On the Bichromatic k-Set Problem

Timothy Chan
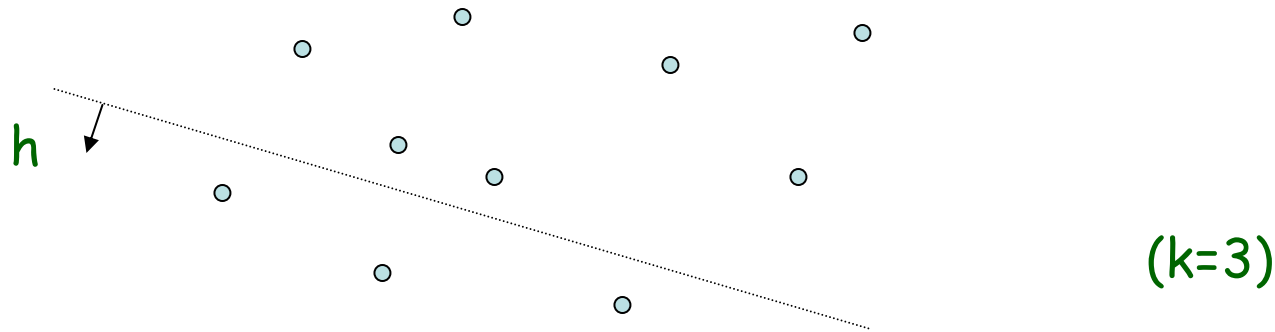
School of CS
U of Waterloo

A STANDARD PROBLEM IN COMBINATORIAL GEOMETRY...

The k-Set Problem (in 2D)

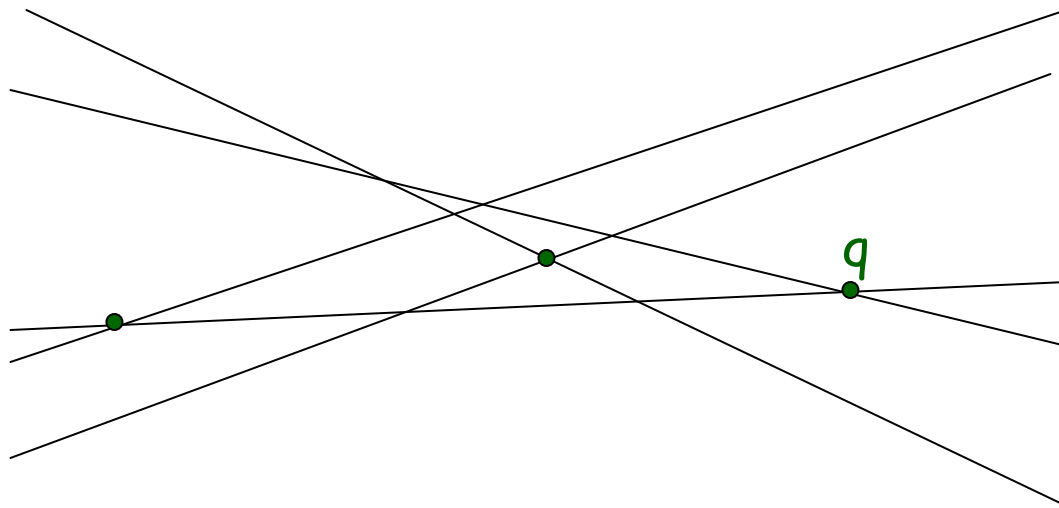- Given n pts P in R^2, a k-set is a subset P ∩ h of size k for some halfplane h

h

(k=3)

- How many k-sets possible as fn of n & k?

(Note: k=n/2 case called halving line problem)

Equiv. Dual Version: k-Level of Lines

- Given n lines in R^2,

  the level of a pt q := # of lines below q

(k=1)

- How many vertices of level k?

Previous Work: Upper Bds

- Lovász'71 $O(n^{3/2})$

- Erdős,Lovász,Simmons,Straus'73 $O(nk^{1/2})$

  (also by Gusfield'79, Edelsbrunner,Welzl'85)

- Pach,Steiger,Szemerédi [FOCS'89] $O(nk^{1/2}/\log^* n)$

- Dey [FOCS'97] $\boxed{O(nk^{1/3})}$

- C. [FOCS'03] $O(nk^{1/2})$

  (a different proof)

Previous Work: Lower Bds

- Erdős,Lovász,Simmons,Straus'73 $\Omega(n \log k)$
- Klawe,Paterson,Pippenger'82 $n2^{\Omega(\sqrt{\log k})}$

  (for pseudolines)

- Tóth [SoCG'00] $n2^{\Omega(\sqrt{\log k})}$

- Conjecture by Erdős et al.'73:

      $o(n^{1+\varepsilon})$??

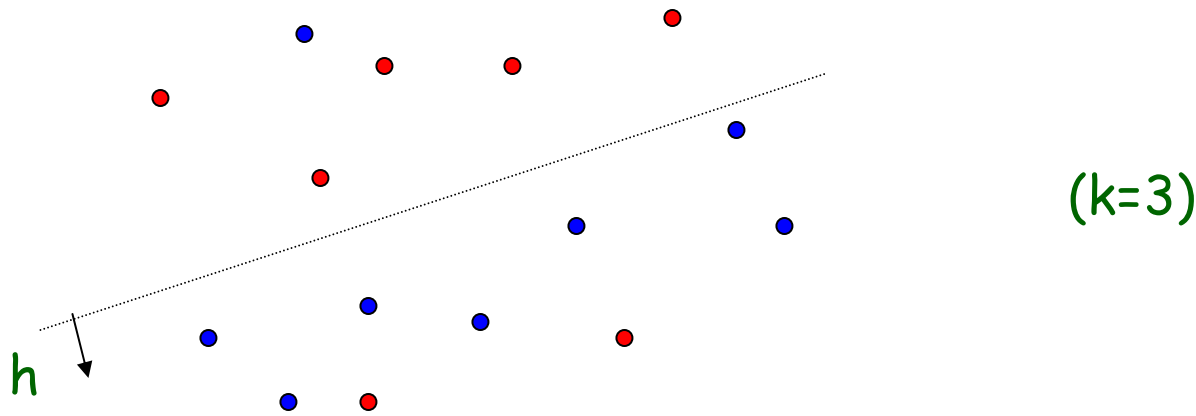# New Result (for the Standard 2D Problem)

New Result (for the Standard 2D Problem)

- Nothing (sorry!)

A "NEW" VERSION OF THE PROBLEM...
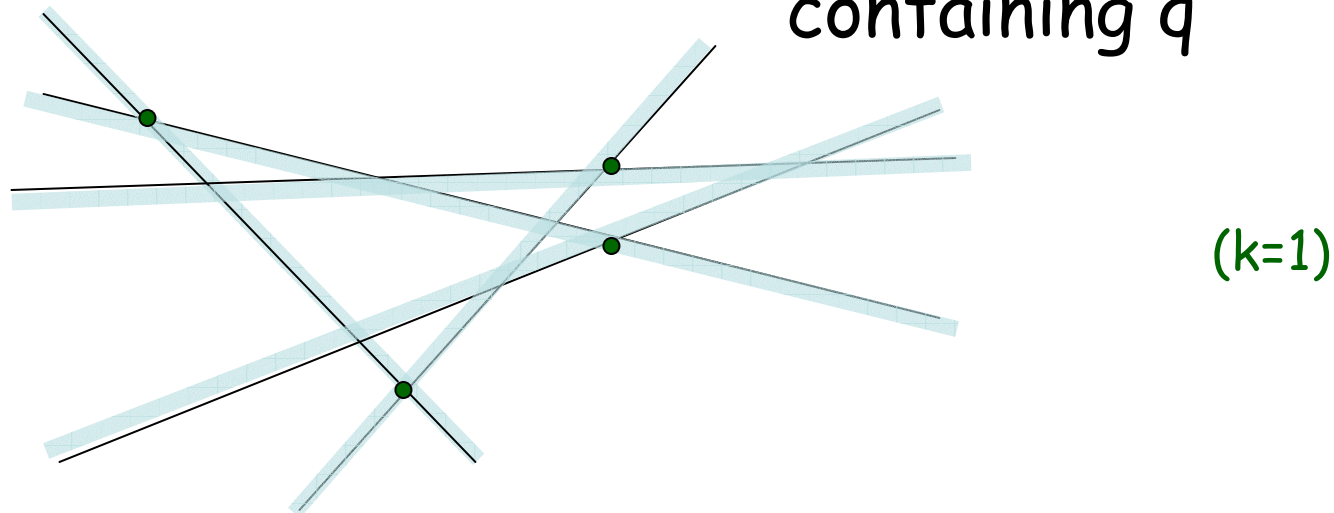
The Bichromatic k-Set Problem (in 2D)

- Given n red pts R & blue pts B in R^2,

  a k-set is a subset $(R \cap h) \cup (B-h)$ of size k for some halfplane h

(k=3)

h

- How many k-sets possible as fn of n & k?

Equiv. Version: k-Level of Halfplanes

- Given n halfplanes in R^2,
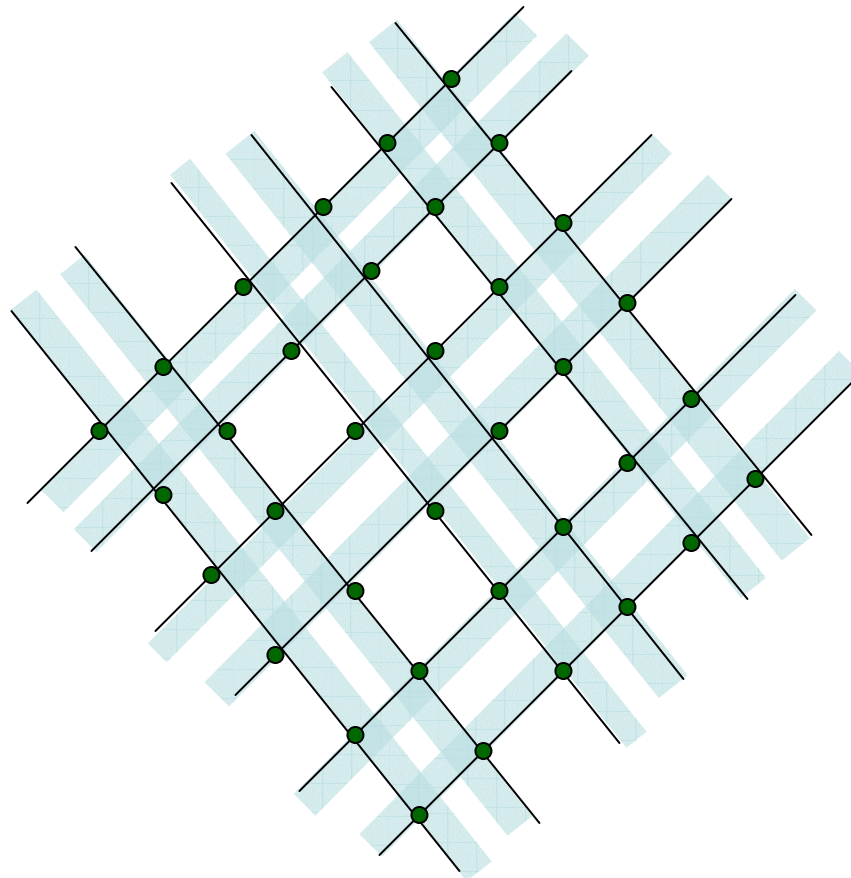 the level of a pt q := # of halfplanes not
  containing q

(k=1)

- How many vertices of level k?

(Appl'ns: optimization with k violated constraints...)

Unfortunately…

- subquadratic in n is not possible!

(k ~ n/2)

One "Easy" Upper Bd

- O(nk)   [e.g. Clarkson,Shor]
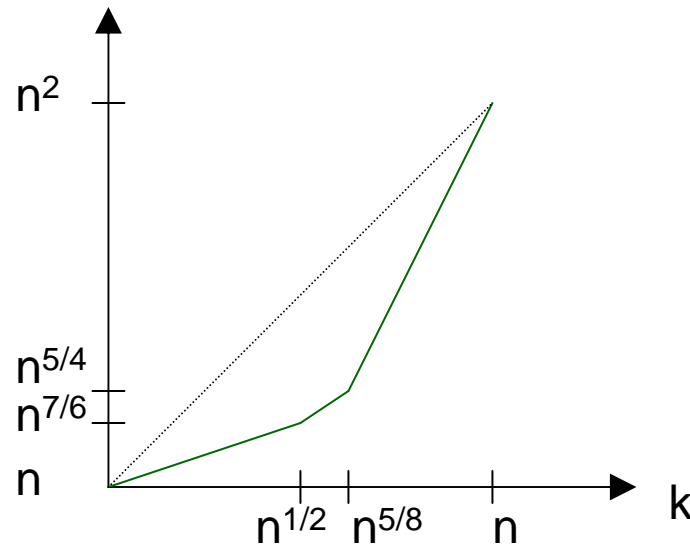
One Previous Work

- Linhart'93: improve consts
- Conjecture by Linhart:
  O(nk) is tight

New Result

New Result

- Linhart is wrong (sorry!)

- $O^*(nk^{1/3} + n^{5/6}k^{2/3} + k^2)$

(Note: * hides n^ε factors)

Rest of Talk

- Rough sketch of the approach
  - Extend Dey's $O(nk^{1/3})$ proof
  - Extend C.'s $O(nk^{1/2})$ proof
  - Combine…

- Results beyond 2D

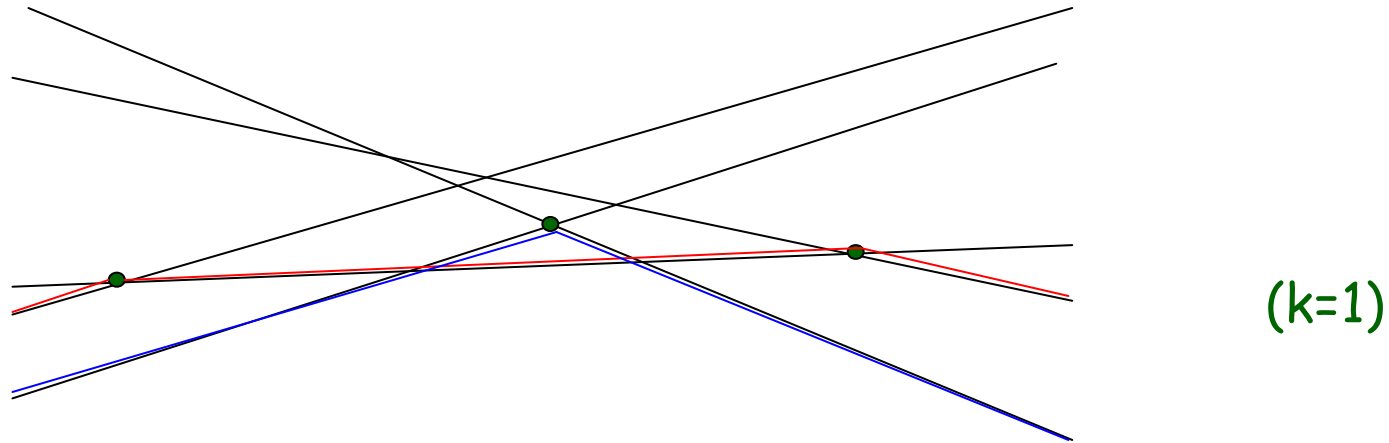# Dey's Proof (for the Standard Problem) [FOCS'97]

- **Crossing Lemma:** A geometric graph with n vertices & X crossings has $O(n + n^{2/3}X^{1/3})$ edges

- **Equivalently:** A set of vertices in an arrangement of n lines with X tangent pairs has cardinality $O(n + n^{2/3}X^{1/3})$
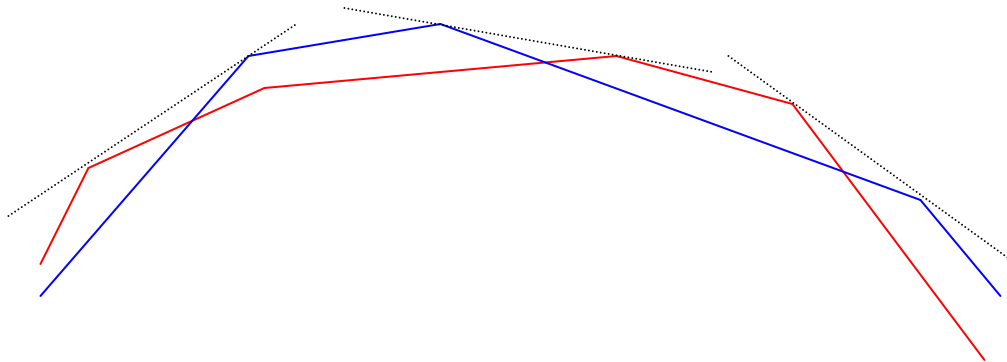

duality

crossing

tangent pair
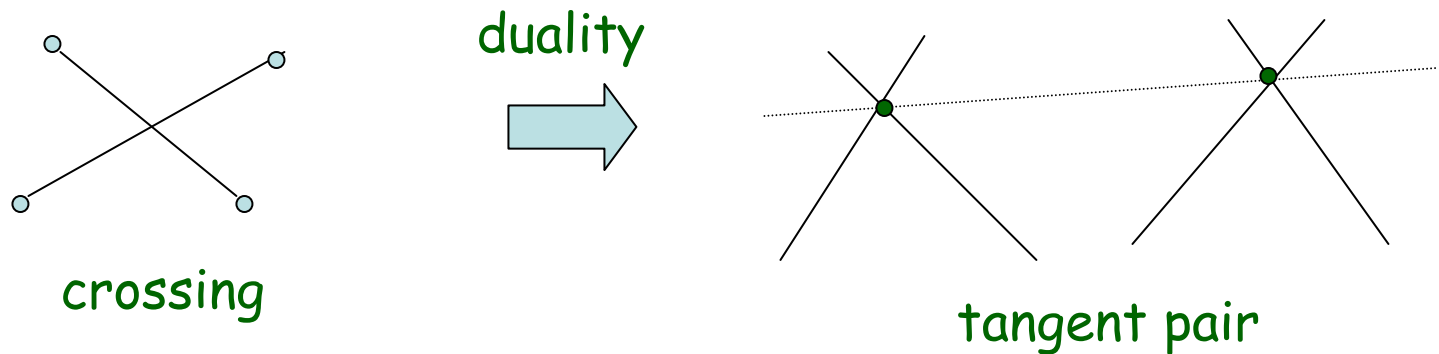
- **To show:** $X = O(nk)$

Dey's Pf (Cont'd)

- Form k+1 concave chains to visit all k-level vertices

(k=1)

- # tangents betwn 2 chains ≤ # intersect'ns + O(1)

$\Rightarrow$ X ≤ total # intersect'ns + $O(k^2)$ = $O(nk)$

New Proof 1

- **Idea:** modify Dey's pf

- This time, # concave/convex chains $= O(k + M)$
  where $M = $ # k-level local x-minima/maxima
  $\Rightarrow$ Total # intersect'ns $= O(nk + M^2)$
  $\Rightarrow X = O(nk + M^2)$
  $\Rightarrow O(n + n^{2/3}X^{1/3}) = O(nk^{1/3} + n^{2/3}M^{2/3})$

- **Known:** $M = O(k^2)$ [Clarkson,Shor/Mulmuley'93]
  $\Rightarrow$ $\boxed{O(nk^{1/3} + n^{2/3}k^{4/3})}$

Refined Proof 1

- **Idea:** divide&conquer *[Agarwal,Aronov,C.,Sharir'98]*

- Divide into $O(n/k)$ slabs each with $O(k)$ relevant lines

$$\Rightarrow O\left(\sum_i (n_i k^{1/3} + n_i^{2/3} M_i^{1/3}) \right)$$
$$= O(n/k \cdot [k^{4/3} + k^{2/3} (k^2/(n/k))^{1/3}])$$
$$= \boxed{O(nk^{1/3} + n^{1/3}k^{5/3})}$$

C.'s Proof (for the Standard Problem) [FOCS'03]

- Let t_i = # vertices at level $k-i, ..., k+i$

- To show: $t_i \lesssim 2i (t_{i+1} - t_i)$

$\Rightarrow t_0 = O(t_k / k^{1/2}) = O(nk^{1/2})$

Proof 2

- Idea: modify C.'s pf

- $O^*(nk^{1/2} + M) = \boxed{O^*(nk^{1/2} + k^2)}$

Combining Proof 1 & 2

- $\boxed{O^*(nk^{1/3} + n^{5/6}k^{2/3} + k^2)}$

BEYOND 2D...

The Standard Problem in 3D: Previous Work

- Bárány,Füredi,Lovász [SoCG'89] $O(n^{2.998})$
- Aronov,Chazelle,Edelsbrunner,
  Guibas,Sharir,Wenger [SoCG'90] $O(n^{8/3}\log^{5/3}n)$
- Eppstein'93 $O(n^{8/3}\log^{2/3}n)$
- Dey,Edelsbrunner [SoCG'93] $O(n^{8/3})$
- Agarwal,Aronov,C.,Sharir'98 $O(nk^{5/3})$
- Sharir,Smorodinsky,Tardos [SoCG'00] $\boxed{O(nk^{3/2})}$

- C. [SODA'05] $O(nk^{1.997})$

- Tóth [SoCG'00] $nk2^{\Omega(\sqrt{\log k})}$

The Bichromatic Problem in 3D: New Result

- $O(nk^{3/2} + n^{0.5034}k^{2.4932} + k^3)$

- Proof approach
  - Extend Sharir,Smorodinsky,Tardos' $O(nk^{3/2})$ proof
  - Extend C.'s $O(nk^{1.997})$ proof
  - Combine...

The Standard Problem in 4D: Previous Work

- Bárány,Füredi,Lovász [SoCG'89]

  + Živaljević,Vrecića'92 $\qquad O(n^{3.99997})$

- Agarwal,Aronov,C.,Sharir'98 $\qquad O(n^2 k^{1.99997})$

- Matoušek,Sharir,Smorodinsky,

  Wagner'06 $\qquad \boxed{O(n^2 k^{88/45})}$

The Bichromatic Problem in 4D: New Result

- $O(n^2 k^{3/2} + n^{1.5034} k^{2.4932} + nk^3)$

The Bichromatic Problem in 4D: New Result

- $O(n^2 k^{3/2} + n^{1.5034} k^{2.4932} + n k^3)$

Note: new even for the standard k-set problem in 4D !!
(beats $O(n^2 k^{88/45})$ for $k \ll n^{0.92}$)

Final Remarks

- New results also for k-level of curved objects, e.g., balls in 2D/3D…

- Open:
  - Improvements in 2D & 3D ??
  - 5D, 6D, … ??
  - More on k-sets for small k ??