Optimal Partition Trees

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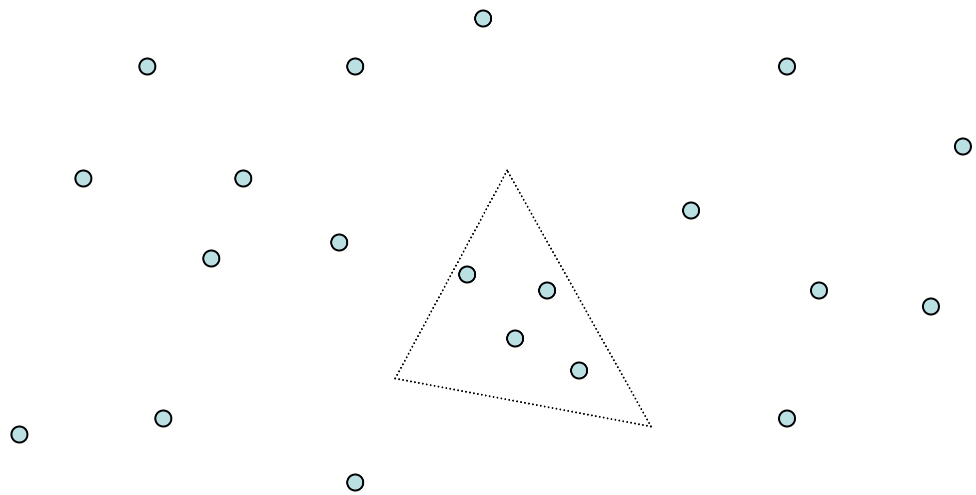
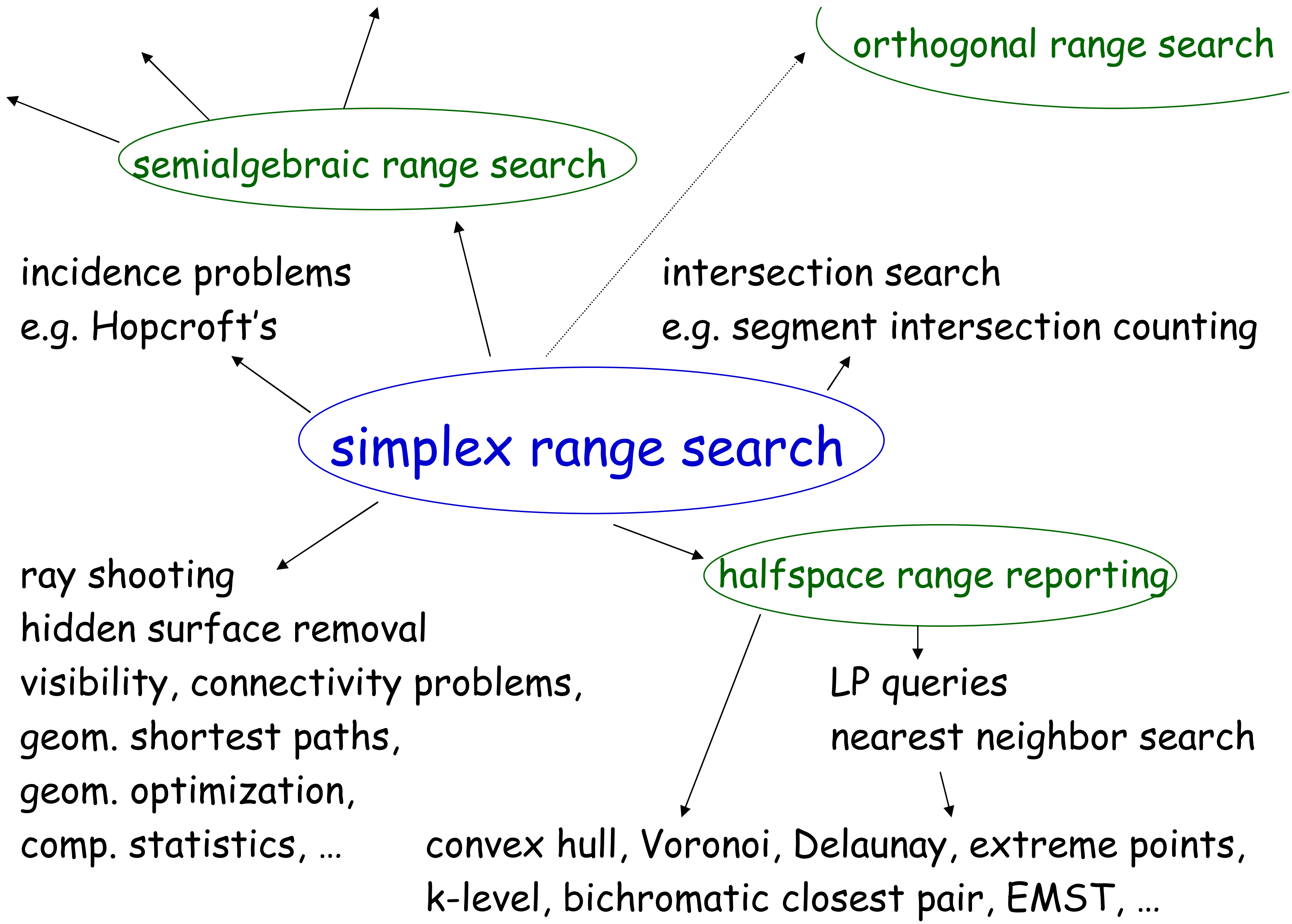
Range Searching



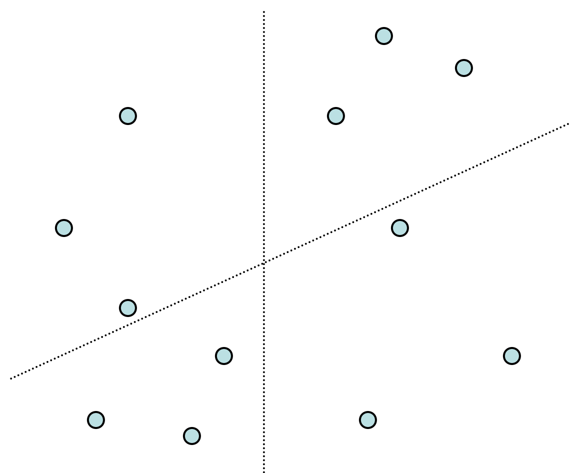
at the “core” of computational geometry…


orthogonal range search
semialgebraic range search
incidence problems
e.g. Hopcroft's
intersection search
e.g. segment intersection counting
simplex range search
ray shooting
hidden surface removal
visibility, connectivity problems,
geom. shortest paths,
geom. optimization,
comp. statistics, …
halfspace range reporting
LP queries
nearest neighbor search
convex hull, Voronoi, Delaunay, extreme points,
k-level, bichromatic closest pair, EMST, …

History of Simplex Range Search

(O(n)-space data structures)

- 2-d: Willard'82 $O(n^{\log_4 3}) \approx O(n^{0.792})$



4-partition

Willard'82 $O(n^{\log_6 4}) \approx O(n^{0.774})$

Edelsbrunner,Welzl'86 $O(n^{\log_2 \phi}) \approx O(n^{0.695})$

History (Cont'd)

- 3-d: F. Yao'83 $O(n^{\log_8 7}) \approx O(n^{0.936})$
  Dobkin,Edelsbrunner'84 $O(n^{0.916})$
  Edelsbrunner,Huber'84 $O(n^{0.909})$
  Yao,Dobkin,Edelsbrunner,Paterson'89 $O(n^{0.899})$
- Higher-d: Yao,Yao'85 $O(n^{\log_2(2^d-1)/d})$

History (Cont'd)

- Haussler, Welzl [SoCG'86]

$O(n^{1-1/[d(d-1)+1]+\varepsilon})$

e.g. 2-d: $O(n^{2/3+\varepsilon})$, 3-d: $O(n^{0.858})$

key idea: random sampling (ε-nets)

History: Turning Point

- Welzl [SoCG'88], Chazelle-Welzl'89

  2-d: $O(n^{1/2} \log n)$ query, $O(n)$ space
  3-d: $O(n^{2/3} \log^2 n)$ query, $O(n \log n)$ space

  $O(n^{1-1/d} \log n)$ query, $O(n)$ space in semigroup model
  (but not "algorithmic")

  spanning trees with low crossing number
  key idea: iterative reweighting

History: Last Stretch

- Chazelle, Sharir, Welzl [SoCG'90]

  $O(n^{1-1/d+\varepsilon})$ query, $O(n^{1+\varepsilon})$ space

  idea: multiple cuttings

- Matoušek [SoCG'91]

  $O(n^{1-1/d} \log^{O(1)} n)$ query, $O(n)$ space

  "partition thm"

  idea: iterative reweighting + cuttings

- Matoušek [SoCG'92]

  $O(n^{1-1/d})$ query, $O(n)$ space

  "final method"

  idea: iterative reweighting + Chazelle's hierarchical cuttings

Loose Ends

Matoušek's final method is great, but…

- has large preprocessing time: $O(n^{1+\varepsilon})$

  (in algorithmic applications, we usually switch back to partition-thm method, which has $O(n \log n)$ preproc.)

- is not good for multilevel data structures

  (Matoušek switched to multiple-cuttings method: cost 1 log in query, 2 log's in space, n^{ε} in preproc. per level,
  or $\log^{O(1)} n$ in query/space/preproc. per level)

- is complicated !!

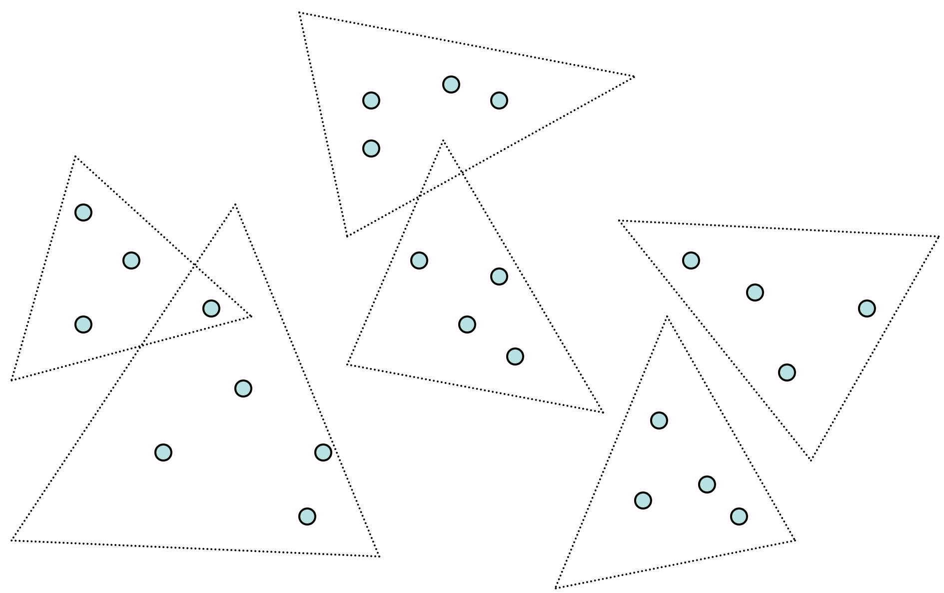
New Result

$O(n^{1-1/d})$ query, $O(n)$ space

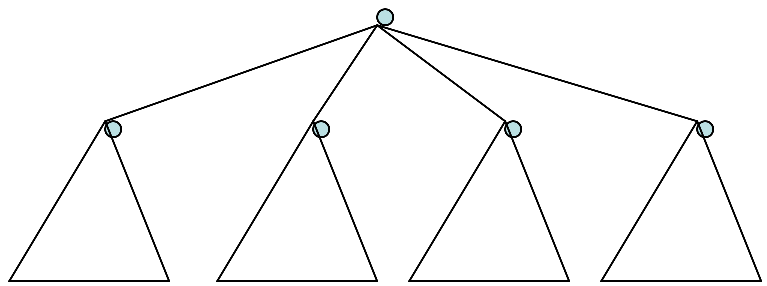
- with $O(n \log n)$ preprocessing time (rand. w.h.p.)
- good for multilevel data structures
  (cost only 1 log in query/space/preproc. per level)
- simpler !!

Recap: Matoušek's Partition Thm

Let P be n-point set in R^d, $t \le n$.
Can partition P into t subsets of $O(n/t)$ points &
enclose each subset in a (simplicial) cell s.t.
max # cells crossed by any hyperplane is $O(t^{1-1/d})$



Matoušek's Partition Tree



recurse

$$Q(n) \;=\; O(t^{1-1/d})\, Q(n/t) \;+\; O(t)$$

set t = large const $\Rightarrow$ $Q(n) = O(n^{1-1/d+\varepsilon})$

set $t = n^{\varepsilon}$ $\Rightarrow$ $Q(n) = \boxed{O(n^{1-1/d} \log^{O(1)} n)}$

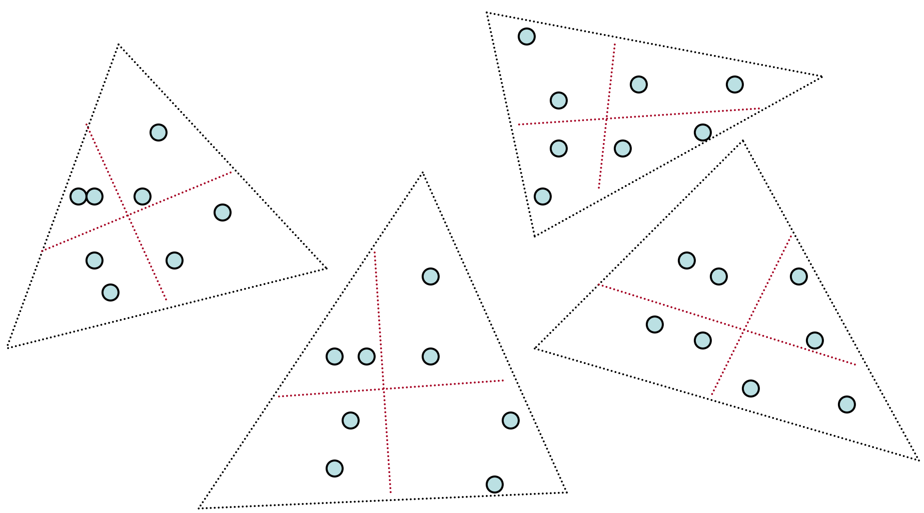
A New "Partition Refinement Thm"

Let P be n-point set in R^d, $bt \le n$.

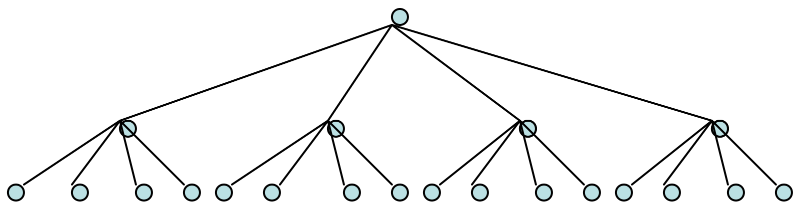
Given a partition with t disjoint cells each with $O(n/t)$ points s.t. max # cells crossed by any hyperplane is ℓ.

Can subdivide each cell into $O(b)$ disjoint subcells each with $O(n/bt)$ points s.t.

max total # subcells crossed by any hyperplane is $O((bt)^{1-1/d} + b^{1-1/(d-1)}\ell + b \log^{O(1)} n)$



The New Partition Tree



build level by level

...

$$\ell(bt) \approx O(\, (bt)^{1-1/d} + b^{1-1/(d-1)}\, \ell(t)\,)$$

$$\text{set } b = \text{large const} \Rightarrow \ell(n) = \boxed{O(n^{1-1/d})}$$

Proof Sketch of Partition Refinement Thm

Preliminaries

- Suffices to work with a finite set of $n^{O(1)}$ "test hyperplanes"
- Cutting Lemma: [Clarkson,Shor/Chazelle,Friedman]
  Given m hyerplanes in R^d & cell Δ containing X vertices,
  can divide Δ into $O(X(r/m)^d + r^{d-1})$ disjoint subcells s.t.
  each subcell is crossed by $O(m/r)$ hyperplanes

The New Algorithm

Idea: iterative reweighting

Initialize multiplicity ("weight") of each hyperplane to 1

For i = t to 1 do:

0. Among the i remaining cells, pick a good cell Δ
1. Apply cutting lemma to subdivide Δ into O(b) subcells
2. Further subdivide Δ s.t. each subcell has O(n/bt) points
3. For each hyperplane h, multiply multiplicity of h by $(1+1/b)^{\lambda(h)}$ where $\lambda(h)$ = # subcells of Δ crossed by h

Analysis

Let M = total multiplicity of all hyperplanes

0. Among the i remaining cells, pick a good cell Δ
   with $X \leq O(M^d/i)$ vertices,
   crossed by $m \leq O(M\ell/i)$ hyperplanes

1. Apply cutting lemma with $r = \min\{ m(b/X)^{1/d}, b^{1/(d-1)} \}$
   to subdivide Δ into $O(X(r/m)^d + r^{d-1}) = O(b)$ subcells

   $\Rightarrow$ # hyperplanes crossing each subcell $\leq O(m/r)$
   $\leq O((X/b)^{1/d} + m/b^{1/(d-1)})$
   $\leq M \cdot O(1/(bi)^{1/d} + \ell/(b^{1/(d-1)}i))$

Analysis (Cont'd)

2. Further subdivide Δ s.t. each subcell has $O(n/bt)$ points
3. For each hyerplane h, multiply multiplicity of h by $(1+1/b)^{\lambda(h)}$ where $\lambda(h)$ = # subcells of Δ crossed by h

$$\sum_h \lambda(h) \le O(bm/r)$$

$$\Rightarrow \text{ increase in } M \le \sum_h [(1+1/b)^{\lambda(h)} - 1]$$
$$\le O(\textstyle\sum_h \lambda(h)/b)$$
$$\le O(m/r)$$
$$\le M \cdot O(\, 1/(bi)^{1/d} + \ell\, /(b^{1/(d-1)} i)\,)$$

Analysis (Cont'd)

- Final value of M

$$\leq \quad n^{O(1)} \prod_{i=t,\dots,1} \left[1 + O\left(1/(bi)^{1/d} + \ell / (b^{1/(d-1)} i) \right)\right]$$

$$\leq \quad n^{O(1)} \exp\left(O\left(t^{1-1/d}/b^{1/d} + \ell \ln t \,/\, b^{1/(d-1)} \right) \right)$$

- Final multiplicity of h $= (1+1/b)^{\text{crossing-number}(h)}$

$$\leq \text{ final value of } M$$

$\Rightarrow$ crossing-number(h) $\leq$ b log(final value of M)

$$\leq \boxed{O\left((bt)^{1-1/d} + b^{1-1/(d-1)} \ell \cancel{\ln t} + b \log n\right)}$$

Concluding Remarks

- Simple tree structure, with const degree, disjoint cells (in 2-d, can make it a BSP tree)
- Optimal crossing number at essentially all levels of tree
- An open problem: halfspace range reporting for odd d with $O(n^{1-1/\lfloor d/2 \rfloor} + k)$ query, $O(n)$ space ?