Loewy Decomposition
of Linear Differential Equations

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Factorization of polynomials, and more general primary decomposition of polynomial ideals, are well known concepts with a long history in commutative algebra. They are important tools for understanding the solution set of the corresponding algebraic equations. Similar notions apply to differential operators, ideals generated by such operators and the linear differential equations corresponding to them. Originating from the work of Beke, Schlesinger and Loewy for linear ordinary differential equations (ode's for short), in recent years their findings have been generalized to certain classes of linear partial differential equations (pde's), mostly in two independent variables. The current status of this newly emerging field is reviewed and the possible further development is outlined. The following topics will be covered.

(1) Loewys Results for Linear ODE's. The original results of Loewy for decomposing ordinary differential operators are presented. To this end, the concept of a reducible equation and its decomposition into completely reducible components is introduced. It is shown how they may be applied for solving the corresponding equation.
(2) Rings of Partial Differential Operators. In order to generalize Loewy's theory to pde's, a more algebraic language is appropriate. The objects of interest are left ideals in the ring $\mathbb{Q}[\partial_x, \partial_y]$ of partial differential operators, and modules over this ring, so-called $\mathcal{D}$-modules. They are generated by Janet bases. According to Kolchin, fundamental properties are its *differential type* and *typical differential dimension*, jointly called its *gauge*. Of fundamental importance are the sum and the intersection of two ideals and how they determine the lattice structure of left ideals in the ring $\mathbb{Q}[\partial_x, \partial_y]$. The decompositions of an ideal are determined by its possible divisors and the respective module of *relative syzygies.*
(3) Equations with Finite-Dimensional Solution Space. It turns out that Loewy's theory for linear ode's may be extended in a fairly straightforward manner to linear pde's with the property that their general solution involves only constants, i. e. if it is a finite-dimensional vector space over constants. The reason is that the corresponding ideals form a sublattice of the lattice of left ideals in the ring of partial differential operators.
(4) Decomposing Second- and Third-Order PDE's. Principal ideals correspond to individual linear pde's. There is an extensive literature on such equations

in the 19^{th} and early 20^{th} century. Various ad hoc methods are described there with no apparent guiding principle behind them. Applying the algebraic theory outlined in preceding parts of the tutorial, most of these results may be obtained in a completeley systematic way. An instructive example of decomposing a third-order pde has been given in Blumberg's dissertation. It is shown how it fits into Loewy's original scheme if intersection ideals are not restricted to being principal.

(5) Summary and Outlook. The results described in this tutorial could mark the opening of a new subfield in the theory of pde's. It has been shown how Loewy's decomposition of linear ode's may be generalized to large classes of pde's if it is supplemented by the proper tools from differential algebra. In this way several ad hoc methods are obtained in a systematic way from general principles. The most obvious continuation is its extension to more than two independent variables, and to general modules over the corresponding ring of differential operators. In order to make these results accessible to practical problem solving, various algorithmic aspects have to be further discussed.

(6) Loewy Decomposition with ALLTYPES. The theory described in preceding sections is accompanied by constructive proofs and algorithms for selected tasks. In order to solve concrete problems, they have been implemented in the computer algebra typesystem ALLTYPES that may be used interactively over the internet. Its application for solving various problems is demonstrated.

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Literature

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