On the Adequacy of Partial Orders for Preference Composition

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Abstract

We identify several anomalies in the behavior of conventional notions of composition for preferences defined by strict partial orders. These anomalies can be avoided by defining a preorder that extends the given partial order, and using the pair of orders to define order composition.

1. Introduction

Much past work on preferences has been built on a formalism in which preference is specified via a strict partial order[1] [2, 4, 5, 1]. One writes $x \succ y$ to describe a preference for x over y. In order to respond to a user's answer preferences (e.g., to return the Pareto-optimal set of answers), one can use the definition of the partial order to test whether a dominating element exists.

Example 1.1: [6] Suppose that a user cares about the price of a car, but not about small differences in price. For example, the user might wish to state "For any given class of car, car A is preferred to car B if the price of A is less than 80% of the price of B." Cars that differ by 20% or less in price are incomparable. It is easy to verify the antisymmetry and transitivity of this relation, making it a strict partial order. It is not *complete*, i.e., not a total order, because some pairs of values are incomparable to each other. □

Example 1.2: Suppose that a hiring manager is trying to decide which person to hire for a job. Each person has a set of qualifications. We might prefer person p to person q if p possesses every qualification possessed by q in addition to at least two qualifications not possessed by q. It is relatively straightforward to verify that this preference relation is a strict partial order. □

Example 1.3: Let G be a directed acyclic graph (DAG). We might define a preference relation that prefers a node p to a node q if there is a path from p to q in G of length at least 2. Again, it is relatively straightforward to verify that this preference relation is a strict partial order. □

Each of these examples has a common flavor: there is a "gap" between the preference relation and the equality relation. In Example 1.1 the gap corresponds to prices that are better but not more than 20% better. In Example 1.2 the gap corresponds to pairs of people where one has exactly one more qualification than the other. In Example 1.3 the gap corresponds to pairs of nodes where one is an immediate predecessor of another. These gaps cause semantic difficulties with existing approaches to composing preferences, as we shall see in Section 2.

Definition 1.1: The *indifference relation* $\sim$ for a binary relation $\succ$ is defined as

$$x \sim y \text{ iff } x \not\succ y \text{ and } y \not\succ x.$$ □

When the indifference relation is transitive, an order is said to be a *strict weak ordering*. Note that none of Examples 1.1, 1.2, or 1.3 is a strict weak ordering. For more examples and discussion of preference relations whose indifference relation is nontransitive, see [2].

2. Composing Preferences

When composing preferences, we will assume that the entities being compared are tuples $(x_1, \ldots, x_n)$ of values, where each x_i is from a domain having a strict partial order $\succ_i$. (The generalization to orders that depend on multiple values is straightforward.) The prioritized composition [4], $\succ = \succ_1 \& \succ_2$, is defined as: $(x_1, x_2) \succ (y_1, y_2)$ iff

$$x_1 \succ_1 y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 \succ_2 y_2)$$

Prioritized composition gives priority to the first preference order, and uses the second order only to break ties in the first order. The Pareto composition [4], $\succ = \succ_1 \otimes \succ_2$, is defined as: $(x_1, x_2) \succ (y_1, y_2)$iff

$$\begin{array}{l} x_1 \succ_1 y_1 \text{ and } x_2 \succ_2 y_2, \text{ or} \\ x_1 \succ_1 y_1 \text{ and } x_2 = y_2, \text{ or} \\ x_1 = y_1 \text{ and } x_2 \succ_2 y_2. \end{array}$$

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[1]A strict partial order is an irreflexive, antisymmetric, and transitive binary relation.

Pareto composition treats the component orders symmetrically. A record must be strictly better than another according to at least one of the orders, and either better or equal according to the other order.

A potential alternative to these notions of composition is defined by Chomicki [1]. He defines a different version of prioritized composition that we will call "triangle composition" to distinguish it from the previously defined notion of prioritized composition.

Definition 2.1: Given two binary relations $\succ_1$ and $\succ_2$, the *triangle composition* $\succ \; = \; \succ_1 \rhd \succ_2$ is defined as $(x_1, x_2) \succ (y_1, y_2)$ iff

$$x_1 \succ_1 y_1 \text{ or } (x_1 \sim_1 y_1 \text{ and } x_2 \succ_2 y_2)$$

where $\sim_1$ is the indifference relation for $\succ_1$. □

Both prioritized composition and Pareto composition define partial orders [4]. indifference relation for the first strict partial order is transitive, the triangle composition is a strict partial order [1]. In general, though, triangle composition does not yield a strict partial order; see Example 2.3 below.

Prioritized composition and Pareto Composition can sometimes give unintuitive results, as illustrated by Examples 2.1 and 2.2 below.

Example 2.1: As in Example 1.1, consider a preference $\succ_1$ on prices defined as preferring one car over another if the price of the first is less than 80% of the price of the second. Let $\succ_2$ represent a preference for red cars over blue cars. According to both $\succ_1 \otimes \succ_2$ and $\succ_1 \& \succ_2$, (75,red) and (100,red) are preferred to (100,blue), while (90,red) is not. This artifact appears to violate notions of monotonicity that are implicit in the application: improving the price should not cause a previously preferred record to become not preferred. □

Example 2.2: As in Example 1.2, consider a preference $\succ_1$ on sets defined as preferring S over T if $T \subseteq S$ and $|S| \geq |T| + 2$. Let $\succ_2$ represent a preference for a larger value for a years-of-experience attribute. According to both $\succ_1 \otimes \succ_2$ and $\succ_1 \& \succ_2$, $(\{a,b,c\},5)$ and $(\{a\},5)$ are preferred to $(\{a\},3)$, while $(\{a,b\},5)$ is not. Again, this example violates the implicit monotonicity in the application: improving the set of qualifications should not cause a previously preferred record to become not preferred. □

Triangle composition can also give problematic results, as illustrated by Examples 2.3 and 2.4.

Example 2.3: Extend Example 2.1, so that according to $\succ_2$, red cars are preferred to blue cars, which are in turn preferred to green cars. Let $\succ$ denote $\succ_1 \rhd \succ_2$. Then according to the definition of triangle composition, we have

$$(100, \text{red}) \succ (90, \text{blue}) \succ (75, \text{green}) \succ (100, \text{red}).$$

Thus, there is a cycle of preferences according to triangle composition. □

Example 2.4: Let $\succ$ denote $\succ_1 \rhd \succ_2$ for the orders defined in Example 2.2. Then according to the definition of triangle composition, we have

$$(\{a\},5) \succ (\{a,b\},4) \succ (\{a,b,c\},3) \succ (\{a\},5).$$

Again, there is a cycle of preferences. □

Cycles of preferences present several obvious problems. The transitive closure of such a relation is not a strict order, and would be considered inconsistent. An evaluation method that discards a tuple in favor of a preferred tuple may get stuck in an infinite loop, even in a finite domain.

3. A Solution

We propose that preference relations be specified as a *pair* of orders. The first order $\succ$ is a strict partial order that expresses the "is better than" relationship, as before. The second order $\sqsupseteq$ is a preorder[2] that extends $\succ$, and is intended to better capture when items are comparable.

Definition 3.1: Let R_1 and R_2 be binary relations. We say R_2 *extends* R_1 if

1. When $(x,y) \in R_1$, $(x,y) \in R_2$ and $(y,x) \notin R_2$.
2. When $(x,y) \in R_1$ and $(y,z) \in R_2$, $(x,z) \in R_1$.
3. When $(x,y) \in R_2$ and $(y,z) \in R_1$, $(x,z) \in R_1$. □

Example 3.1: The standard order $\leq$ on the real numbers extends the strict order $<$ on the reals, because if $x < y$ we know both $x \leq y$ and $y \not\leq x$, and since $x < y \leq z$ and $x \leq y < z$ each imply $x < z$. □

Example 3.2: Let $\subset$ denote the standard proper-subset relation on finite sets, and let $<$ denote the relation that says $S_1 < S_2$ if the cardinality of S_1 is less than that of S_2. $<$ *does not* extend $\subseteq$. Even though the first condition of Definition 3.1 is satisfied, the other two are not. □

Definition 3.2: Given a strict partial order $\succ$ and a preorder $\sqsupseteq$ that extends $\succ$, we write $>$ to denote the pair of orders $(\succ, \sqsupseteq)$. We call $>$ a *paired order*. $x > y$ is defined to be $x \succ y$, and $x \geq y$ is defined to be $x \sqsupseteq y$. □

The novel aspect of using pairs of orders is apparent when we compose orders. We define analogs of prioritized and Pareto composition for paired orders.

[2] A preorder is reflexive and transitive, but not necessarily antisymmetric.

Definition 3.3: Let $>_1$ and $>_2$ be paired orders. The prioritized composition $>_1$ & $>_2$ is defined as the pair $(\succ, \sqsupseteq)$, where $\succ$ is defined on pairs of values by

$$x_1 >_1 y_1 \text{ or } (x_1 \geq_1 y_1 \text{ and } x_2 >_2 y_2)$$

and $\sqsupseteq$ is defined by

$$x_1 >_1 y_1 \text{ or } (x_1 \geq_1 y_1 \text{ and } x_2 \geq_2 y_2)$$ □

Lemma 3.1: The prioritized composition $(\succ, \sqsupseteq)$ of two paired orders $>_1 = (\succ_1, \sqsupseteq_1)$ and $>_2 = (\succ_2, \sqsupseteq_2)$ is a paired order.

Proof. We need to verify that $\succ$ is a strict partial order, that $\sqsupseteq$ is a preorder, and that $\sqsupseteq$ extends $\succ$.

- $\succ$ is irreflexive, since $x \succ_1 x$ or $(x \sqsupseteq_1 x$ and $x \succ_2 x)$ is false, each $\succ_i$ being irreflexive.
- $\succ$ is transitive. If $x \succ_1 y$ and $y \succ_1 z$ then transitivity follows from the transitivity of $\succ_1$. If $(x \sqsupseteq_1 y$ and $x \succ_2 y)$ and $(y \sqsupseteq_1 z$ and $y \succ_2 z)$ then transitivity follows from the transitivity of both $\sqsupseteq_1$ and $\succ_2$. The remaining cases follow similarly, using the fact that $\sqsupseteq_1$ extends $\succ_1$.
- $\succ$ is antisymmetric. If $x \succ y$ and $y \succ x$ then $x \succ x$ by transitivity, violating irreflexivity.
- $\sqsupseteq$ is reflexive. This is an easy consequence of the reflexivity of both $\sqsupseteq_1$ and $\sqsupseteq_2$.
- $\sqsupseteq$ is transitive. The argument is similar to that for the transitivity of $\succ$.
- $\sqsupseteq$ extends $\succ$. The first condition of Definition 3.1 follows from the fact that $\sqsupseteq_2$ extends $\succ_2$. The second and third conditions follow from $\sqsupseteq_2$ extending $\succ_2$ and $\sqsupseteq_1$ extending $\succ_1$. □

Definition 3.4: Let $>_1$ and $>_2$ be paired orders. The Pareto composition $>_1 \otimes >_2$ is defined as the pair $(\succ, \sqsupseteq)$, where $\succ$ is defined on pairs of values by

$$(x_1 >_1 y_1 \text{ and } x_2 \geq_2 y_2) \text{ or } (x_1 \geq_1 y_1 \text{ and } x_2 >_2 y_2)$$

and $\sqsupseteq$ is defined by

$$x_1 \geq_1 y_1 \text{ and } x_2 \geq_2 y_2$$ □

Lemma 3.2: The Pareto composition of two paired orders is a paired order.

Proof. Similar to the proof of Lemma 3.1. □

Conventional prioritized composition and Pareto composition are a special cases of Definition 3.3 and Definition 3.4, respectively, with $\sqsupseteq_1$ being the union of $\succ_1$ and the equality relation. For transitive indifference relations, triangle composition is a special case of Definition 3.3, as shown in the following lemma.

Lemma 3.3: For a strict partial order $\succ$ with indifference relation $\sim$, define $\sqsupseteq$ as $\succ \cup \sim$. If $\sim$ is transitive, then $(\succ, \sqsupseteq)$ is a paired order, and prioritized composition of such paired orders according to Definition 3.3 is equivalent to triangle composition.

Proof sketch. The main idea for both transitivity and for showing that $\sqsupseteq$ extends $\succ$ is to demonstrate that if $x \succ y$ and $y \sim z$, then $x \succ z$. This is achieved by elimination. $x \sim z$ is not possible because then $x \sim y$ would hold by the transitivity and symmetry of $\sim$, violating $x \succ y$. $z \succ x$ is not possible because transitivity of $\succ$ would imply $z \succ y$, contradicting $y \sim z$. □

For cases where $\sim$ is not transitive, we can address the deficiencies highlighted in Section 2 by supplying a suitable preorder $\sqsupseteq$ for the given strict partial order, and using Definition 3.3 to construct a prioritized composition.

Example 3.3: Consider the order of Example 1.1, and define $p_1 \sqsupseteq_1 p_2$ to be true if price p_1 is less than or equal to p_2. It is relatively simple to see that $\sqsupseteq_1$ is a preorder that extends the strict partial order $p_1 \succ_1 p_2$ defined by price p_1 being less than 80% of p_2.

Revisiting Example 2.1, suppose we instead used the paired order $(\succ, \sqsupseteq) = (\succ_1, \sqsupseteq_1)\&(\succ_2, \succ_2 \cup =)$ to define the preferences. Then even though $90 \not\succ_1 100$, we do have $90 \sqsupseteq_1 100$, and so (90,red) is now preferred to (100,blue) because blue $\succ_2$ red. Note that (90,red) is still not preferred to (100,red).

Our choice also eliminates the problems of Example 2.3 in which triangle composition led to a cycle of preferences. Because $\succ$ is a partial order, such cycles do not occur. □

Example 3.4: Consider the order of Example 1.2, and define $S_1 \sqsupseteq_1 S_2$ to be true if $S_2 \subseteq S_1$. $\sqsupseteq_1$ is a preorder that extends the strict partial order $\succ_1$.

Revisiting Example 2.2, suppose we instead used the paired order $(\succ, \sqsupseteq) = (\succ_1, \sqsupseteq_1)\&(\succ_2, \succ_2 \cup =)$ to define the preferences. Then even though $\{a, b\} \not\succ_1 \{a\}$, we do have $\{a, b\} \sqsupseteq_1 \{a\}$, and so $(\{a, b\}, 5)$ is now preferred to $(\{a\}, 3)$ because $5 \succ_2 3$. Note that $(\{a, b\}, 5)$ is still not preferred to $(\{a\}, 5)$.

In Example 2.4, triangle composition led to a cycle of preferences. Because $\succ$ is a partial order, such cycles do not occur. □

Example 3.5: Consider a variant of Example 3.3 in which $p_1 \sqsupseteq_1 p2$ is true if price p_1 is less than 90% of p_2 or if $p_1 = p_2$. $\sqsupseteq_1$ is a preorder that extends $\succ_1$, and so $(\succ, \sqsupseteq_1)$ a paired order. There would still be a "gap" between the prioritized composition and the equality relation, and problems like those of Example 2.1 would remain. □

As Examples 3.3 and 3.5 show, there may be multiple preorders that extend a given partial order. (Conventional prioritized composition represents a third option for the preorder in this example.) Some of these preorders may retain the gap between the strict partial order and equality. The following theorem implies that there is a unique maximal preorder that extends any given partial order, and that therefore minimizes the gap with equality.

Theorem 3.4: Let S be a collection of preorders $\sqsupseteq_i$, each of which extend a strict partial order $\succ$. Let $\sqsupseteq$ denote the transitive closure of the union of all $\sqsupseteq_i$ in S. Then $\sqsupseteq$ is a preorder that extends $\succ$.

Proof. The transitivity of $\sqsupseteq$ is trivial, and reflexivity follows from the reflexivity of each $\sqsupseteq_i$. To verify the first condition of Definition 3.1, suppose $x \succ y$. Then $x \sqsupseteq y$ since $x \sqsupseteq_i y$ holds for each i. Suppose that, contrary to Definition 3.1 $y \sqsupseteq x$. Then consider the values of x and y with the shortest finite sequence $y_1, \ldots, y_n$ having the property that

$$y \sqsupseteq_{i_1} y_1 \sqsupseteq_{i_2} y_2 \sqsupseteq_{i_2} \cdots \sqsupseteq_{i_n} y_n \sqsupseteq_{i_{n+1}} x$$

for some values $i_1, \ldots, i_{n+1}$. If $n = 0$, we would have $y \sqsupseteq_{i_1} x$ which would contradict the assumption that each $\sqsupseteq_i$ extends $\succ$. If $n > 0$, then observe that since $x \succ y$ and $y \sqsupseteq_{i_1} y_1$, $x \succ y_1$ because $\sqsupseteq_{i_1}$ extends $\succ$. Then x and y_1 would be values with a shorter sequence of the above form, contradicting the assumption that we started with a shortest sequence.

The second and third conditions of Definition 3.1 follow from the assumption that each component preorder extends $\succ$. For example, if $x \succ y$ and $y \sqsupseteq z$ then for some $y_1, \ldots, y_n$ and some $i_1, \ldots, i_{n+1}$,

$$x \succ y \sqsupseteq_{i_1} y_1 \sqsupseteq_{i_2} y_2 \sqsupseteq_{i_2} \cdots \sqsupseteq_{i_n} y_n \sqsupseteq_{i_{n+1}} z.$$

which implies

$$x \succ y_1 \sqsupseteq_{i_2} y_2 \sqsupseteq_{i_2} \cdots \sqsupseteq_{i_n} y_n \sqsupseteq_{i_{n+1}} z.$$

because $\sqsupseteq_{i_1}$ extends $\succ$. We can iterate this inference process to obtain $x \succ z$. □

Corollary 3.5: There is a unique maximal preorder that extends a given partial order. □

Theorem 3.6: The preorder $\sqsupseteq$ defined in Example 3.3 is the maximal preorder that extends the $\succ$ relation of Example 1.1 on the real numbers.

Proof. Suppose to the contrary that some preorder R that extends $\succ$ properly contains the $\sqsupseteq$ relation. Since $\sqsupseteq$ contains all pairs (x, y) with $x \leq y$, R must contain some pair (a, b) with $a > b$. Let $c = \frac{a+b}{2*0.8}$. Then $b < 0.8 * c$ and so $b \succ c$, while $a > 0.8 * c$ meaning that $a \not\succ c$. But if R extends $\succ$, then aRb and $b \succ c$ must imply $a \succ c$. This contradiction implies that no such R can exist. □

Theorem 3.7: The preorder $\sqsupseteq$ defined in Example 3.4 is the maximal preorder that extends the $\succ$ relation of Example 1.2, assuming that sets are finite and elements are drawn from an infinite domain.

Proof. Suppose to the contrary that some preorder R that extends $\succ$ properly contains the $\sqsupseteq$ relation. Since $\sqsupseteq$ contains all pairs (x, y) with $x \supseteq y$, R must contain some pair (s_1, s_2) with $s_2 - s_1 \neq \emptyset$. Let $s_3 = s_1 \cup \{a, b\}$, where neither a nor b is in $s_1 \cup s_2$. (It is always possible to find such a and b since the element domain is infinite, and $s_1 \cup s_2$ is finite.) Then $s_3 \succ s_1$ while $s_3 \not\succ s_2$. But if R extends $\succ$, then $s_3 \succ s_1$ and $s_1 R s_2$ must imply $s_3 \succ s_2$. This contradiction implies that no such R can exist. □

Consider the $\succ$ relation defined on nodes in a DAG, requiring one node to be at least two nodes upstream of the other, as in Example 1.3. One might guess that the maximal preorder $\sqsupseteq$ extending $\succ$ is the (reflexive) ancestor relationship, so that $n_1 \sqsupseteq n_2$ if $n_1 = n_2$ or n_1 is upstream of n_2. However, this guess is not always correct. For example, if the DAG has no paths of length 2 or more, then $\succ$ is empty and all nodes are related to each other via the maximal preorder. This example suggests that it would be unwise to force the user to always select the maximal preorder $\sqsupseteq$; the user may intend a weaker relation.

4. Conclusions

We have shown how semantic anomalies can be avoided by defining a preorder that extends the given partial order, and using the pair of orders to define order composition. In some preference logics, one also defines a pair of relations: a strict order $\succ$ and an explicit indifference relation $\approx$ such that $\approx$ is symmetric and disjoint from $\succ$ [3]. Our preorders $\sqsupseteq$ are not definable as $\succ \cup \approx$, in general, because $\sqsupseteq - \succ$ is typically not symmetric.

We propose that in future work, authors using strict partial orders to define preferences also specify an appropriate partner preorder.

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