Speeding up the Four Russians Algorithm by About One More Logarithmic Factor

Timothy Chan

U of Waterloo

The Problem:
Boolean Matrix Multiplication (BMM)

Given $n \times n$ Boolean matrices A, B,
compute $C = \{c_{ij}\}$ where

$$c_{ij} = \bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj})$$

Previous Alg'ms

trivial	$O(n^3)$ time
Strassen'69	$O(n^{2.81})$
⋮	
⋮	
⋮	
Coppersmith&Winograd'87	$O(n^{2.376})$
Stothers'10	$O(n^{2.373})$
Vassilevska Williams,STOC'12	$O(n^{2.3729})$
Le Gall'14	$O(n^{2.372864})$

Today

a new BMM alg'm which is. . .

SLOWER!

but is purely combinatorial. . .

Previous Combinatorial BMM Alg'ms

- Arlazarov&Dinic&Kronrod&Faradzhev'70
  (the "4 Russians")
  - $\boxed{O(n^3/\log n)}$ time
  - technique: table lookup
  - refinement: + word ops (bitwise-or)

    $\Rightarrow$ $O(n^3/(w \log n))$ on w-bit word RAM
    ($w \geq \log n$)

    $\Rightarrow$ $\boxed{O(n^3/\log^2 n)}$

Previous Combinatorial BMM Alg'ms

- Arlazarov&Dinic&Kronrod&Faradzhev'70 (the "4 Russians")
  - inspired many other combinatorial alg'ms...
    - $*$ all-pairs shortest paths (APSP) in $O^*(n^3/\log^2 n)$ time [Fredman'75; ...; C.,STOC'07; Han&Takaoka'12]
    - $*$ LCS & edit distance for bounded alphabet in $O(n^2/\log n)$ [Masek&Paterson'80]
    - $*$ max unweighted bipartite matching in $O(n^{5/2}/\log n)$ [Alt et al.'91; Feder&Motwani'91]
    - $*$ regular expression matching in $O(nP/\log n)$ [Myer'92]
    - $*$ integer 3SUM in $O^*(n^2/\log^2 n)$ [Baran&Demaine&Pătraşcu'05]

      $\vdots$

Previous Combinatorial BMM Alg'ms

- Arlazarov&Dinic&Kronrod&Faradzhev'70 (the "4 Russians")
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    - $*$ unweighted undirected APSP in $O(mn/\log n)$ for $m \gg n \log n$ [C.,SODA'06]
    - $*$ transitive closure in $O(mn/\log n)$
    - $*$ min-plus convolution in $O^*(n^2/\log^2 n)$ [Bremner et al.'06]
    - $*$ CFL reachability in $O(n^3/\log^2 n)$ [Chaudhuri'08]
    - $*$ k-cliques in $O(n^k/\log^{k-1} n)$ [Vassilevska'09]
    - $*$ diameter of real-weighted planar graphs in $O^*(n^2/\log n)$ [Wulff-Nilsen'10]

$\vdots$

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  - inspired many other combinatorial alg'ms...
    - discrete Fréchet distance decision in $O^*(n^2/\log n)$ [Agarwal et al.,SODA'13]
    - Klee's measure problem for integers in $O^*(n^{d/2}/\log^{d/2-2} n)$ [C.,FOCS'13]
    - continuous Fréchet distance decision in $O^*(n^2/\log n)$ [Buchin et al.,"4 Soviets walk the dog...",SODA'14]
    - real 3SUM in $O^*(n^2/\log n)$ [Grønlund&Pettie,FOCS'14] etc. etc. etc.
  - notation: O^* ignores $\log\log$ factors

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    - $*$ real 3SUM in $O^*(n^2/\log n)$ [Grønlund&Pettie,FOCS'14] etc. etc. etc.
  - pattern: mostly speedup of $\approx$ 1 or 2 logs, until ...

Previous Combinatorial BMM Alg'ms

- Bansal&Williams [FOCS'09]
  - $O^*(n^3/(w \log^{7/6} n))$ on w-bit word RAM
    or $\boxed{O^*(n^3/\log^{2.25} n)}$
  - need "advanced" techniques: regularity lemmas

New Combinatorial BMM Alg'm

- Our result
  - $O^*(n^3/(w \log^2 n))$ on w-bit word RAM

    $\Rightarrow \boxed{O^*(n^3/\log^3 n)}$
  - completely "elementary" techniques:
    table lookup + word ops (as in the 4 Russians)
    + an embarrassingly simple divide&conquer!

Setup

- Suffice to solve rectangular matrix multiplication:

  multiply a Boolean $n \times d$ matrix A & a Boolean $d \times n$ matrix B

  in $O(n^2)$ time, for $d \approx^* \log^3 n$

- Re-formulation (with $m = n$):

  given m red & n blue Boolean vectors in $\{0, 1\}^d$, report all pairs of red & blue vectors with Boolean inner product $= 1$

Preliminaries: Sparse Version of the 4 Russians Alg'm

- Lemma: If fraction of 1's in the blue vectors is $\leq \beta$, then problem can be solved in time

$$O^*\left(\beta \frac{dmn}{\log^2 n} + \text{lower-order terms}\right)$$

- Proof: let $b = 0.01 \log_d n$
  - precompute inner product of the m red vectors with every possible vector v w. $\leq b$ 1's
    $\Rightarrow\ O(dmd^b) = O(dmn^{0.01})$ time
  - store answer for each v in $O(m/w)$ words
  - break the n blue vectors into $\approx \beta dn/b$ chunks each w. $\leq b$ 1's & do table lookup for each chunk
    $\Rightarrow\ O(\beta dn/b \cdot m/w)$ time Q.E.D.

The Main Divide&Conquer Alg'm

0. if fraction of 1's in the blue vectors is $\leq \beta$ then apply Lemma & return $\Rightarrow\ O^*\left(\beta\frac{dmn}{\log^2 n} + \cdots\right)$
1. find coord. position w. $\geq \beta n$ 1's in the n blue vectors; w.l.o.g., say it's 1st coord. $\Rightarrow\ O(dm + dn)$
2. recurse on all red vectors w. 1st coord. 0 & all blue vectors $\Rightarrow\ T_{d-1}((1-\alpha)m, n)$
3. recurse on all red vectors w. 1st coord. 1 & all blue vectors w. 1st coord. 0 $\Rightarrow\ T_{d-1}(\alpha m, (1-\beta)n)$
4. report all pairs betw'n all red vectors w. 1st coord. 1 & all blue vectors w. 1st coord. 1

THAT'S IT!

The Recurrence

$$\boxed{T_d(m,n) \le \begin{cases} T_{d-1}((1-\alpha)m, n) + T_{d-1}(\alpha m, (1-\beta)n) \\ \;+ O(dm + dn) \;\; \text{for some } \alpha\text{, or} \\ O^*\left(\beta \frac{dmn}{\log^2 n} + \cdots\right) \end{cases}}$$

for any choice of β

How to solve it: guess!

Solving the Recurrence (Cont'd)

- try induction hypothesis like

$$T_d(m,n) \le (1+\delta)^d dnm^{1-\varepsilon} + \beta \frac{dmn}{\log^2 n} + \cdots$$

- do the boring math... (read the paper!)
- choosing $\delta \approx 1/d$, $\varepsilon \approx^* 1/\log m$, $\beta \approx^* 1/\log m$ will work...

$$\Rightarrow T_d(n,n) \le O(n^2) + O^*\left(\frac{dn^2}{\log^3 n}\right) = O(n^2)$$

for $d \approx^* \log^3 n$ Q.E.D.

Final Remarks

1. How to define combinatorial alg'ms??

2. Practical or not?

3. Our technique is currently limited to BMM only. . .

Final Remarks

4. Our divide&conquer alg'm is inspired by a result of Impagliazzo&Lovett&Paturi&Schneider'14 on the dominance problem:

   given n red & n blue points in R^d,
   report all K pairs of red & blue points (p, q) s.t.
   $p_i > q_i \ \forall i = 1, \ldots, d$

   in near $n^{2-\Omega(1/(c^{15} \log c))} + K$ time for $d = c \log n$

   – our better analysis: $n^{2-\Omega(1/(c \log^2 c))} + K$

   – also implies a simple combinatorial alg'm for APSP in $O^*(n^3/\log^2 n)$ time (but worse than Williams' non-comb. $n^3/2^{\Omega(\sqrt{\log n})}$ alg'm [STOC'14])