Improved Deterministic Algorithms for Linear Programming in Low Dimensions

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The Problem: LP

maximize $c_1 x_1 + \cdots + c_d x_d$

subject to

$$a_{11} x_1 + \cdots + a_{1d} x_d \leq b_1$$

$$\vdots$$

$$a_{n1} x_1 + \cdots + a_{nd} x_d \leq b_n$$

over real variables $x_1, \ldots, x_d$

The Problem: LP

- interested in runtime of alg'ms as a function of n & d (not bit complexity)

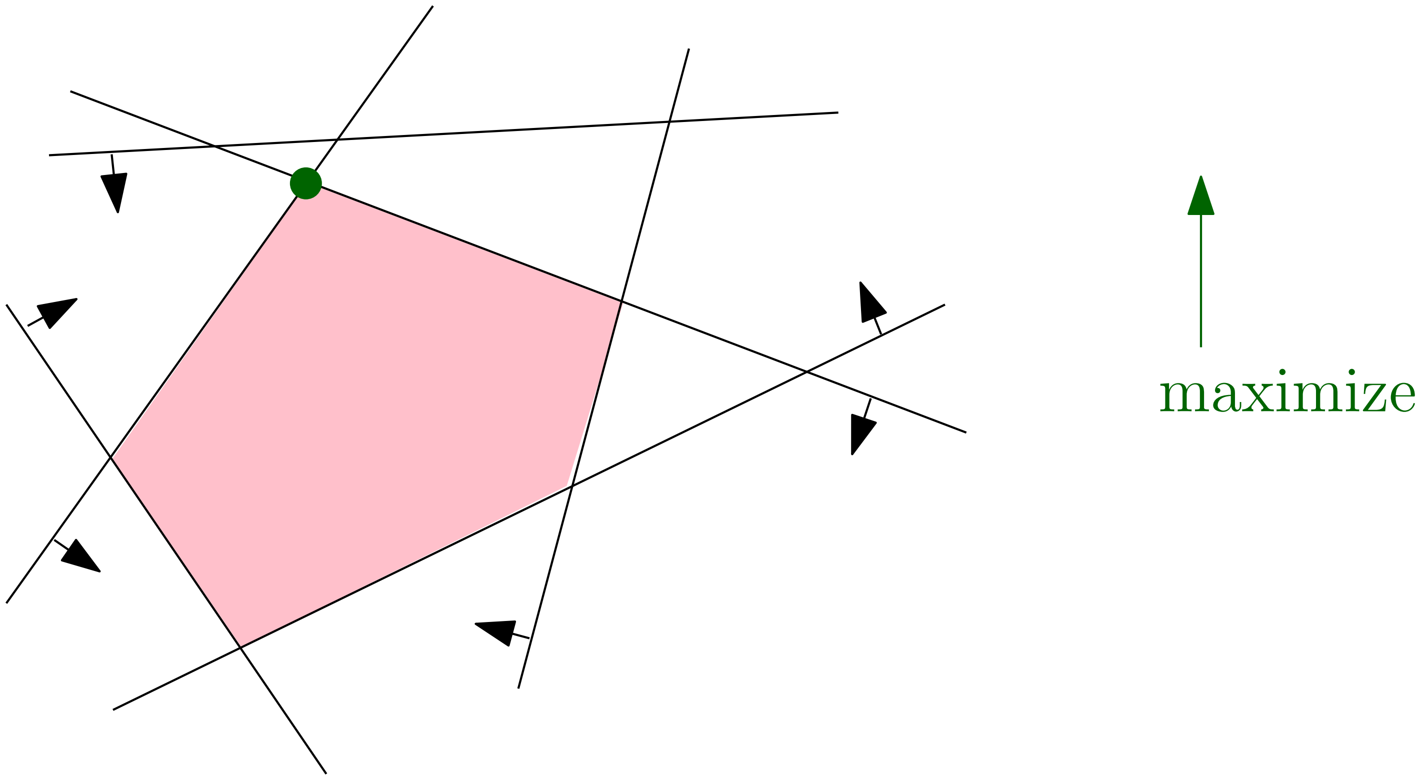
- Big Open Q: $\exists$ strongly polynomial alg'm??

- Ex: simplex method is exponential, with upper bound about $\binom{n}{\lfloor d/2 \rfloor} = O(n/d)^{d/2}$

  whereas ellipsoid method & interior-point methods (from Karmarkar'84 to … Lee–Sidford'15) aren't applicable

The Problem: LP

Our Focus: when d is small



History: Prune&Search Alg'ms

- $d = 2$ or 3: Dyer'82 / Megiddo [FOCS'82]: $\boxed{O(n)}$ time

$$T_2(n) = T_2(3n/4) + O(n)$$
$$T_3(n) = T_3(15n/16) + O(n)$$

- any constant d: Megiddo'84: $\boxed{O(n)}$ time

$$T_d(n) = T_d((1 - 1/2^{2^{d-1}})n) + O(2^d T_{d-1}(n))$$
$$\Rightarrow \boxed{2^{O(2^d)} n}$$

History: Prune&Search Alg'ms

- Clarkson'86 / Dyer'86: $\boxed{O(3^{d^2})n}$ time
- Dyer–Frieze'89: $\boxed{\widetilde{O}(d)^{3d}n}$ time (but randomized)
- Agarwal–Sharir–Toledo'93: $\boxed{\widetilde{O}(d)^{10d}n}$ (deterministic)

History: Rand. Sampling Alg'ms

- Clarkson [FOCS'88]: (recursive version)

  $$T_d(n) \approx (d+1)\cdot(T_d(d\sqrt{n}) + O(dn))$$

  (with base case $T_d(d^2) = O(d)^{d/2}$ by simplex method)

  $\Rightarrow$ $\boxed{O(d^2n) + O(d)^{d/2}}$ (rand.)

- Clarkson [later]: (iterative reweighting version)

  $$T_d(n) \approx d\log n \cdot O(dn + T_d(d^2))$$

  $\Rightarrow$ roughly the same

both simple!

History: Rand. Incremental Alg'ms

- Seidel [SoCG'90]:

  $T_d(n) \;=\; T_d(n-1) + O((d/n)\cdot T_{d-1}(n))$

  $\Rightarrow \boxed{O(d!n)}$ (rand.) very simple!

- Kalai [STOC'92] / Matoušek–Sharir–Welzl [SoCG'92]:

  $\boxed{2^{O(\sqrt{d\log n})}n}$ (rand.)

- combined with rand. sampling alg'ms

  $\Rightarrow \boxed{O(d^2n) + 2^{O(\sqrt{d\log d})}}$ (rand.) current record

(Hansen–Zwick [STOC'15]: $2^{O(\sqrt{d})}$ rand. for $n = O(d)$)

History: Back to Deterministic Alg'ms

- Chazelle–Matoušek [SODA'93]: derandomize Clarkson's recursive rand. sampling alg'm by designing an ε-net alg'm (via "ε-approximations", method of conditional probabilities, & a clever merge&reduce technique)

  $\Rightarrow$ $\boxed{\widetilde{O}(d)^{7d} n}$ (det.)

- Brönnimann–Chazelle–Matoušek [FOCS'93]: throw in "sensitive ε-approximations"

  $\Rightarrow$ $\boxed{\widetilde{O}(d)^{5d} n}$ (det.) current record. . . till now

Today: New Deterministic Alg'ms

- much simpler derandomization of Clarkson's recursive rand. sampling alg'm (without ε-approximations, method of conditional probabilities, merge&reduce, ...)
  $\Rightarrow$ $\boxed{\widetilde{O}(d)^{3d} n}$ (det.) $\longleftarrow$

- combined with a new variant of Clarkson's iterative reweighting alg'm $\Rightarrow$ $\boxed{\widetilde{O}(d)^{2d} n}$ (det.)

- throw in combinatorial bounds on $(\leq k)$-levels
  $\Rightarrow$ $\boxed{\widetilde{O}(d)^{d} n}$ (det.)

- new ε-net alg'm (by throwing back in sensitive ε-approximation, method of conditional probabilities, ... & a new merge&reduce)
  $\Rightarrow$ $\boxed{\widetilde{O}(d)^{d/2} n}$ (det.) (new current record)

Review of Clarkson's Random Sampling Alg'm (Recursive Version)

LP(H), given set H of n halfspaces in $\mathbb{R}^d$:

1. choose a subset R of H
2. repeat:
3. recursively compute $p =$ LP(R)
4. add $\{$all halfspaces of H violated by $p\}$ to R

Claim: # repeats $\leq d + 1$

Proof: let $B^* =$ the d halfspaces defining optimal sol'n p^*

each iteration adds ≥ 1 halfspace of B^* to R

(if not, p inside $\cap B^* \Rightarrow p$ worse than p^*: contradiction!) $\square$

Review of Clarkson's Random Sampling Alg'm (Recursive Version)

LP(H), given set H of n halfspaces in $\mathbb{R}^d$:

1. choose a subset R of H $\leftarrow$ how?
2. repeat $d + 1$ times:
3. recursively compute $p =$ LP(R)
4. add {all halfspaces of H violated by p} to R

ε-Nets

Def: $R \subset H$ is an ε-net iff $\forall p \in \mathbb{R}^d$,

p violates $> \varepsilon n$ of $H \quad \Rightarrow \quad p$ violates ≥ 1 of R

Fact: $\exists$ ε-net R of size $\widetilde{O}(d/\varepsilon)$

Proof:

- call $\{$all halfspaces violated by $p\}$ a "violation set"
- want R to hit all violation sets of size $> \varepsilon n$
- # diff. violation sets $= m \ = \ O(\binom{n}{d}) \ = \ O(n/d)^d$
- just take random sample of size $O((1/\varepsilon) \log m)$ $\quad \square$

Alternate Proof: run greedy hitting set alg'm $\quad \square$

Review of Clarkson's Random Sampling Alg'm (Recursive Version)

LP(H), given set H of n halfspaces in $\mathbb{R}^d$:

1. choose ε-net R of size $\widetilde{O}(d/\varepsilon)$ by sampling
2. repeat $d+1$ times:
3. recursively compute $p = \text{LP}(R)$
4. add {all halfspaces of H violated by p} to R

By Def: p violates none of $R \Rightarrow p$ violates $\leq \varepsilon n$ of H

$$\Rightarrow \quad T_d(n) \approx (d+1) \cdot (T_d(\sout{(d/\varepsilon) + d\varepsilon n}) + O(dn))$$

$$d\sqrt{n}$$

Chazelle–Matoušek's Derandomization

- gave complicated alg'm to compute ε-net R of size $\widetilde{O}(d/\varepsilon)$ in $\widetilde{O}(d^3/\varepsilon^2)^d n)$ time (det.)

- set $\varepsilon \approx 1/(Cd^2)$

  $\Rightarrow\ T_d(n) \approx (d+1)\cdot(T_d(\underbrace{\sout{(d/\varepsilon)+d\varepsilon n}}_{n/(Cd)}) + \widetilde{O}(d)^{7d} n)$

  $\Rightarrow\ \boxed{\widetilde{O}(d)^{7d} n}$ (det.)

- New Obs: can afford ε-net of much larger size...

New Simple Derandomization

LP(H), given set H of n halfspaces in $\mathbb{R}^d$:

1. divide H into groups of size b;
   compute ε-net of each group by greedy hitting set alg'm;
   $R =$ union of these ε-nets
2. repeat $d + 1$ times:
3. recursively compute $p =$ LP(R)
4. add {all halfspaces of H violated by p} to R

New Simple Derandomization

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set $\varepsilon = 1/(Cd^2)$, $b = \widetilde{\Theta}(d^4)$

size of ε-net $= s = (n/b) \cdot \widetilde{O}(d/\varepsilon) \le n/(C'd)$

time to compute ε-net $= (n/b) \cdot O(b/d)^d = \widetilde{O}(d)^{3d} n$

$\Rightarrow T_d(n) \approx (d+1) \cdot (T_d(\sout{s + d\varepsilon n}) + \widetilde{O}(d)^{3d} n)$

$n/(C''d)$

$\Rightarrow \boxed{\widetilde{O}(d)^{3d} n}$ (det.)

Conclusions

- simpler, even compared to Megiddo's det. alg'm
- throw in a few more ideas $\Rightarrow$ $\boxed{\widetilde{O}(d)^{d/2}n}$ (det.)
- one barrier: for the base case $n \approx d^2$, can we beat $O(\binom{n}{\lfloor d/2 \rfloor}) = O(d)^{d/2}$ det. time?
- generalize to many LP-type problems (with $\widetilde{O}(d)^d n$ det. time)
- $2^{O(d)}n$ det. alg'm??