# Equiprojective Polyhedra

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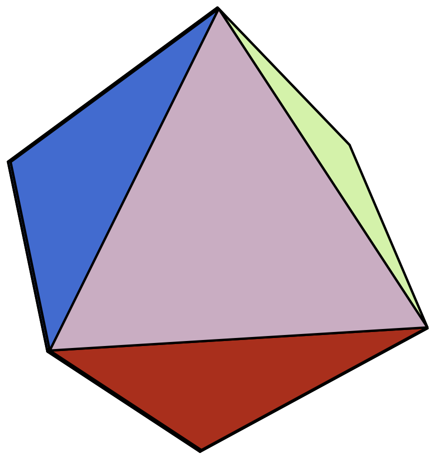
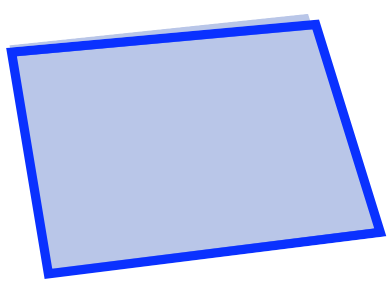
University of Waterloo

# Equiprojective Polyhedra



convex polyhedron

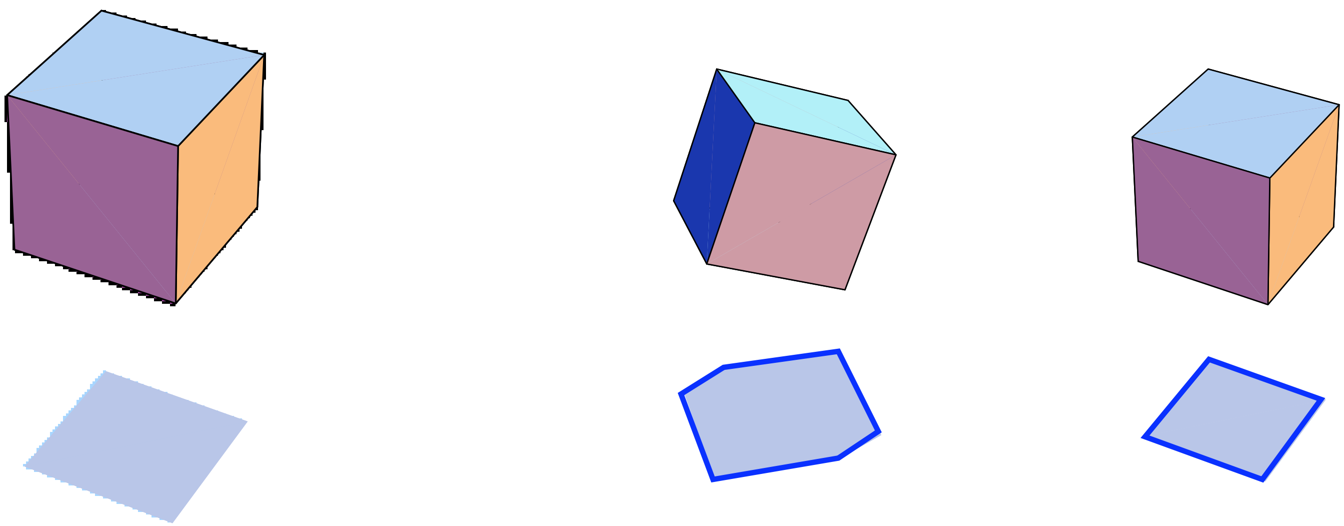


orthographic projection
= shadow

size of shadow boundary

equiprojective = constant size shadow boundary

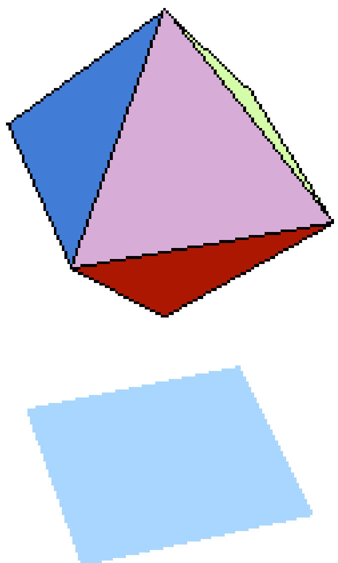
# The Cube is Equiprojective

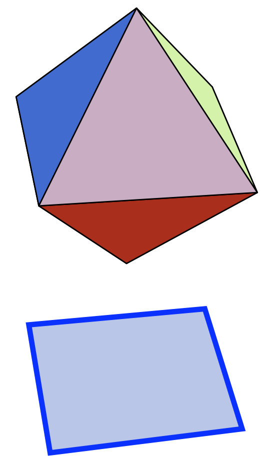


this projection
doesn’t count

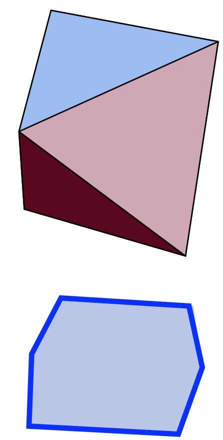
the shadow of a cube always has 6 sides

# The Octahedron is Not Equiprojective



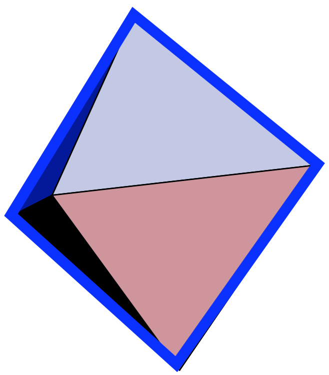


4-sided

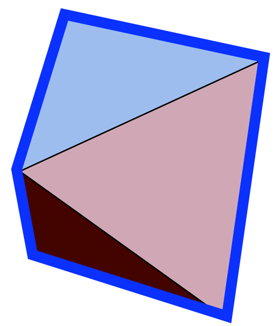


6-sided

# The Octahedron is Not Equiprojective

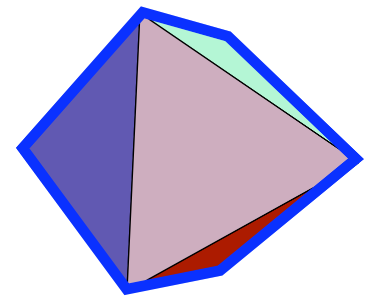


4-sided



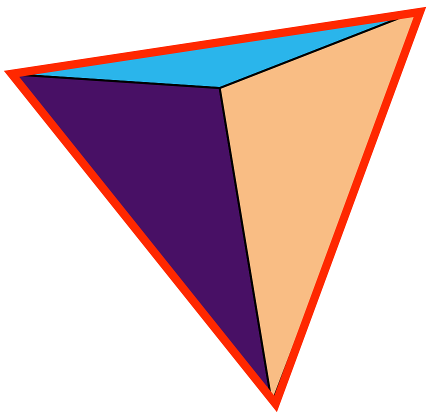
5-sided

this projection
doesn’t count

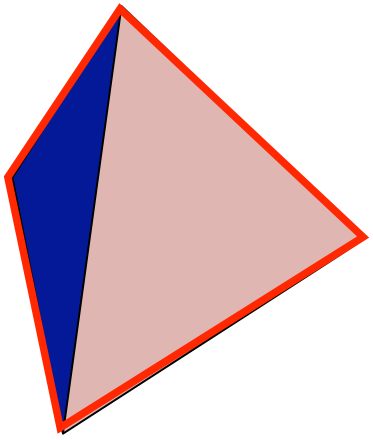


6-sided

# The Tetrahedron is Not Equiprojective



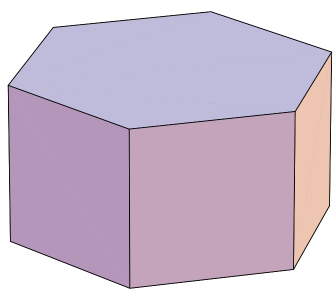
3-sided



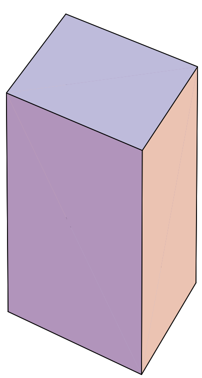
4-sided

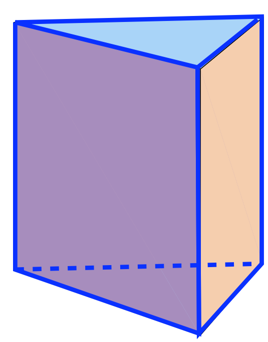
# Definition

Convex polyhedron $P$ is *k*-equiprojective if all orthogonal projections, except those parallel to faces of $P$, are $k$-gons.



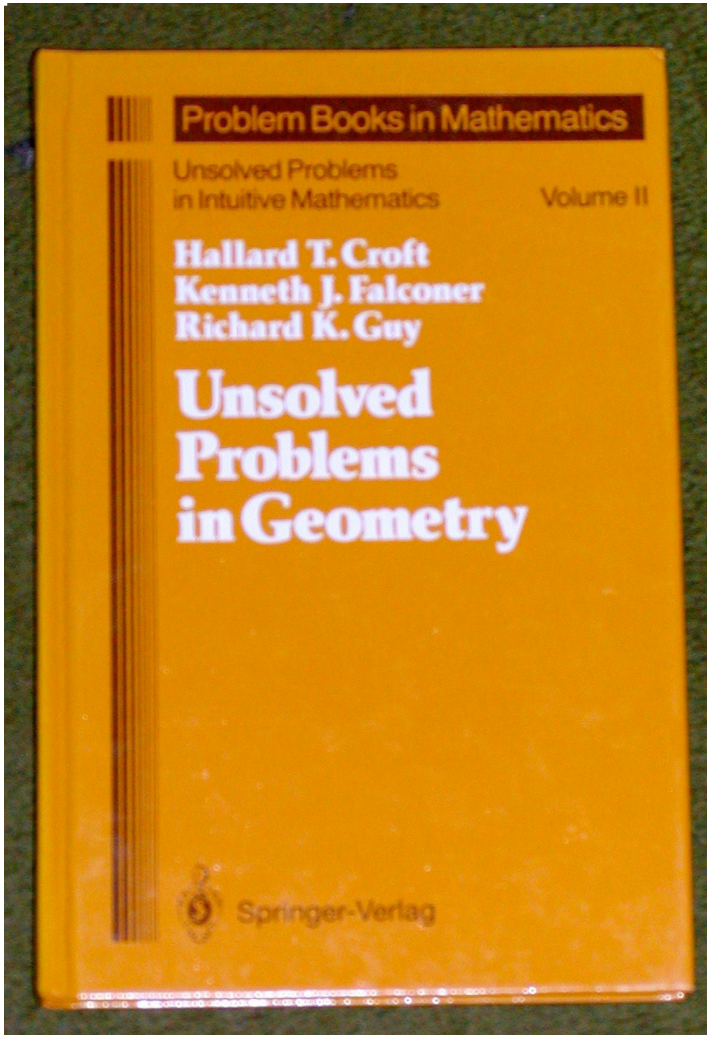
Any $k$-gonal prism is $(k+2)$-equiprojective





# History

Croft, Falconer, Guy,
*Unsolved Problems in Geometry*,
1991.



# History

G.C. Shephard, Twenty problems on convex polyhedra. II, Mathematical Gazette 52, 359-367, 1968.

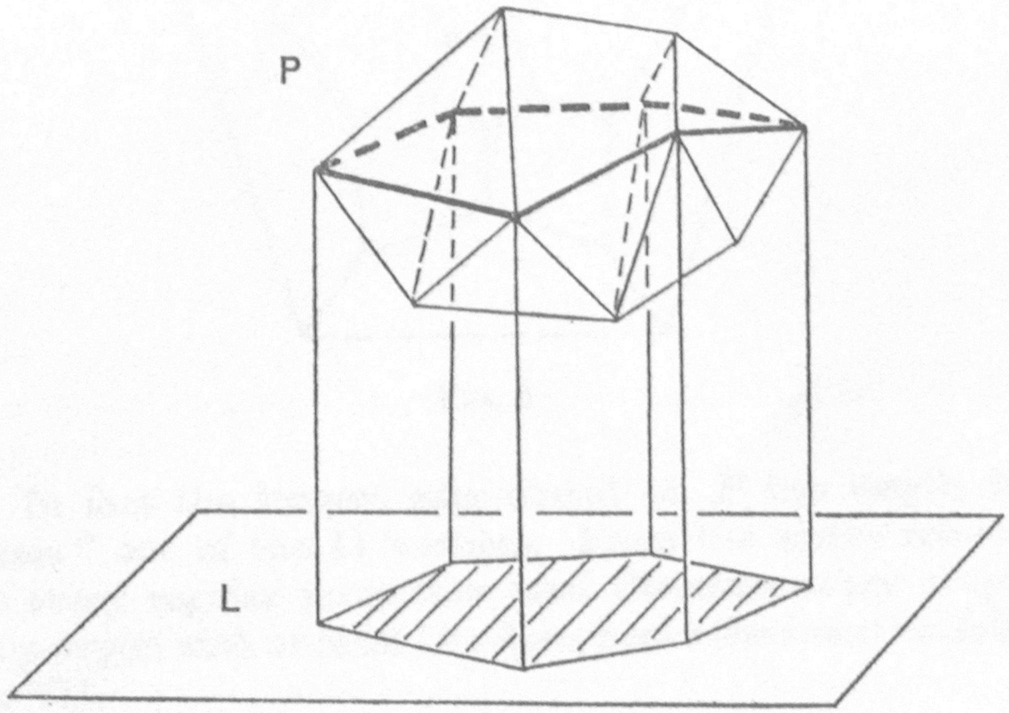



FIG. 7

# History

G.C. Shephard, Twenty problems on convex polyhedra. II, Mathematical Gazette 52, 359-367, 1968.

Some convex polyhedra $P$ have the remarkable property that every regular projection of $P$ is an $n$-gon for some fixed value of $n$. For example, every regular projection of a cube is a hexagon, and every regular projection of a right triangular prism is a pentagon. Such convex polyhedra, which are called *equiprojective*, are easy to construct, but the following problem (which is probably not very difficult) is unsolved:

IX. *Devise a method for constructing every equiprojective convex polyhedron.*

# Results

- a characterization of equiprojective polyhedra
- a linear time recognition algorithm

# Projections

Is $P$ equiprojective?

Could try all combinatorially distinct projections.

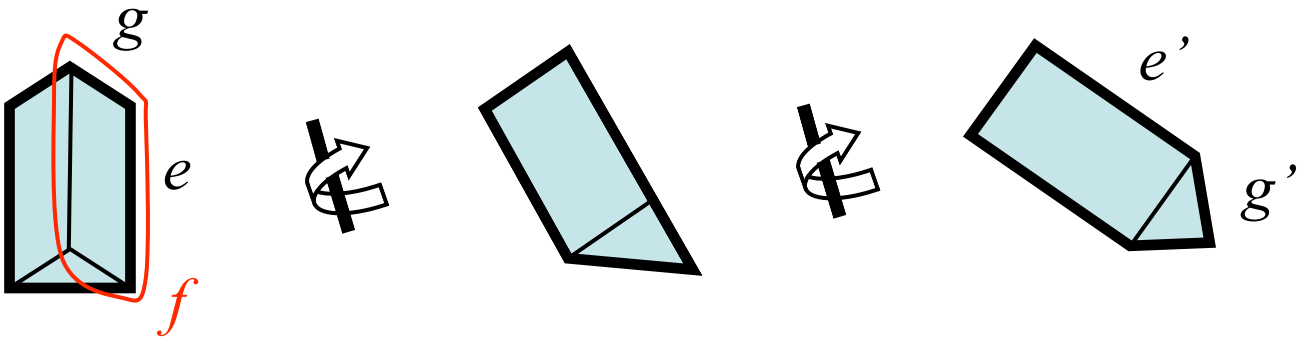
How many? $O(n^2)$

H. Plantinga and C.R. Dyer, Visibility, occlusion, and the aspect graph, International J. Computer Vision, 1990.

Open. Is there a fast algorithm to find projections with min and max number of edges?

# Idea of Our Characterization

Crucial rotation: when one face becomes parallel to the projection

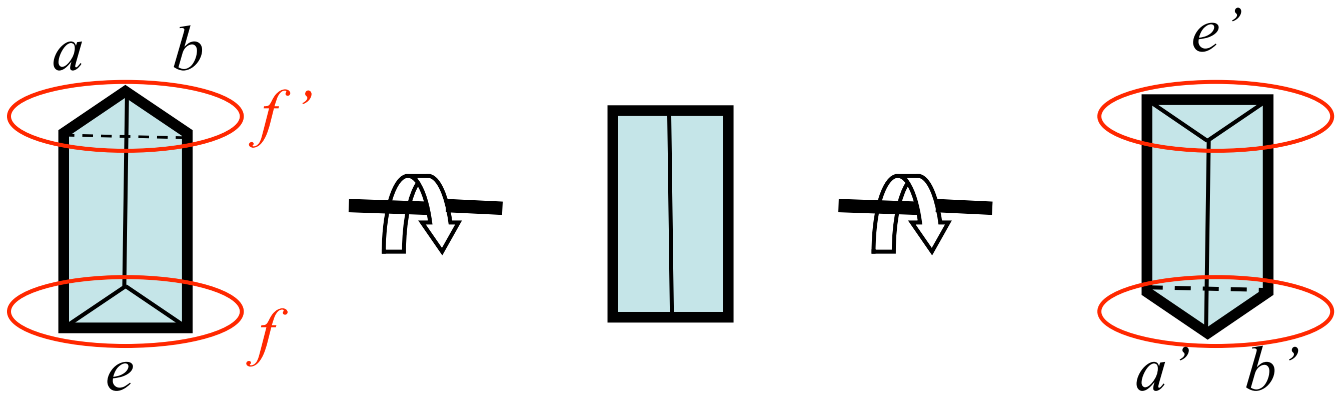



edge *e* on face *f* compensated by edge *e'* on face *f*

edge *g* on face *f* compensated by edge *g'* on face *f*

# Idea of Our Characterization

Crucial rotation: when one face becomes parallel to the projection

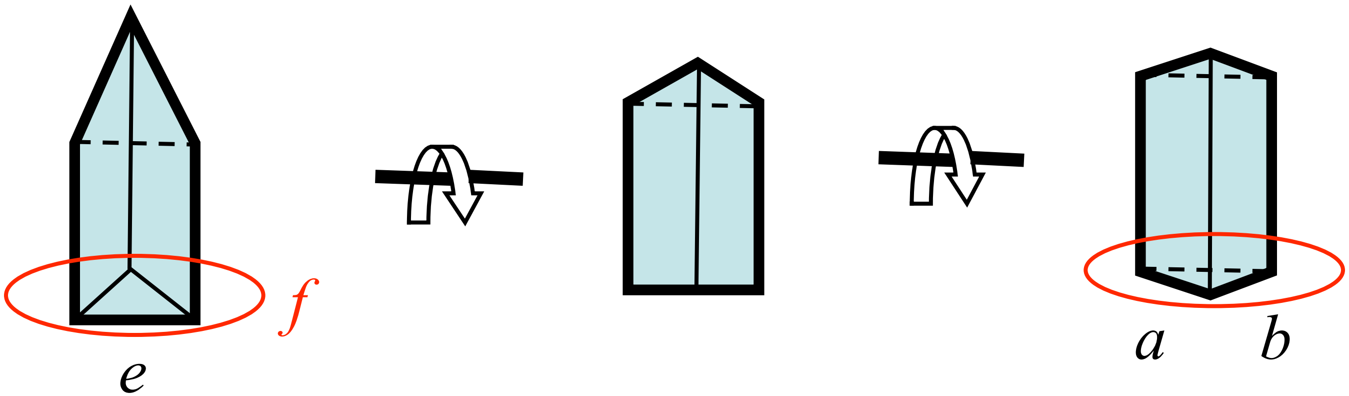



edge *e* on face *f* compensated by edge *e’* on face *f ’*

edge *a* on face *f* compensated by edge *a’* on face *f ’*
edge *b* on face *f* compensated by edge *b’* on face *f ’*

# Idea of Our Characterization

Crucial rotation: when one face becomes parallel to the projection

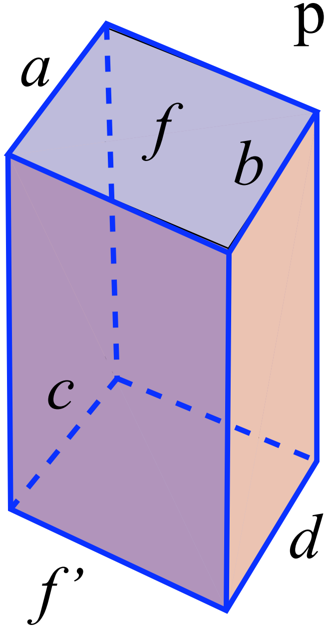



edges on face *f* don't compensate each other

# Compensation

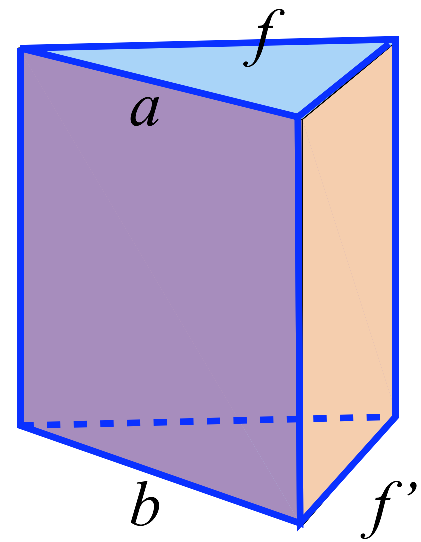
Definition. For edge $e$ in face $f$, $(e, f)$ is an *edge-face duple*.

Definition. Edge-face duples $(e, f)$ and $(e', f')$ are *parallel* if $e$ is parallel to $e'$ and $f$ is parallel to or equal to $f'$.




parallel to $(a, f)$:

$(b, f)$
$(c, f')$
$(d, f')$



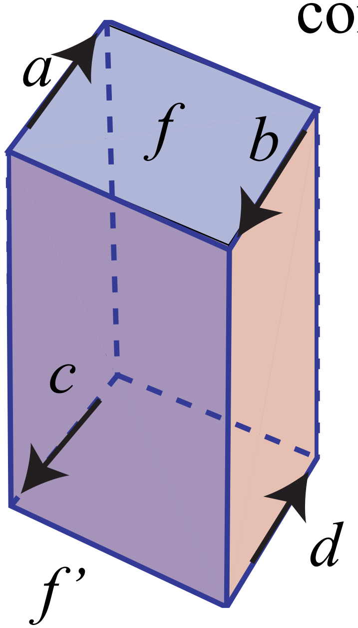

parallel to $(a, f)$:

$(b, f')$

Number of parallel duples: 0, 1, 2, or 3

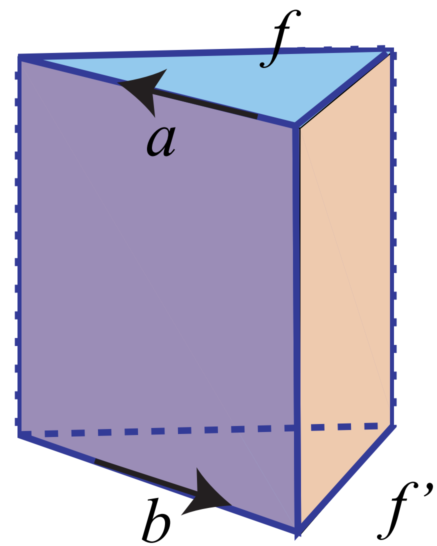
# Compensation

Definition. Parallel edge-face duples ($e$, $f$) and ($e'$, $f'$) *compensate* each other if their directions are opposite.




compensating ($a$, $f$):

($b$, $f$)
($c$, $f'$)




compensating ($a$, $f$):

($b$, $f'$)

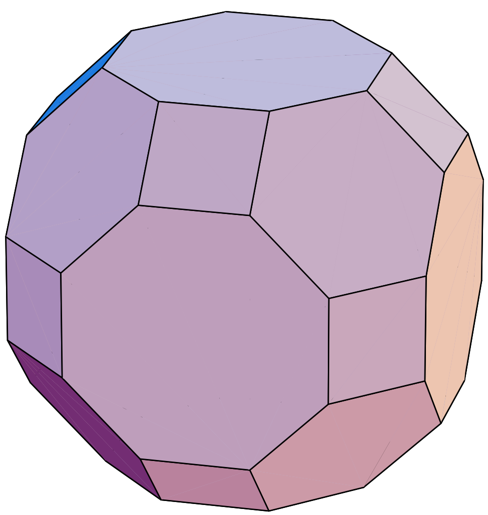
Number of compensating duples: 0, 1, or 2

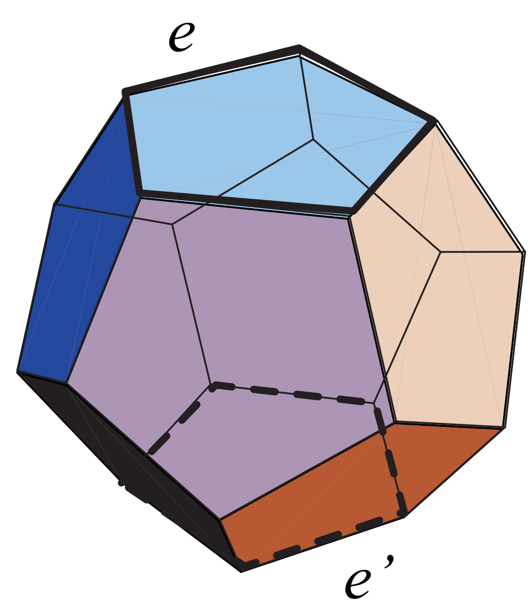
# Our Characterization

Theorem. Convex polyhedron *P* is equiprojective

iff

its edge-face duples can be partitioned into compensating pairs.





# Classifying Equiprojective Polyhedra

Simple: every edge-face duple compensated in the same face

Face-compensating: every edge-face duple compensated in
an opposite parallel face

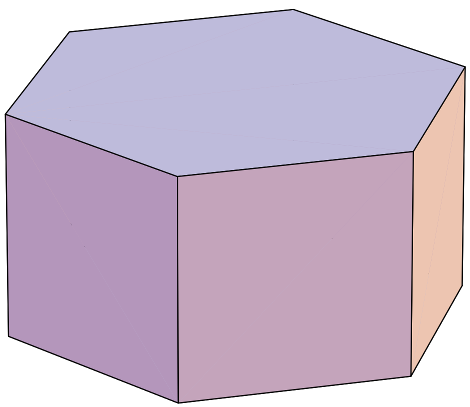
General: none of the above

simple == every face consists of parallel pairs of edges

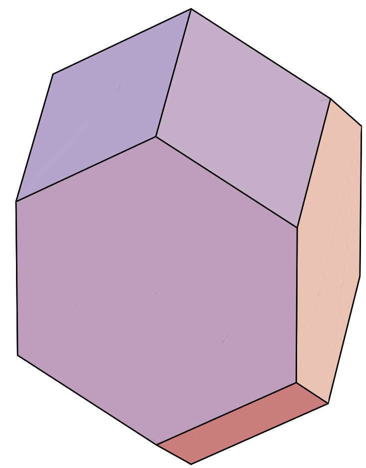
== zonohedron

# Zonohedra

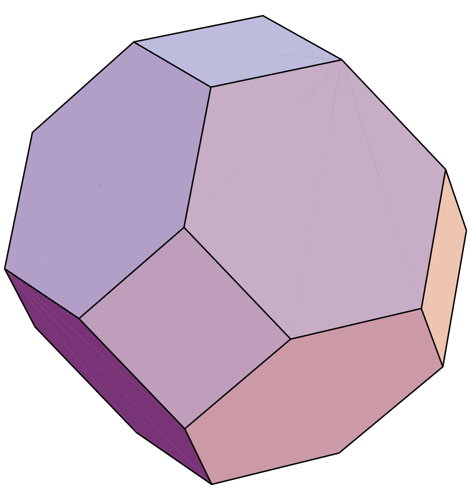
zonohedron: every face has parallel pairs of edges
(equivalently, a Minkowski sum of vectors)



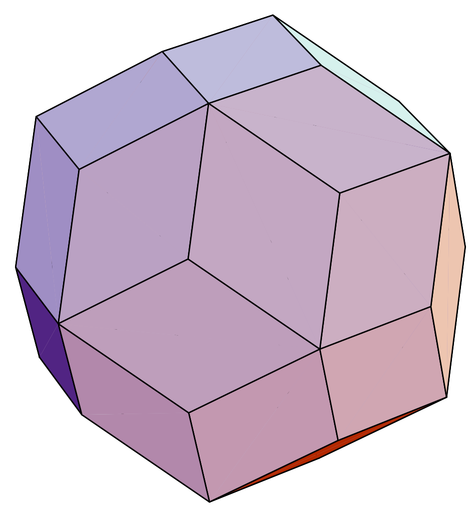
even
prism



extended-
rhombic-
dodecahedron



truncated
octahedron



rhombic
tricontahedron

# Classifying Equiprojective Polyhedra

Simple: every edge-face duple compensated in the same face

Face-compensating: every edge-face duple compensated in an opposite parallel face

General: none of the above

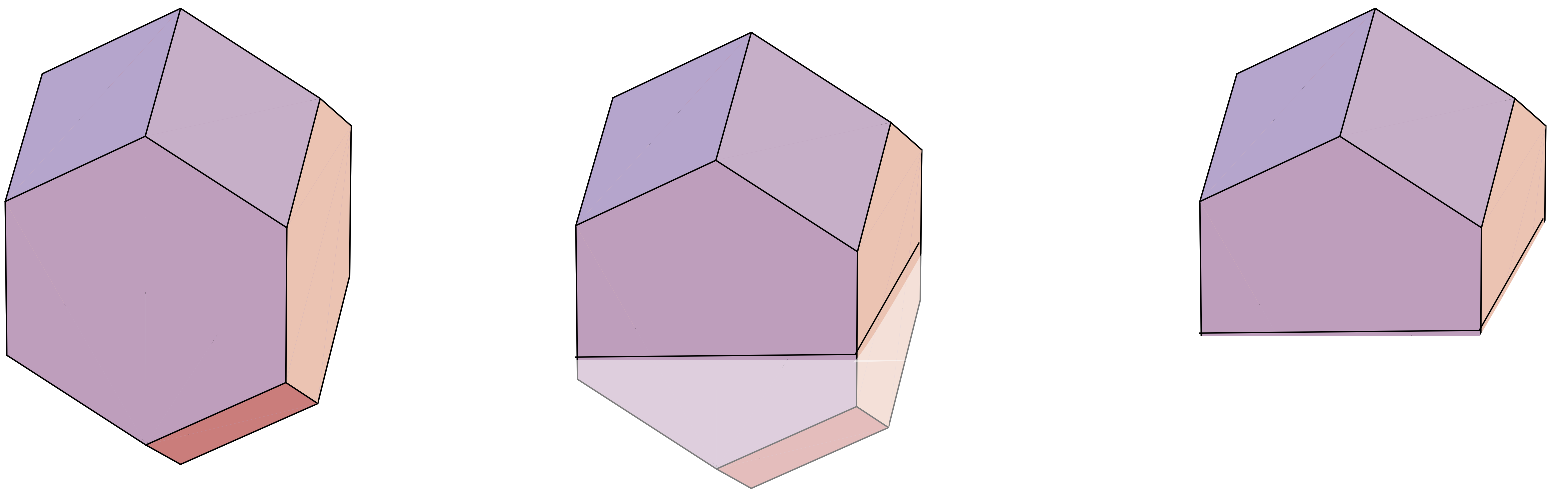
simple == zonohedron ==> face-compensating

Open: face-compensating ==> simple ?

# Classifying Equiprojective Polyhedra

General: not simple or face-compensating

Construction I: chopping

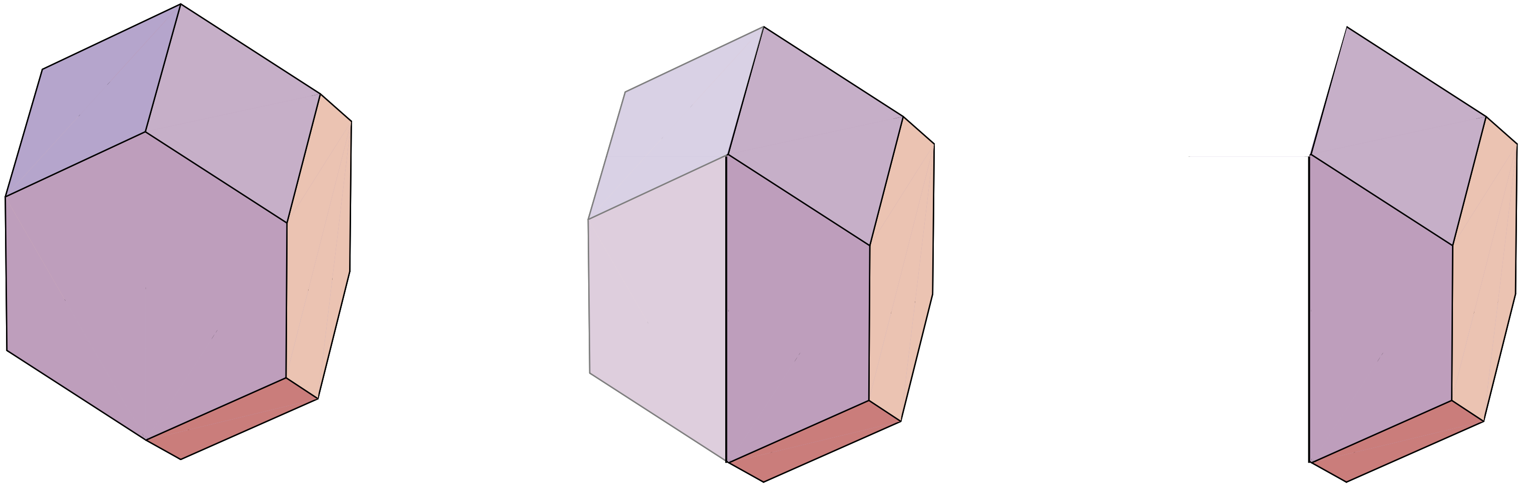


Each face is compensated by itself or by a parallel face.

# Classifying Equiprojective Polyhedra

General: not simple or face-compensating

Construction I: chopping

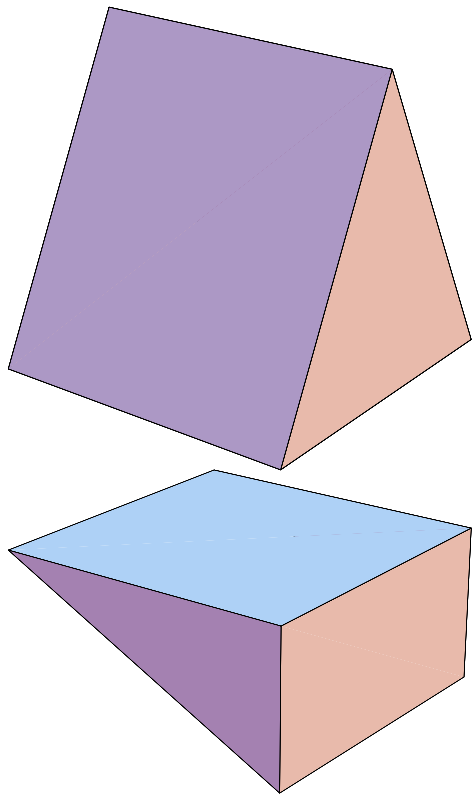


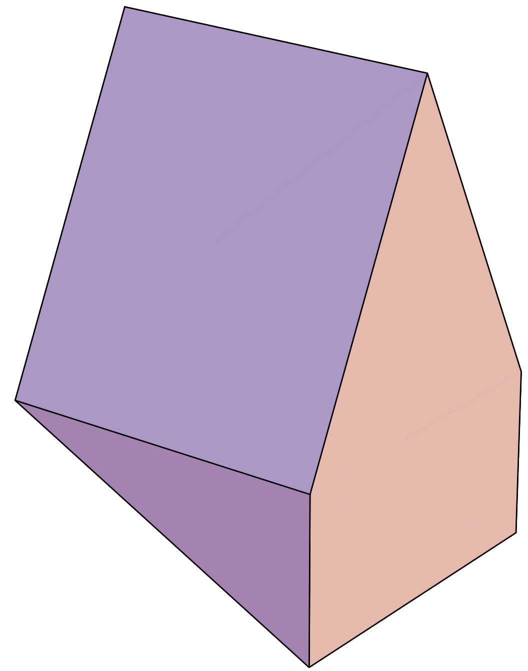
Each face is compensated by itself or by a parallel face.

# Classifying Equiprojective Polyhedra

General: not simple or face-compensating

Construction II: abutting

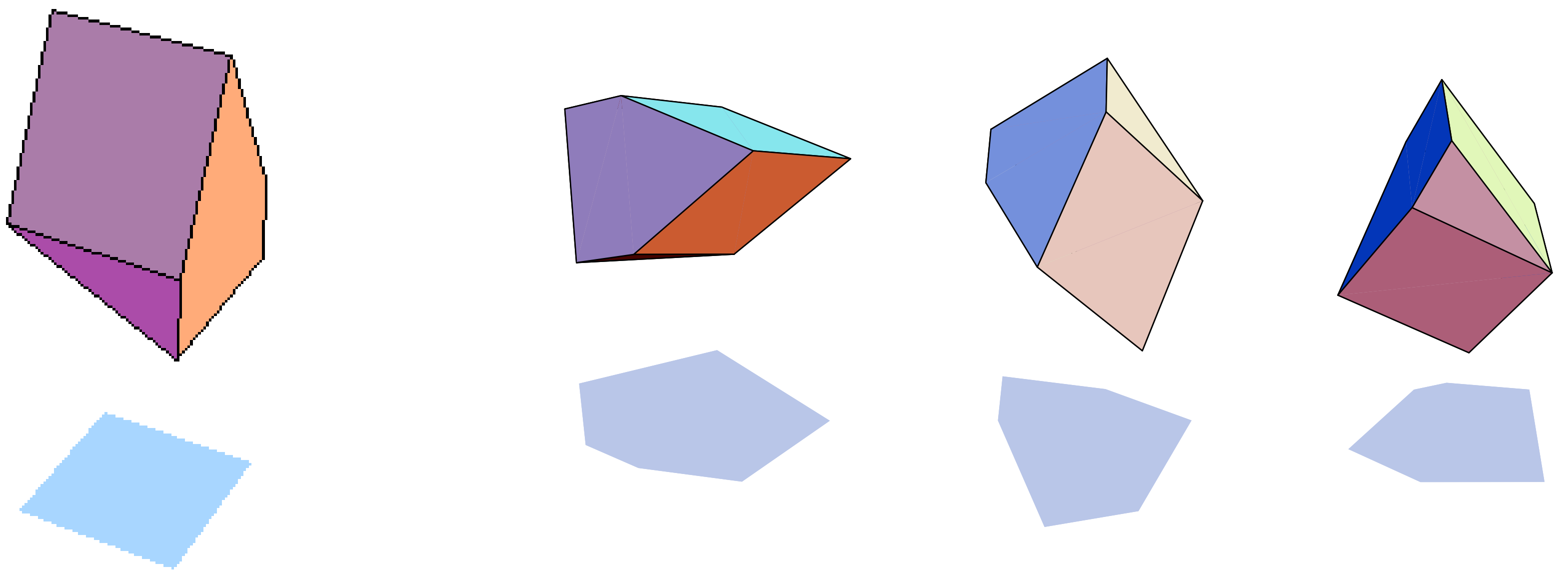




Not

Each face is compensated by itself or by a parallel face.

# Classifying Equiprojective Polyhedra



# Classifying Equiprojective Polyhedra

Open: Is there an algorithm to generate all equiprojective polyhedra?

# Algorithm to Recognize Equiprojective Polyhedra

Use

Theorem. Convex polyhedron $P$ is equiprojective

iff

its edge-face duples can be partitioned into compensating pairs.

# Algorithm to Recognize Equiprojective Polyhedra

1. find parallel faces, and parallel edge-face duples
2. try to partition each family of parallel edge-face duples into compensating pairs

---

1. can be done in linear time: need to overlay two planar maps corresponding to the generalized Gauss map and its negative
2. each family has 1, 2, 3, or 4 duples, so this is easy

# Proof of Theorem

Theorem. *P* is equiprojective iff its edge-face duples can be partitioned into compensating pairs.

Proof. <==

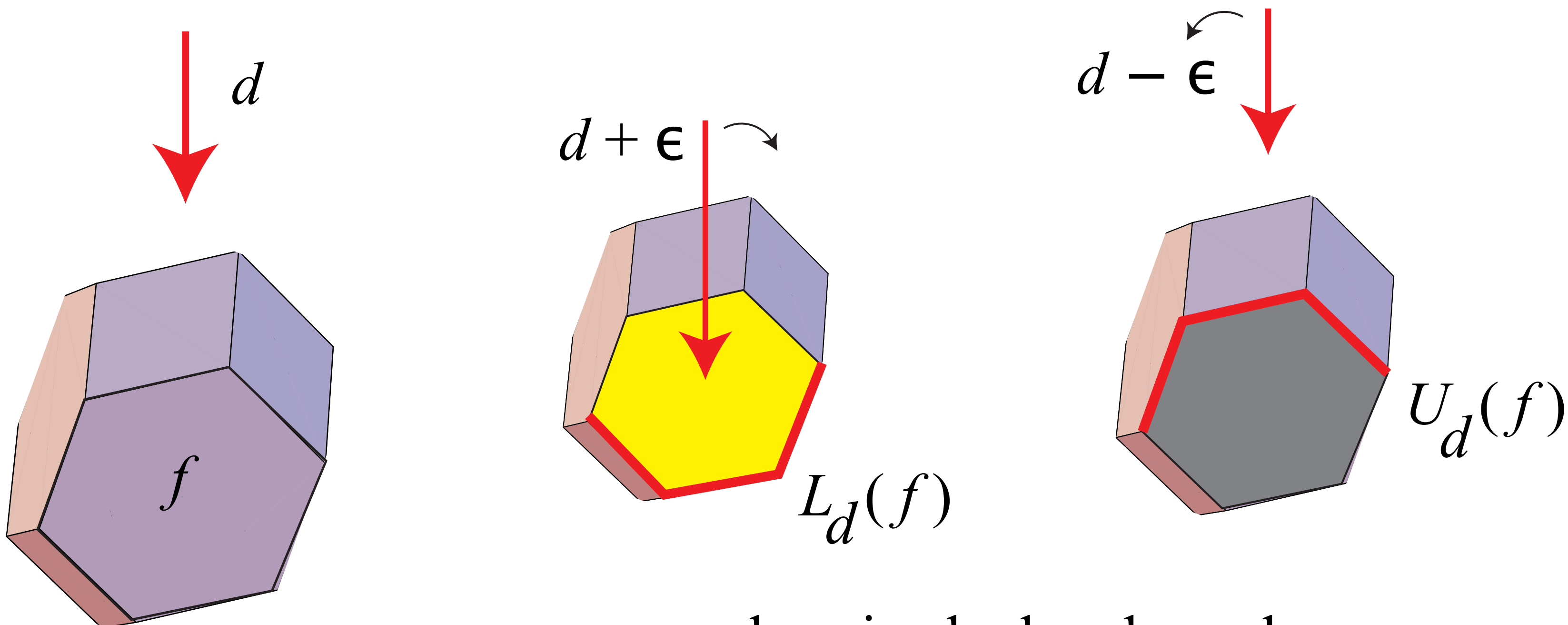
Suppose edge-face duples partitioned into compensating pairs. Prove any two projections of *P* have the same size.

Claim. Can rotate such that we always have at most one face (and its parallel partner) parallel to projection direction.

Thus suffices to study case where one face (and its parallel partner) rotate through the projection direction.

# Proof of Theorem

One face rotates through the projection direction.



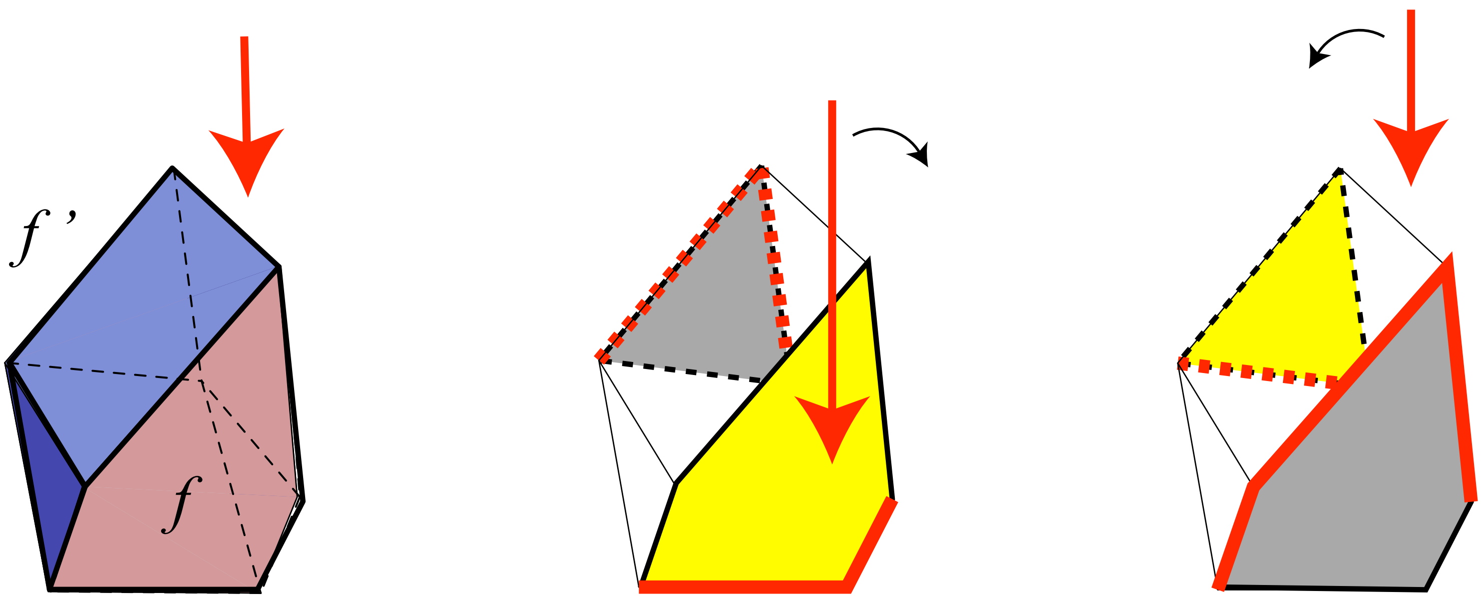

edges in shadow boundary

for $(e, f)$ and $(e', f)$ a compensating pair

$e$ is in the shadow boundary iff $e'$ is not

# Proof of Theorem

face *f* and parallel partner *f '* rotate through projection direction




edges in shadow boundary

for (*e*, *f* ) and (*e'*, *f '* ) a compensating pair

*e* is in the shadow boundary iff *e'* is not

# Proof of Theorem

Theorem. *P* is equiprojective iff its edge-face duples can be partitioned into compensating pairs.

Proof. ==> (via contrapositive)

Suppose edge-face duples *cannot be* partitioned into compensating pairs.

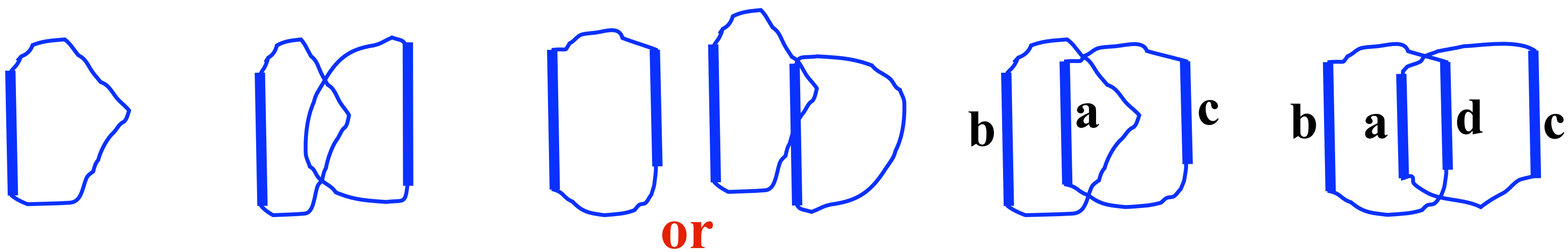
Find two projections of *P* with different sizes.

Consider a family of parallel edge-face duples that *cannot be* partitioned into compensating pairs.

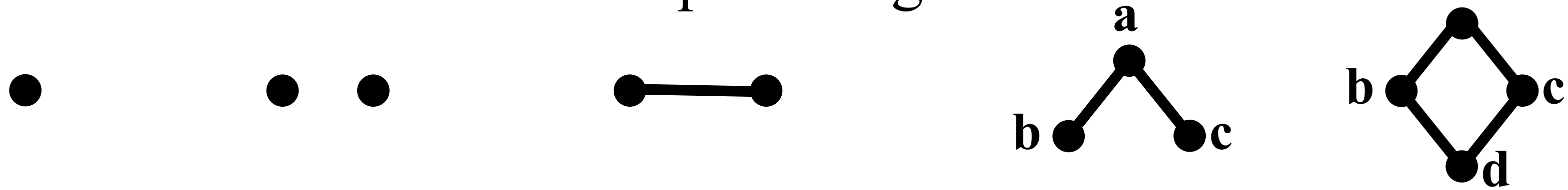
# Proof of Theorem

Consider a family of parallel edge-face duples that *cannot be* partitioned into compensating pairs.

parallel family



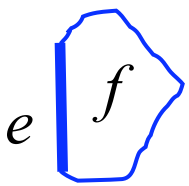

compensating



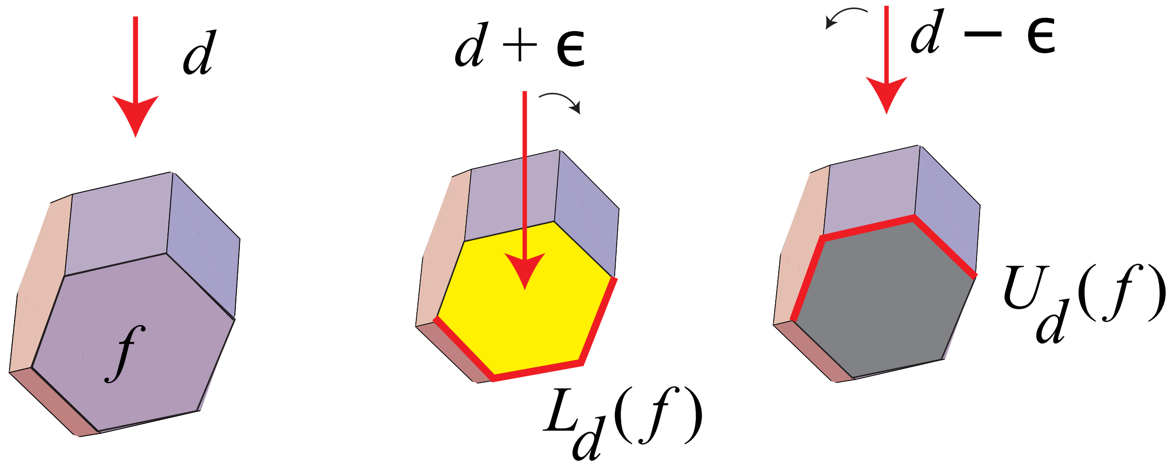

Case 1 Case 2 Case 4 Case 4 Case 5

# Proof of Theorem

Case 1

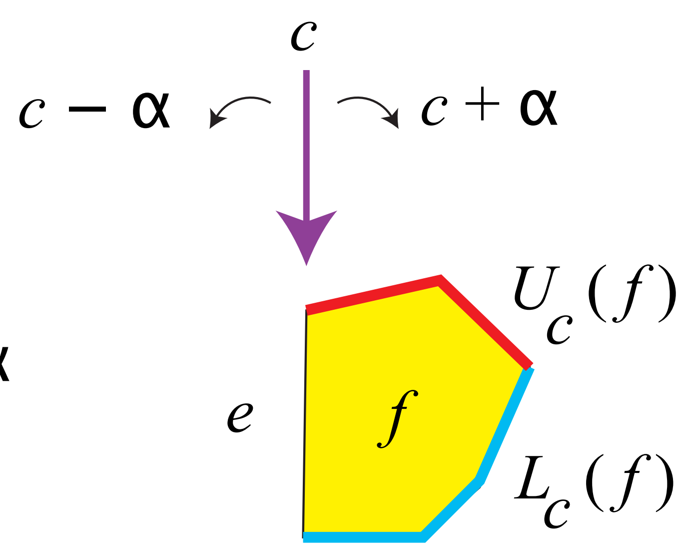



Recall:




Find a direction $d$ in plane of $f$

s.t. $|L_d(f)| \neq |U_d(f)|$

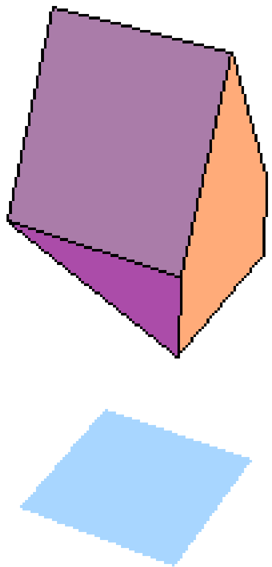



Use $c - \alpha$ or $c + \alpha$

Other cases similar.

# Non-Convex
# Equiprojective Polyhedra

# The End



# The End