theorem 2.1 we get Theorem 2.2.

References

[1] D. H. Lehmer. Interesting series involving the central binomial coefficient. *Amer. Math. Monthly* **92** (1985), 449–457.

[2] Jin-Hua Yang and Feng-Zhen Zhao, Sums involving binomial coefficients, *Journal of Integer Sequences* **9** (2006), Article 06.4.2.

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