

# Cauchy’s Theorem and Edge Lengths of Convex Polyhedra

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**Abstract.** In this paper we explore, from an algorithmic point of view, the extent to which the facial angles and combinatorial structure of a convex polyhedron determine the polyhedron—in particular the edge lengths and dihedral angles of the polyhedron. Cauchy’s rigidity theorem of 1813 states that the dihedral angles are uniquely determined. Finding them is a significant algorithmic problem which we express as a spherical graph drawing problem. Our main result is that the edge lengths, although not uniquely determined, can be found via linear programming. We make use of significant mathematics on convex polyhedra by Stoker, Van Heijenoort, Gale, and Shepherd.

## 1 Introduction

Cauchy proved that convex polyhedra are rigid in the sense that if the faces were metal plates, and the edges were hinges, no flexing would be possible. Although most presentations (e.g. see [6, 1]) don’t point this out, Cauchy actually proved a stronger result: that the dihedral angles of a convex polyhedron are completely determined by the facial angles and the combinatorial structure (the list of faces, edges, vertices, and their containments). In other words, Cauchy’s proof makes no use of the edge lengths. See [26] or [3] for presentations where this is explicit.

In this paper we examine the relationships among these four attributes of convex polyhedra: the facial angles and combinatorial structure, which we consider as givens; and the dihedral angles and the edge lengths, which we consider as unknowns. Dihedral angles are uniquely determined, by Cauchy’s Theorem; edge lengths are not, as the family of boxes demonstrates. This paper addresses the question: Are there efficient algorithms to find dihedral angles/edge lengths for given facial angles and combinatorial structure?

Our main result is an algorithm for the case of edge lengths (see Section 4). We give a linear system expressing the edge lengths in terms of facial angles and combinatorial structure. Correctness depends on a new characterization (in Section 3) of the edge lengths, facial angles, and dihedral angles of convex polyhedra. This is related to recent work on “local” or “easily checkable” conditions for convex polyhedra [17, 10, 20].

It is a major open question to devise an efficient algorithm to find the dihedral angles corresponding to given facial angles and combinatorial structure—i.e. to

devise an algorithm for Cauchy’s theorem. It is usually assumed that edge lengths are given as well. Such an algorithm can be used to find the unique polyhedron formed by “folding up” a surface satisfying Alexandrov’s conditions [3]. (See [9] for further explanation.) In 1998 Sabitov ([21], or see the sketch in [9]) gave a finite algorithm for Cauchy’s theorem. His algorithm uses the edge lengths heavily and crucially, and involves trying out all roots of the “volume polynomial”. The algorithm takes an exponential number of steps, where solving a high-degree polynomial is counted as one step. Recently Bobenko and Izvestiev ([5], see also [19]) gave an algorithm for the more general Alexandrov’s problem that, although it is not polynomial time, is effective in practice and has been implemented.

As noted, both these algorithms assume edge lengths are given—an assumption that Cauchy did not make. Our work fills in this gap: it is possible to compute edge lengths from facial angles and combinatorial structure and then proceed with these algorithms.

The present work falls under the general topic of reconstructing polyhedra. For other results and open problems in this area see [14, 8, 15].

We assume standard definitions about convex polyhedra and graphs, see e.g. [6]. By the “combinatorial structure” of a convex polyhedron we mean the list of faces, edges, and vertices and their containments. Equivalently, the combinatorial structure is a 3-connected planar graph, from which we can uniquely determine a *combinatorial embedding*, i.e. a cyclic list of the edges around each vertex that can be used to draw the graph in the plane without edge crossings.

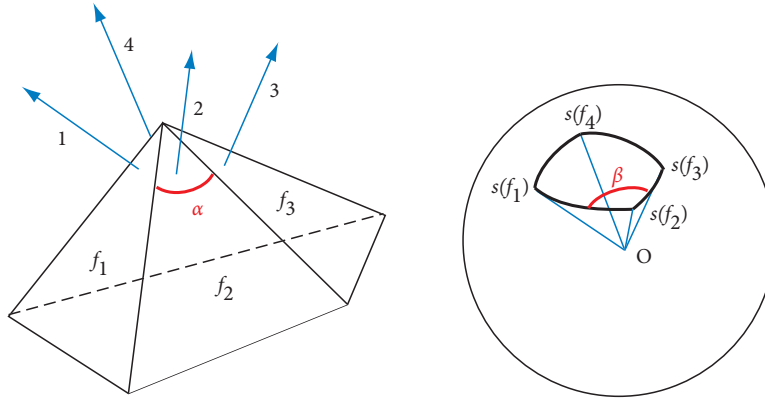
## 2 The Spherical Dual

In this section we describe the spherical dual of a convex polyhedron, which we will use in the next section for our characterization of the edge lengths, facial angles, and dihedral angles of convex polyhedra.

In a 1968 paper Stoker [26] gave some generalizations of Cauchy’s rigidity theorem. He pointed out explicitly that Cauchy’s proof does not use edge lengths, and transformed Cauchy’s problem in a way that abolishes edge lengths. This transformation is known as the spherical dual or “Gauss map” (see, e.g. [2]).

Given a convex polyhedron  $P$ , map it to the unit sphere as follows. Translate the sphere so that the origin is an interior point. Consider any point  $p$  on the unit sphere as a vector. Because the origin is inside the polyhedron, this vector  $p$  is the outward normal to a unique supporting plane  $\pi_p$  of the polyhedron. Label the point  $p$  on the sphere according to the vertex/edge/face of the polyhedron that lies in the plane  $\pi_p$ . Thus each face  $f$  of  $P$ , having a single supporting plane, maps to a single point  $s(f)$  on the sphere. Each edge  $e$  of  $P$  has a one-dimensional space of supporting planes and maps to a geodesic arc  $s(e)$ ; if  $f$  and  $g$  are the two faces on either side of  $e$ ,  $s(e)$  is the shorter arc of the great circle through  $s(f)$  and  $s(g)$ . Each vertex  $v$  of  $P$  maps to a convex spherical polygon bounded by the arcs corresponding to the edges incident to  $v$ . See Figure 1.

Let  $G$  be the skeleton graph of the polyhedron  $P$ . Mapping  $P$  to the sphere produces an embedded graph  $s(G)$  with vertices  $\{s(f) : f \text{ a face of } P\}$ , and edges



**Fig. 1.** The Gauss map and the correspondence of facial angles,  $\beta = \pi - \alpha$ .

$\{s(e) : e \text{ an edge of } P\}$  and the cyclic order of edges around vertices induced by the order of edges around faces in  $G$ . Clearly  $s(G)$  is the dual graph of  $G$ .

We now consider what the Gauss map does to facial and dihedral angles. Let  $e$  and  $e'$  be two incident edges of a face  $f$  of  $P$ . Let  $\alpha$  be the facial angle between them. Let  $\beta$  be the angle between  $s(e)$  and  $s(e')$  at the point  $s(f)$ . Then, as Stoker [26] shows,  $\beta$  and  $\alpha$  are supplementary angles, i.e.  $\beta = \pi - \alpha$ . Let  $f$  and  $g$  be two faces of  $P$  joined at edge  $e$ . Let  $\gamma$  be the dihedral angle between  $f$  and  $g$ , and let  $\delta$  be the length of the arc  $s(e)$  (measured in radians as an angle from the origin). Then  $\gamma$  and  $\delta$  are supplementary angles [26].

In our situation, we know the embedded graph  $G$  and the facial angles. Applying the Gauss map, we know the embedded graph  $s(G)$  and its facial angles. Dihedral angles of the original map to edge lengths in the spherical dual, and Cauchy's theorem says that these edge lengths are unique. Under this transformation the algorithmic form of Cauchy's theorem becomes: Given an embedded 3-connected planar graph and an angle between each consecutive pair of edges incident on a vertex, find a drawing of the graph on the sphere with non-crossing geodesic arcs for the edges, and with the specified angles. Supplements of arc lengths in the drawing provide the dihedral angles for the original problem.

No efficient algorithm is known for this problem, but it is connected to quite a body of work in graph drawing. The problem of drawing a graph in the plane with specified angles was first considered by Vijayan [27] and later proved NP-hard by Garg [12]. Drawing on the sphere might be harder, but all our angles are convex, which should be easier. See also [4] and [23] for the case of triangulated graphs. Spherical drawing of triangulated graphs has been addressed in the graphics community in the context of spherical parameterization, see in particular [24].

A closely related problem is to efficiently represent a 3-connected planar graph as the skeleton graph of a convex polyhedron [7].

### 3 Conditions for Existence of a Convex Polyhedron

In this section we give conditions on the edge lengths, facial angles, and dihedral angles that are necessary and sufficient for the existence of a convex polyhedron. In some sense this is solved by the “local” or “easily checkable” conditions for convex polyhedra [17, 10, 20] (see below for more details); however, our goal is to give conditions that separate the role of edge lengths and dihedral angles.

**Theorem 1.** *A 3-connected planar graph (with its unique combinatorial embedding) and given edge lengths and convex facial and dihedral angles are those of a convex polyhedron iff*

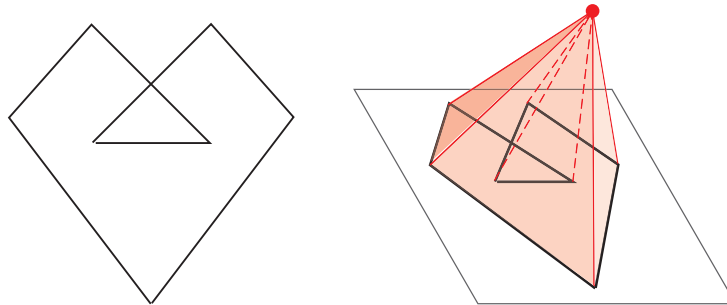
- (1) *the edges around every face form a simple convex polygon*
- (2) *in the spherical dual, the arcs of the edges around every face form a simple convex spherical polygon*

We will prove Theorem 1 using a 1952 result of Van Heijenoort [13] on locally convex manifolds. We first discuss algorithmic ramifications for computing edge lengths and/or dihedral angles. Note that Condition (1) depends on facial angles and edge lengths; Condition (2) depends on facial angles and dihedral angles—recall that dihedral angles correspond to arc lengths in the spherical dual. We were unable to use the conditions to help us find dihedral angles, but we can use them to find edge lengths. We separate Condition (1) into a part (1a) depending only on facial angles, and a part (1b) depending on both facial angles and edge lengths and expressible via linear inequalities. Facial angles determine edge directions. More precisely, if we choose a unit vector in the plane for the direction of one edge of face  $f$ , then the facial angles determine unit direction vectors  $d(e) \in R^2$  for each edge  $e$  as we go around  $f$  from the initial edge. Condition (1b) is that the sum of these unit vectors times the appropriate edge lengths is the zero vector, i.e. that the sequence of edges closes up to form a polygon.

What are the conditions on facial angles? It is not sufficient that every facial angle be convex, i.e. in the interval  $(0, \pi)$ : the polygon in Figure 2, from [17], has only convex angles, but is not a *simple* convex polygon, nor would any other choice of edge lengths make it so. To guarantee a simple convex polygon, we impose Condition (1a): that the facial angles around a face of  $k$  edges sum to  $\pi(k - 2)$ . We therefore arrive at the following conditions:

- (1a) for each face  $f$  of  $k$  edges,  $\pi(k - 2) = \sum\{\alpha : \alpha \text{ a facial angle of } f\}$ .
- (1b) for each face  $f$ ,  $\sum_{e \in f} l(e)d(e) = (0, 0)$  where  $l(e)$  is the length of edge  $e$  and  $d(e)$  is its direction, relative to some initial choice for one edge.

**Lemma 1.** *Condition (1) is equivalent to Conditions (1a) and (1b).*



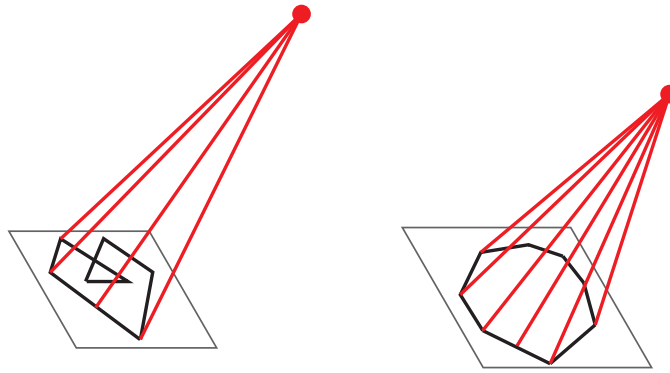
**Fig. 2.** Convex angles do not always make a simple polygon (left), or a simple polyhedron (right).

*Proof.* The main part of this is proved by Vijayan [27]. The proof is easy; we outline it for completeness. Clearly (1) implies (1a) and (1b). For the other direction, (1b) implies that we have a sequence of line segments that closes up to form a cycle. We prove by induction on  $k$ , the number of segments, that a simple convex polygon is formed. This is obvious for  $k = 3$ , and easy for  $k = 4$ . Consider  $k > 4$ . If there are two consecutive angles with sum greater than  $\pi$  we can eliminate the edge between them by extending the two neighbouring edges, and apply induction. If all pairs of consecutive angles have sum at most  $\pi$  then  $2\sum\alpha \leq k\pi$ . Applying condition (1a),  $2\pi(k-2) \leq \pi k$ , so  $2k-4 \leq k$ , so  $k \leq 4$ .

Translated to the sphere, Condition (2) seems symmetric with Condition (1) except that it involves arc lengths (corresponding to dihedral angles) in place of edge lengths. It seems tantalizing to express Condition (2) using subconditions analogous to those above, and thus obtain an algorithm to find dihedral angles.

Condition (1b)—that following the sequence of edges around a face “closes up” the polygon—can be transferred to the spherical situation, though it is computationally more difficult for the following reason. In the plane the edges around a face provide successive translations, so we get a linear system for the edge lengths; however, on the sphere the edges around a face provide successive rotations, so we get a non-linear system. This is not the main difficulty. In the plane, the remaining condition (1a) did not depend on edge lengths, but on the sphere it does, as we now show. Recall that condition (1a) precluded a polygon “wrapping around” more than once as in Figure 2 (left). The same issue arises on the sphere. As described by Mehlhorn et al. [17], the example of Figure 2 can be extended to three dimensions by adding two vertices, one above the plane and one below, with triangular faces joining each of these vertices to each edge of the polygon. Figure 2 (right) shows the new upper vertex. The resulting object is combinatorially a bipyramid with a 7-sided base; each face is a triangle; and each facial angle and each dihedral angle is in the range  $(0, \pi)$ . In the spherical dual the face corresponding to the upper vertex is a spherical polygon that wraps around twice and intersects itself.

In the plane, condition (1a) excluded such “wrapping around” by requiring that the sum of facial angles be  $\pi(k - 2)$  for any face of  $k$  edges. On the sphere no condition on facial angles alone will suffice: we give an example (prior to the Gauss map) of two spherical polygons with the same facial angles, exactly one of which is simple. Consider the example of Figure 2 with the upper vertex far away from the plane of the rest of the Figure, and with many acute triangles incident to it. With these same acute angles we can instead make the dihedral angles larger and connect to a simple polygon—see Figure 3.



**Fig. 3.** The same facial angles at a vertex can form a simple or non-simple cone

In the remainder of this section we give a proof of Theorem 1 using the following result of Van Heijenoort. The terms used in the theorem are defined just below. Our situation is more specialized in that we have a piece-wise linear manifold, which, as we shall see, makes the topological conditions straightforward.

**Theorem 2.** [13] *If a 2-dimensional manifold  $M$  is*

- (i) *mapped into  $R^3$  by a locally topological mapping  $f$*
- (ii) *locally convex under  $f$*
- (iii) *absolutely convex at a point*
- (iv) *complete under  $f$*

*then  $f(M)$  is the boundary of a 3-dimensional convex set.*

Van Heijendoort defines the manifold  $M$  to be *complete* if “every bounded infinite subset of  $M$  has an accumulation point in  $M$ ”. “Bounded” in this case means that the distances are bounded, using the metric induced by the mapping of  $M$  into  $R^3$ .  $M$  is *locally convex* under  $f$  if every point  $p$  of  $M$  has a neighbourhood  $N$  s.t.  $f(N)$  lies on the boundary of a convex body  $K$ . *Absolute convexity* means that, in addition, there is a support plane of  $K$  at  $f(p)$  that contains no other point of  $K$ .

*Proof (of Theorem 1).* The forward direction is clear. For the other direction, assume conditions (1) and (2) hold. We need to prove Van Heijenoort’s conditions. The embedded 3-connected planar graph drawn on the surface of a sphere provides a manifold. We begin by assigning vertex coordinates. Arbitrarily choose coordinates for one vertex  $v$  and directions for two consecutive edges incident with that vertex, forming the correct facial angle for face  $f$  between them. The plane of face  $f$  is now determined. So are the coordinates of the vertices around face  $f$ . From these, and the dihedral angles, we get the planes of the faces adjacent to face  $f$ . Continuing in this way, we obtain coordinates for all the vertices as we expand outward from the initial choices. We claim that these coordinates are well-defined—i.e. that they are independent of the order in which we expand outward. Two paths to a vertex provide a cycle, so it suffices to show that every cycle closes up. Conditions (1) and (2) give this for facial cycles in the graph and its dual, and any other cycle is a sum of facial cycles, which gives the result.

This gives us a mapping of the vertices to points, and the edges to line segments in  $R^3$ . By condition (1) every face of the graph is mapped to a simple planar convex polygon in  $R^3$ . We thus have a piece-wise linear mapping of a manifold into  $R^3$ , and conditions (i) and (iv) of Van Heijenoort’s theorem follow.

We turn to conditions (ii) and (iii). Our Condition (2) ensures local convexity at every vertex. Local convexity at an interior point of an edge follows from the fact that no dihedral angle is larger than  $\pi$ . Local convexity at an interior point of a face is obvious. Thus condition (ii) holds. Finally, only an unbounded object can be locally convex at every point but not absolutely convex anywhere, giving condition (iii). Thus by Van Heijenoort’s Theorem we have the boundary of a piece-wise linear 3-dimensional convex set—i.e. a convex polyhedron.

### 3.1 Background: Local Conditions for Convexity

Although we found Van Heijenoort’s conditions most useful, there is more recent, more algorithmic work on conditions for a polyhedron to be convex. In this section we briefly describe such work by Mehlhorn et al. [17], Devillers et al. [10], and Rybnikov [20]. The conditions of Mehlhorn et al. involve checking if a ray from a point that lies on the “inside” of the plane through every face intersects only one face. The conditions of Devillers et al. are that all dihedral angles be convex and that the projection of the *seam* to the  $x$ - $y$  plane be a convex polygon. The *seam* consists, roughly speaking, of the edges that are extreme with respect to some plane perpendicular to the  $x$ - $y$  plane.

The idea of specializing Van Heijenoort’s conditions to piece-wise linear manifolds is due to Rybnikov. In 3-dimensions it is clear that it suffices to check local convexity at vertices. Rybnikov’s result [20], which he proves using Van Heijenoort’s higher dimensional extension [13], is that to check convexity of piece-wise linear hypersurfaces in  $n$  dimensions, it suffices to check local convexity at the  $(n - 3)$ -dimensional faces. Rybnikov gives a convexity-testing algorithm; the main step is to transform the local convexity test at an  $(n - 3)$ -dimensional face to a convexity test for a [possibly self-intersecting] polygon, for which he gives a straight-forward algorithm (Devillers et al. [10] also give an algorithm for this.)

## 4 Determining Edge Lengths

In this section we consider the following problem: given the combinatorial structure of a convex polyhedron and given the facial angles, find edge lengths for the polyhedron. The edge lengths are not unique, even discounting scaling. For example, a cube can be stretched along any of its three axes. Non-uniqueness is discussed in section 4.4. It turns out to be equivalent to “indecomposability”, a notion introduced by Gale [11], and studied by Shephard [25], Meyer [18], and McMullen [16] among others.

We will make use of the conditions for the existence of a convex polyhedron from the previous section, which were expressed in terms of facial angles, dihedral angles, and edge lengths. Recall that the only condition involving edge lengths was Condition (1b); we will express that condition in terms of linear inequalities.

In section 4.2 we consider the version of the problem where the dihedral angles are known, and we apply duality to give a characterization of when a polyhedron exists with given facial and dihedral angles.

### 4.1 An LP Formulation

Let  $V$ ,  $E$  and  $F$  be the vertices, edges and faces of the graph, respectively. For each face  $f$ , choose one edge  $e_0$  and choose a unit-length direction vector  $d_f(e_0) \in \mathbb{R}^2$  for it. Based on this choice, the facial angles determine unit direction vectors  $d_f(e)$  for all the edges  $e$  in clockwise order around the face  $f$ . Note that an edge is in two faces, and may be assigned totally different edge direction vectors in those two faces. The question of whether there exist edge lengths satisfying condition (1b) is equivalent to feasibility of the following linear system in variables  $\lambda(e)$ ,  $e \in E$ .

$$\begin{aligned} \forall e \in E \quad \lambda(e) &> 0 \\ \forall f \in F \quad \sum_{e \in f} \lambda(e) d_f(e) &= (0, 0) \end{aligned} \tag{3}$$

**Theorem 3.** *Suppose a convex polyhedron exists with given facial angles and combinatorial structure. Then its edge lengths satisfy (3) and any solution to (3) gives edge lengths of such a polyhedron.*

The problem of finding edge lengths is thus solvable via linear programming algorithms [22]. Note that we need an algebraic model of computing to go from facial angles to  $d(e)$ . Linear programming, however, is only solvable in polynomial time in the bit complexity model, so we cannot claim a polynomial time algorithm to find edge lengths. Still, the simplex method should be practical. Note also that solving the linear system says nothing about whether the input facial angles and combinatorial structure are those of a convex polyhedron.



## 4.2 With Dihedral Angles

The above method computes direction vectors for edges within the plane of each face. If we have dihedral angles, we can compute true 3-D direction vectors for edges. We make an initial choice of coordinates for one vertex, and direction vectors for two edges consecutively incident at the vertex, ensuring that the angle between the two vectors matches the required facial angle. Based on these initial choices, we can compute direction vectors for all edges in 3-D. For edge  $e = (u, v) \in E$ , let  $d(e) \in R^3$  be the direction vector of the edge from  $u$  to  $v$ . Note that we (arbitrarily) choose an order  $(u, v)$  or  $(v, u)$  to do this. For face  $f \in F$ , distinguish  $cw(f)$ , the edges of face  $f$  whose vector  $d(e)$  is directed clockwise around  $f$ , and  $ccw(f)$ , the edges of face  $f$  whose vector is directed counter-clockwise around  $f$ . The linear system becomes:

$$\begin{aligned} \forall e \in E \quad \lambda(e) > 0 \\ \forall f \in F \quad \sum_{e \in cw(f)} \lambda(e)d(e) - \sum_{e \in ccw(f)} \lambda(e)d(e) = (0, 0, 0) \end{aligned} \quad (4)$$

**Theorem 4.** *There exists a convex polyhedron with given face and dihedral angles and given combinatorial structure iff conditions (1a) and (2) hold, and the linear system (4) is feasible.*

Our purpose in this section is to give duality conditions for feasibility of (4), but we mention first that it is possible to test conditions (1a) and (2) in polynomial time—see the work referenced in section 3.1.

Duality theory gives a characterization of when the linear system (4) is feasible. The linear system has the form  $Ax = b, x > 0$ . By Stiemke's Transposition Theorem (see Schrijver [22, p. 95]), there is a solution  $x$  iff for any  $y, yA \geq 0$  implies  $yb = 0$ . Translating into our situation, we have a dual variable  $\nu(f) \in R^3$  for each face  $f \in F$ . For edge  $e = (u, v)$  let  $f_r(e)$  be the face to the right of  $e$  and let  $f_l(e)$  be the face to the left of  $e$ . The dual linear system is:

$$\forall e \in E \quad d(e) \cdot (\nu(f_r(e)) - \nu(f_l(e))) \geq 0 \quad (5)$$

A change of variables gives more intuitive conditions. For each edge  $e$  let  $\nu(e) = \nu(f_r(e)) - \nu(f_l(e))$ . Formula (5) becomes  $d(e) \cdot \nu(e) \geq 0$ . We can recover the  $\nu(f)$  vectors from the  $\nu(e)$  vectors so long as the sum of the  $\nu(e)$ 's is 0 around any dual cycle. Let  $\bar{F}$  be the faces of the dual graph. We obtain:

**Theorem 5.** *Given an embedded 3-connected planar graph with specified facial and dihedral angles s.t. conditions (1a) and (2) hold, either there exists a corresponding convex polyhedron OR there are vectors  $\nu(e) \in R^3, e \in E$  s.t.*

$$\forall e \in E \quad d(e) \cdot \nu(e) \geq 0 \quad (6)$$

$$\forall \bar{f} \in \bar{F} \quad \sum_{e \in \bar{f}} \nu(e) = 0 \quad (7)$$

with strict inequality in (6) for at least one edge  $e$ . Furthermore, *NOT BOTH* the polyhedron and the vectors can exist.

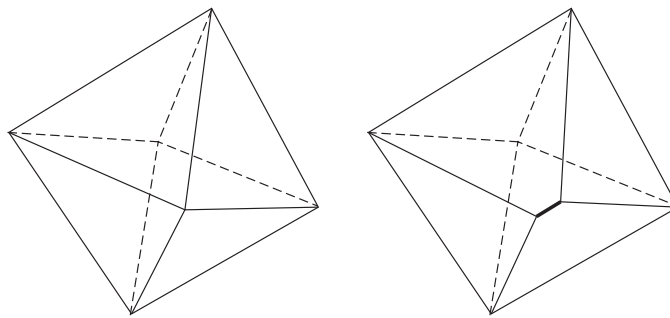
*Proof.* Straightforward: If there is no convex polyhedron then there are vectors  $\nu(f) \in R^3, f \in F$  s.t. (5) holds and with strict inequality for at least one edge  $e$ . Performing a change of variables as described above, gives vectors  $\nu(e) \in R^3, e \in E$  s.t. (6) and (7) hold, and with strict inequality in (6) for at least one edge.

Conversely, if vectors  $\nu(e) \in R^3, e \in E$  exist s.t. (6) and (7) hold, and with strict inequality in (6) for at least one edge, then define vectors  $\nu(f) \in R^3$  for each  $f \in F$  as follows. Begin by choosing one  $f_0 \in F$  and setting  $\nu(f_0) = (0, 0, 0)$ . Then use the formula  $\nu(e) = \nu(f_r(e)) - \nu(f_l(e))$  to define  $\nu : F \rightarrow R^3$ . Note that  $\nu$  is well-defined by (7). From (6) we obtain (5), so there is no convex polyhedron satisfying the requirements.

### 4.3 Example

Recall that in Section 3 we gave conditions (1a), (1b) and (2) for the existence of a convex polyhedron with specified combinatorial structure, edge lengths, and facial and dihedral angles. That section contained an example to show that the “convexity” condition (1a) was necessary. In this section we show that condition (1b) is necessary by giving an example where Conditions (1a) and (2) hold but the linear system (3) is not feasible.

The construction starts with an octahedron, which has facial angles of  $\frac{\pi}{3} = 60^\circ$  and dihedral angles of  $\cos^{-1}(-\frac{1}{3}) \approx 109.47^\circ$ . Split one vertex and add a new edge  $e$  as shown in Figure 4. The four new facial angles are  $\frac{2}{3}\pi = 120^\circ$ . All other facial and dihedral angles stay the same. Consistent edge direction vectors exist, and all convexity conditions are satisfied. The linear system (3) is not feasible: in order for edge  $e$  to have positive length while maintaining the specified angles, the square visible in Figure 4 as the silhouette of the octahedron must become a rectangle—but this destroys the bottom half of the octahedron.



**Fig. 4.** An octahedron (left) and the addition of one new edge (right) making an example where angle convexity conditions hold, but no feasible edge lengths exist.

#### 4.4 Relation to Decomposability of Polyhedra

The current Section 4 has been about the existence of a convex polyhedron with specified combinatorial structure and facial and dihedral angles. There is a considerable body of work on the related uniqueness question: given a convex polyhedron, can we preserve all facial and dihedral angles but alter edge lengths (other than by scaling). In this subsection we briefly summarize this work.

For polytopes  $P$  and  $Q$ , Gale [11] defined  $Q \leq P$  if for every direction  $u$ , the extreme set of  $Q$  in direction  $u$  has dimension less than or equal to the dimension of the extreme set of  $P$  in direction  $u$ . In particular, this means that any face of  $Q$  has a corresponding face of  $P$  with the same normal; however, the combinatorial structure may be different in that a face of  $P$  may have shrunk to an edge or vertex of  $Q$ , and an edge of  $P$  may have shrunk to a vertex. Thus this concept seems at first glance to be more general than the uniqueness of edge-lengths question mentioned above. But in fact the notions are equivalent.

Gale [11] defined a convex polyhedron  $P$  to be *decomposable* if  $P$  can be expressed as a Minkowski sum,  $P = R + S$  where neither  $R$  nor  $S$  is homothetic to (i.e. a scaled translated version of)  $P$ . Shephard [25] proved that a convex polyhedron  $P$  is decomposable iff there is a convex polyhedron  $Q \leq P$  that is not homothetic to  $P$ . In fact he proved a stronger thing, that such a  $Q$  can be used in a decomposition of  $P$ . We will use Shephard's result to relate uniqueness of edge lengths to the relation  $\leq$ .

**Lemma 2.** *For convex polyhedron  $P$ , the following are equivalent:*

- (i)  $P$  is decomposable
- (ii) there is a convex polyhedron  $Q \leq P$  that is not homothetic to  $P$
- (iii) there is a convex polyhedron  $R$  with the same combinatorial structure as  $P$  and the same facial and dihedral angles, but with different non-zero edge lengths (not just re-scaled)

*Proof.* Equivalence of (i) and (ii) is Shephard's result. Clearly, (iii) implies (ii). Suppose (ii). By Shephard's result  $P = Q + S$  where neither  $Q$  nor  $S$  is homothetic to  $P$ . Then  $Q + \frac{1}{2}S$  satisfies (iii).

Meyer [18] followed up on Shephard's work, giving a characterization of decomposable polyhedra, and McMullen [16] later reproved Meyer's results—they consider the space of polyhedra  $Q \leq P$ , parameterizing in terms of the face offsets for the specified face normals and prove that this space is a cone, whose extreme rays correspond to the indecomposable polyhedra  $\leq P$ .

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