On Levels in Arrangements of Surfaces in Three Dimensions*

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Abstract

A favorite open problem in combinatorial geometry is to determine the worst-case complexity of a *level* in an arrangement. Up to now, nontrivial upper bounds in three dimensions are known only for the linear cases of planes and triangles. We propose the first technique that can deal with more general surfaces in three dimensions. For example, in an arrangement of n "pseudo-planes" or "pseudo-spherical patches" (where the main criterion is that each triple of surfaces has at most two common intersections), we prove that there are at most $O(n^{2.997})$ vertices at any given level.

1 Introduction

Given an arrangement of n surfaces in $\mathbb{R}^d$, the *level* of a point $p \in \mathbb{R}^d$ is the number of surfaces strictly below p. Combinatorial and computational geometers have been baffled by the following simple, basic question:

> Consider the number of vertices in the arrangement that have level equal to k. How large can this number be, asymptotically as a function of n and k?

This question has relevance to the analysis of algorithms for a number of fundamental geometric problems [14, 20, 22, 24].

Here is a summary of what is known:

- The most famous case—dually related to the so-called *k-set problem*—concerns lines in 2-d. The early papers by Lovász [18] and Erdős *et al.* [16] showed that every arrangement of n lines has at most $O(n\sqrt{k})$ vertices at level k, and furthermore, there exist arrangements of lines with $\Omega(n \log k)$ such vertices. Major improvements did not come for more than twenty years, until Dey [12] and Tóth [28] improved the upper bound to $O(nk^{1/3})$ and the lower bound to $n2^{\Omega(\sqrt{\log k})}$.

- For planes in 3-d, Bárány, Füredi, and Lovász [5] were to first to obtain a nontrivial, subcubic upper bound of $O(n^{3-1/343}) = O(n^{2.9971})$. After a series of improvements (in chronological order, [4, 15, 13, 1]), Sharir, Smorodinsky, and Tardos [25] gave the current best upper bound of $O(nk^{3/2})$. The current lower bound is $nk2^{\Omega(\sqrt{\log k})}$ [28].

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- For planes in 4-d, Sharir [23] obtained the most recent upper bound $O(n^{4-1/18})$ (improving a previous result by Matoušek *et al.* [21]). By a technique of Agarwal *et al.* [1], the bound can be made sensitive to k, namely, $O(n^2 k^{2-1/18})$.

- For hyperplanes in a fixed dimension $d > 3$, Bárány *et al.*'s proof in combination with the multicolored Tverberg theorem [29, 6] yielded an upper bound of $O(n^{d-\alpha_d})$ for a very small $\alpha_d = 1/(2d)^d$. As Agarwal *et al.* [1] observed, the bound can be made sensitive to k, namely, $O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - \alpha_d})$.

- The more general case of nonlinear curves in 2-d has also been studied extensively. Previous proofs for arrangements of lines, including Dey's $O(nk^{1/3})$ result, can be adapted to arrangements of *pseudo-lines*, where each pair of curves intersects at most once [26]. For more general families, however, new techniques are required. Tamaki and Tokuyama [27] were the first to suggest the approach of "cutting" curves and obtained an $O(n^{23/12})$ upper bound on the number of vertices of any level in an arrangement of n *pseudo-parabolas*, where each pair intersects at most twice. The bound for pseudo-parabolas has eventually been reduced to $O(nk^{1/2} \log k)$, after a series of improvements ([7, 2, 8, 19, 10] in chronological order). In 2005, the author [8] proposed a new, simple approach that yielded an $\widetilde{O}(nk^{1-1/2s})$ upper bound for general curve families, where each pair intersects at most a constant s number of times, and the $\widetilde{O}$ notation hides small, inverse-Ackermann-like factors; further improvements were also given for even values of s and for specific curve families (for example, graphs of fixed-degree polynomials in one variable). A follow-up paper [10] obtained yet more improvements.

- For polyhedral surfaces comprising $O(n)$ triangles in 3-d, Agarwal *et al.* [1] gave an $\widetilde{O}(n^2 k^{7/9})$ upper bound, which was improved to $\widetilde{O}(n^2 k^{2/3})$ by Katoh and Tokuyama [17].

In this paper, we give upper bounds for nonlinear surfaces in 3-d. The surface families that can be handled by our proof are fairly general, and include a certain definition of *pseudo-planes* and *pseudo-triangles*, where the main condition is that each triple of surfaces intersects at most once, and *pseudo-spherical patches*, where the main condition is that each triple of surfaces intersects at most twice (see Section 2.3 for the precise general requirement and specific definitions used). For these particular surface families, our upper bound is $O(n^{3-1/286.97}) = O(n^{2.9966})$,which can be made k-sensitive, namely, $O(nk^{1.9966})$ for pseudo-planes and $\widetilde{O}(n^2 k^{0.9966})$ for the other families, by Agarwal *et al.*'s observations [1]. It should be emphasized that while this barely subcubic bound may not look very impressive, it is the first demonstration that a nontrivial result is possible, and with any luck, improvements might subsequently follow.[1] (The reader is asked to take a historical perspective and compare our result with Bárány *et al.*'s for planes [5].) Rather than the specific bound, the proof technique itself should be regarded as the main contribution.

Why can't previous techniques be adapted to handle general surfaces? The existing upper-bound proofs for planes [4, 5, 13, 15, 25] and triangles [1, 17] in 3-d are all similar in that they are based (in part) on Lovász's original approach [18]. It is not unthinkable that this approach could work for pseudo-planes, although finding an appropriate extension of Lovász's main lemma was posed by Agarwal *et al.* [1] some time ago and is still unanswered. In any case, an extension of this approach to more general surfaces in 3-d (or even general curves in 2-d) is less imaginable due to a variety

[1] In fact, the bound presented here is already an improvement over the $O(n^{3-1/705.48}) = O(n^{2.9986})$ bound that was originally announced in the conference version of this paper [9].

of obstacles (for starters, Lovász's lemma is typically applied to points in dual space, but no point-surface duality is known). We therefore need a proof that is completely different from all previous proofs for planes. The author's new technique for curves in 2-d [8] turns out to be just what is needed, but the extension to 3-d is not easy and requires a number of additional ideas, together with an intricate charging argument.

2 The Proof Plan

We may assume (by perturbation arguments) that the given arrangement is nondegenerate. To make it easier to understand, we first describe our proof plan for the case of *pseudo-planes*, which we formally define as surfaces satisfying the following requirement:

> The surfaces are graphs of total bivariate functions such that (i) the intersection of each pair is an x-monotone curve, and (ii) the intersection of each triple is a single point.

Note that a more common definition of pseudo-planes would omit condition (i). Later in Section 2.3, we will consider generalizations that would relax both conditions and allow for certain non-pseudo-plane families.

2.1 The Previous Approach

As mentioned, the main approach is from the author's previous paper [8]. The idea is to extend the problem by looking at the number of vertices at nearby levels. A nontrivial upper bound is then obtained by solving a recurrence/difference equation.

Given an arrangement of n pseudo-planes in $\mathbb{R}^3$ and an integer k, let t_i be the number of vertices in the arrangement with levels in the range $(k-i, k+i)$. Let $\Delta t_i = t_{i+1} - t_i$ be the number of vertices with level $k-i$ or $k+i$. Our problem is to bound t_1, the number of vertices in the arrangement at level k. (A more common version of the problem is to bound the combinatorial complexity of the k-*level*, defined as the boundary of the set of all points in $\mathbb{R}^3$ with level at most k; the two versions are known to be asymptotically equivalent—e.g., see [3].)

In the previous proof in 2-d [8], we bound t_i in terms of Δt_i by a simple charging argument, yielding an inequality of the form

$$t_i \;\leq\; c_0 i \Delta t_i \;+\; \text{[some overhead term]}. \tag{1}$$

(In the pseudo-line case, $c_0 = 2$.) With the base case $t_n = O(n^2)$, this recurrence gives $t_i = O(n^{2-1/c_0} i^{1/c_0})$, implying a subquadratic bound for t_1.

We would like to derive a similar inequality in 3-d (this time, with $t_n = O(n^3)$). Unfortunately, there seems no obvious way to make the original charging argument work in 3-d. Further ideas are needed...

2.2 Reduction to 2-d

As it turns out, the difficulties arising in 3-d can be resolved by not working in 3-d at all. More precisely, we consider the 2-d subarrangement inside each of the n given surfaces, prove a general inequality in 2-d, and then "sum up" to get the 3-d inequality. (At least one previous proof, by

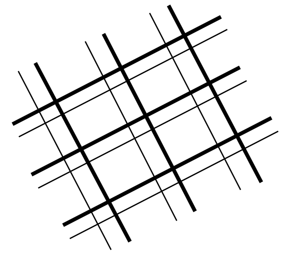

Figure 1: A "bad" example of a bichromatic arrangement, with all vertices shown having the same level. In all figures, unbold lines are red and bold lines are blue.

Sharir *et al.* [25], was also in part based on summing up contributions from various 2-d subproblems, though in a very different way; our idea here was indirectly inspired by Sharir *et al.*'s proof.)

The subarrangement within a given surface σ is a pseudo-line arrangement when projected to the xy-plane. In studying this subarrangement, we need to classify each pseudo-line γ as one of two types, "red" or "blue", depending on whether within σ, points above γ in the y-direction are above or below the surface defining γ in the z-direction.

We now need to prove an inequality in 2-d that more generally applies to a bichromatic arrangement.

Definition 2.1 Consider an arrangement of pseudo-lines in $\mathbb{R}^2$, where each curve is colored red or blue. For a red curve γ, we say that a point p *violates* γ if p is strictly above γ; for a blue curve γ, p *violates* γ if p is strictly below γ. Define the *level* of p to be the number of curves violated by p. Call an intersection of two curves a *monochromatic vertex* if the two curves have the same color; call it a *bichromatic vertex* otherwise. Call a vertex at level in the range $(k-i, k+i)$ an *interior* vertex, and a vertex at level $k-i$ or $k+i$ a *boundary* vertex.

Let $t_i^{\rm mo}$ and $t_i^{\rm bi}$ be the number of interior monochromatic and bichromatic vertices respectively, and $\Delta t_i^{\rm mo} = t_{i+1}^{\rm mo} - t_i^{\rm mo}$ and $\Delta t_i^{\rm bi} = t_{i+1}^{\rm bi} - t_i^{\rm bi}$ be the number of boundary monochromatic and bichromatic vertices respectively.

We would like to bound the number of interior vertices ($t_i^{\rm mo}$ or $t_i^{\rm bi}$) in terms of the number of boundary vertices ($\Delta t_i^{\rm mo}$ or $\Delta t_i^{\rm bi}$), as in (1). Unfortunately, this is not possible in the bichromatic setting, because as Figure 1 indicates, both $t_i^{\rm mo}$ and $t_i^{\rm bi}$ can be quadratic in the worst case.

To overcome this problem, we need yet another idea: charge interior monochromatic vertices ($t_i^{\rm mo}$) not only to boundary vertices ($\Delta t_i^{\rm mo}$ and $\Delta t_i^{\rm bi}$) but also to interior bichromatic vertices ($t_i^{\rm bi}$). Intuitively, if $t_i^{\rm mo}$ is large, then $t_i^{\rm bi}$ would be large too, as the example from Figure 1 seems to suggest. We thus aim to prove the following:

Theorem 2.2 (Main Inequality) *For any bichromatic arrangement of n pseudo-lines in $\mathbb{R}^2$,*

$$t_i^{\rm mo} \;\le\; (c_1 i + O(1))(\Delta t_i^{\rm mo} + \Delta t_i^{\rm bi}) + (c_2 + O(1/i))t_i^{\rm bi} + O(n_i i),$$

where n_i denotes the number of curves that have level in $[k-i, k+i]$ at $x = \infty$ or $x = -\infty$, and c_1 and c_2 are specific constants with $c_2 < 2$.

In our 3-d application, interior bichromatic vertices are fortunately rarer than interior monochromatic vertices, by a factor of 2. We claim that the above inequality implies a subcubic bound for a level in 3-d:

Corollary 2.3 *For any arrangement of n pseudo-planes in $\mathbb{R}^3$, the number of vertices at level k is $O(n^{3-1/c_0})$, where $c_0 = \frac{3c_1}{2-c_2}$, and c_1 and c_2 are the constants in the main inequality.*

Proof: We apply Theorem 2.2 to the subarrangement inside each given surface, with the color scheme described above. Observe that the level of each vertex v in the 2-d (bichromatic) subarrangement is equal to the level of v in the 3-d arrangement. Each vertex v lies in three subarrangements. We claim that it is monochromatic in two of them and bichromatic in one of them: Consider the lower envelope of the three surfaces $\sigma_1, \sigma_2, \sigma_3$ that define v. One edge of the envelope—say, $\sigma_1 \cap \sigma_2$—is to the left of v, and the other two are to the right of v, or vice versa. Then v is monochromatic in the subarrangement inside σ_1 and the subarrangement inside σ_2, but bichromatic inside σ_3.

Therefore, summing the left- and right-hand sides of the main inequality over all the n subarrangements gives the following upper bound:

$$2t_i \;\le\; (3c_1 i + O(1))\Delta t_i + (c_2 + O(1/i))t_i + O(n^2 i) \qquad \Longrightarrow \qquad t_i \;\le\; (c_0 i + O(1))\Delta t_i + O(n^2 i).$$

So, $t_i \;\le\; (c_0 i + O(1))(t_{i+1} - t_i) + O(n^2 i)$, i.e., $t_i \;\le\; (1 + \frac{1}{c_0 i + O(1)})\, t_{i+1} + O(n^2)$. As base case, $t_n = O(n^3)$. Solving this recurrence is straightforward (as described in the appendix of [8]) and yields $t_i = O(n^{3-1/c_0} i^{1/c_0})$. □

In Section 3, we will establish the 2-d main inequality for a certain choice of constants c_1 and c_2.

2.3 Generalization to Other Surface Families

The preceding proof plan can be adapted to handle other types of surfaces in 3-d besides pseudo-planes. The precise requirement for the surfaces is as follows:

($*$) The n given surfaces are graphs of total bivariate functions such that the 2-d subarrangement within each surface (when projected to xy-plane) forms a collection of $O(n)$ x-monotone curve segments in a family with subquadratic cutting number.

Here, a family of curve segments has *subquadratic cutting number* if any N curve segments in the family can be cut into $O(N^{2-\kappa})$ pseudo-segments (where each pair intersects at most once), for some constant $\kappa > 0$.

The restriction to graphs of total functions is not crucial, since we can add near-vertical extensions to make them total.

The cutting number in 2-d arrangements was introduced by Tamaki and Tokuyama [27], who proved the first nontrivial results for pseudo-parabolas (graphs of total univariate functions that pairwise intersect twice); the current best bound for pseudo-parabolas has $\kappa \approx 1/2$ [2, 19]. Other curve families known to have subquadratic cutting number include pseudo-parabolic segments (x-monotone curve segments that pairwise intersect at most twice) with $\kappa = 1/3$ [7], and graphs of univariate degree-s polynomial functions with $\kappa \approx 1/2^{s-1}$ [7, 19].

Some examples of surface families that satisfy ($*$) thus include the following:

- *pseudo-triangles*, which we formally define as a collection of the graphs of n bivariate functions such that after near-vertical extensions are added, each 2-d subarrangement forms a family of $O(n)$ pseudo-segments;

- *pseudo-spherical patches*, which we formally define as a collection of the graphs of n bivariate functions such that after extensions are added, each 2-d subarrangement forms a family of $O(n)$ pseudo-parabolic segments;
- surfaces each having equation $z = p(x) + ay$ for some degree-s polynomial $p(x)$ and constant a.

(In these three examples, we have $\kappa = 1$, $\kappa \approx 1/2$, and $\kappa \approx 1/2^{s-1}$ respectively.)

We can apply the same reduction to 2-d (in the proof of Corollary 2.3, we should consider the lower envelope of $\sigma_1, \sigma_2, \sigma_3$ only locally around v). All that is needed is a main inequality for 2-d curve families with subquadratic cutting number. It is not obvious how to obtain such an inequality for general curves directly without blowing up the coefficient c_2 beyond 2. One idea is to cut the curves first (which increases the number of endpoints) and then prove the same main inequality for pseudo-segments (with n_i redefined as the number of pseudo-segment endpoints with level in $[k-i, k+i]$). The overhead term increases (to $O(n^{3-\kappa}i)$), but this can still yield a subcubic result. However, for better results, we follow instead an idea from the previous paper [8] of not explicitly cutting the curves but charging features to certain "lenses":

Definition 2.4 If two curves γ_1 and γ_2 intersect more than once, the part of $\gamma_1 \cup \gamma_2$ between two consecutive intersection points is called a *lens*. We say that a lens is *i-light* if every vertical line segment inside the lens intersects at most i red curves and at most i blue curves.

We will prove the following generalization of the main inequality for arbitrary 2-d x-monotone curves:

Theorem 2.5 (Main Inequality) *For any bichromatic arrangement of n curves that are graphs of total univariate functions,*

$$t_i^{\rm mo} \;\le\; (c_1 i + O(1))(\Delta t_i^{\rm mo} + \Delta t_i^{\rm bi}) + (c_2 + O(1/i))t_i^{\rm bi} + O(n_i i + |\Lambda_i|),$$

where Λ_i denotes the collection of $(c_3 i)$-light lenses, n_i denotes the number of curves that have level in $[k-i, k+i]$ at $x = \infty$ or $x = -\infty$, and c_1, c_2, c_3 are specific constants with $c_2 < 2$.

The same result would then hold for curve segments (with n_i replaced by $O(n)$), since we can add near-vertical (upward if red, downward if blue) extensions at the endpoints to make the functions total—this increases $|\Lambda_i|$ by at most $O(ni)$, as this generates at most $O(i)$ extra lenses in Λ_i incident to each endpoint.

The collection of lenses Λ_i has depth $O(i)$, i.e., a point can be lie on at most $O(i)$ lenses in Λ_i. By a random sampling technique from the previous paper [8, Lemma 4.1] (see also [27]), we know that $|\Lambda_i| = O(n^{2-\kappa} i^{\kappa})$. The new recurrence in 3-d becomes

$$t_i \;\le\; (c_0 i + O(1))\Delta t_i + O(n^{3-\kappa} i^{\kappa}).$$

The solution of this recurrence (e.g., see the appendix of [8]) leads to:

Corollary 2.6 *For any arrangement of n surfaces in $\mathbb{R}^3$ satisfying $(*)$, the number of vertices at level k is $O(n^{3-1/c_0})$ if $\kappa > 1/c_0$, and $O(n^{3-\kappa})$ if $\kappa < 1/c_0$. Here, $c_0 = \frac{3c_1}{2-c_2}$, where c_1 and c_2 are the constants in the main inequality.*

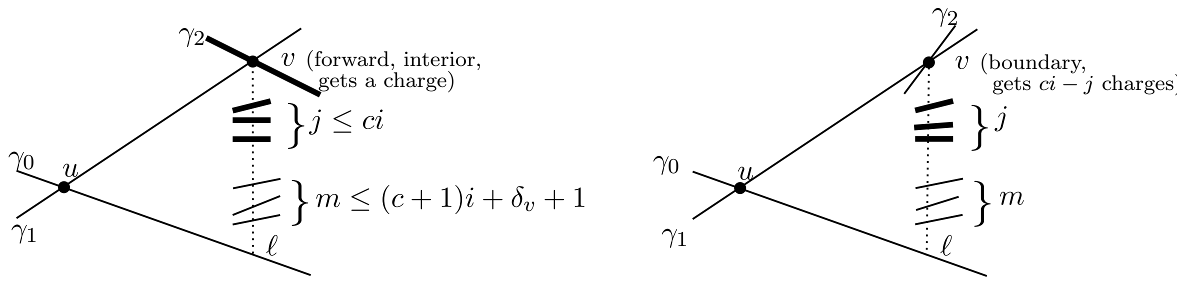

Figure 2: The basic charging scheme.

3 Proof of the 2-d Main Inequality

It remains to prove the (generalized) main inequality, Theorem 2.5. This is done in the next three subsections.

3.1 The Basic Charging Scheme

We start with a natural but more involved extension of the author's charging argument for monochromatic 2-d arrangements [8].

Let c be a positive constant, to be determined later. To avoid special cases, we treat the $O(n_i)$ points with level in $[k-i, k+i]$ at $x = \pm\infty$ as boundary vertices. We also pad the arrangement with $(c+2)i+1$ extra curves below and $(c+2)i+1$ extra curves above all the given curves of both colors, without creating any new intersections. We introduce some notation and terminology:

Definition 3.1

- Call a vertex *exceptional* if it is a vertex of some lens in Λ_i, and *ordinary* otherwise.
- For any point v, let δ_v be the number such that the level of v is $k + \delta_v$.
- For a point v and a curve γ, let $\chi_{v,\gamma}$ be $+1$ if v violates γ, and -1 otherwise.
- Given a vertex u defined by curves γ_0 and γ_1 and a vertex v defined by curves γ_1 and γ_2, we say that v is *forward w.r.t. u* if γ_2 lies between γ_0 and γ_1 at a vertical line slightly to the right (resp. left) of v, assuming that v is to the right (resp. left) of u. Otherwise, v is *backward w.r.t. u*. (Figure 2 (left) shows an example of a forward vertex v and Figure 2 (right) shows an example of a backward vertex v.)
- Given curves γ_0 and γ_1 of the same color and given a vertical line ℓ, let $m(\gamma_0, \gamma_1, \ell)$ be the number of curves of the same color as γ_0, γ_1 that lie between γ_0 and γ_1 at ℓ; let $j(\gamma_0, \gamma_1, \ell)$ be the the number of curves of the opposite color as γ_0, γ_1 that lie between γ_0 and γ_1 at ℓ.
- We say that $(\gamma_0, \gamma_1, \ell)$ is *within range* if $j(\gamma_0, \gamma_1, \ell) \leq ci$ and both $\delta_{\gamma_0 \cap \ell}, \delta_{\gamma_1 \cap \ell} \in [-i, +i]$. Otherwise, $(\gamma_0, \gamma_1, \ell)$ is *out of range*.

Observation 3.2 *Given curves γ_0 and γ_1 of the same color and a vertical line ℓ intersecting γ_1 at the point v, if $(\gamma_0, \gamma_1, \ell)$ is within range, then*

$$m(\gamma_0, \gamma_1, \ell) \leq j(\gamma_0, \gamma_1, \ell) + i + \chi_{v,\gamma_0}\delta_v + 1 \leq (c+1)i + \chi_{v,\gamma_0}\delta_v + 1 \leq (c+2)i + 1.$$

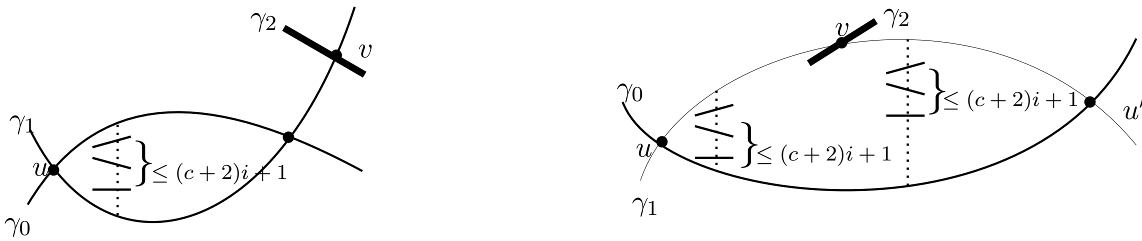

Figure 3: If more than one vertex on γ_0 were to send a charge to v, then a lens in Λ_i would be formed.

Proof: W.l.o.g., say γ_0 is below v and is red (the other cases are similar). By the definition of levels in a bichromatic arrangement, $m(\gamma_0, \gamma_1, \ell) - j(\gamma_0, \gamma_1, \ell) \le \delta_v - \delta_{\gamma_0 \cap \ell} + 1$. Since $\delta_v, \delta_{\gamma_0 \cap \ell} \in [-i, +i]$, we have $m(\gamma_0, \gamma_1, \ell) \le j(\gamma_0, \gamma_1, \ell) + \delta_v + i + 1 \le (c+1)i + \delta_v + 1 \le (c+2)i + 1$. □

We now describe a scheme of charging interior monochromatic vertices to interior bichromatic vertices and boundary vertices (see Figure 2).

Definition 3.3 Suppose u is an ordinary interior monochromatic vertex u defined by curves γ_0 and γ_1, and v is a vertex defined by curves γ_1 and γ_2 and lies on the vertical line ℓ.

- For an interior bichromatic vertex v, we say that u *sends a charge to* v if
  - (A1) $(\gamma_0, \gamma_1, \ell')$ is within range for all vertical lines ℓ' between u and v, and
  - (A2) v is forward w.r.t. u.
- For a boundary (monochromatic or bichromatic) vertex v, we say that u *sends* $ci - j(\gamma_0, \gamma_1, \ell)$ *charges to* v if (A1) holds (regardless of whether (A2) holds).

Remarks: Note that if u sends a charge to v, then γ_0 cannot cross γ_1 between u and v, because otherwise u would define a lens that is $((c+2)i+1)$-light (by condition (A1), with Observation 3.2) and would thus be exceptional for a sufficiently large $c_3 > c+2$ (see Figure 3 (left)). For a similar reason, we cannot have v receiving a charge from both a vertex u left of v and another vertex u' right of v (see Figure 3 (right)). Consequently, at most one vertex on γ_0 can send a charge to v.

Lemma 3.4

(i) *Each ordinary interior monochromatic vertex u sends at least $2ci$ charges.*

(ii) *Each interior bichromatic vertex v receives at most $4(c+1)i + O(1)$ charges.*

(iii) *Each boundary (monochromatic or bichromatic) vertex v receives at most $2c(c+2)i^2 + O(i)$ charges.*

Proof:

(i) Suppose u is defined by γ_0 and γ_1. Imagine moving a vertical sweep line ℓ from left to right, starting at u. As ℓ passes through a bichromatic forward vertex on $\gamma_0 \cup \gamma_1$ (w.r.t. u) while $(\gamma_0, \gamma_1, \ell)$ stays within range, the "counter" $j(\gamma_0, \gamma_1, \ell)$ increases by 1 and a charge is sent from u to that vertex. (On the other hand, as ℓ passes through a bichromatic backward vertex, the counter decreases by 1.) As soon as $(\gamma_0, \gamma_1, \ell)$ gets out of range, i.e., $j(\gamma_0, \gamma_1, \ell)$ reaches ci or ℓ passes through a boundary vertex on $\gamma_0 \cup \gamma_1$, terminate the sweep. In the latter case, $ci - j(\gamma_0, \gamma_1, \ell)$ charges are sent from u to the boundary vertex. Thus, at least ci charges are sent from u during this left-to-right sweep. Similarly, at least ci charges are sent during a right-to-left sweep.

(ii) Suppose v is defined by red curve γ_1 and blue curve γ_2. By Observation 3.2, if v receives a charge from a vertex defined by γ_1 and a red curve γ_0 below v, then $m(\gamma_0, \gamma_1, \ell) \le (c+1)i + \delta_v + 1$. Thus, there are at most $(c+1)i+\delta_v+1$ candidates for γ_0 below v, and by a symmetric argument at most $(c+1)i - \delta_v + 1$ candidates for γ_0 above v, yielding a total of at most $2(c+1)i + 2$ charges received by v from vertices on γ_1. Similarly, there are at most $2(c+1)i + 2$ charges from vertices on γ_2.

(iii) Suppose v is defined by γ_1 and γ_2. W.l.o.g., say γ_1 is red. By Observation 3.2, $j(\gamma_0, \gamma_1, \ell) \ge m(\gamma_0, \gamma_1, \ell) - i - \delta_v - 1$ for red curves γ_0 below v. Thus, v receives at most the following number of charges from vertices defined by γ_1 and red curves below v:

$$\sum_{m=0}^{\lfloor (c+1)i+\delta_v+1 \rfloor} \min\{ci - (m - i - \delta_v - 1),\, ci\} \;\le\; (i + \delta_v)(ci) + (\lfloor ci \rfloor + \cdots + 1) + O(i)$$
$$\le\; (i + \delta_v)(ci) + \frac{c^2 i^2}{2} + O(i).$$

By a symmetric argument, the number of charges received by v from vertices defined by γ_1 and curves above v is at most $(i - \delta_v)(ci) + \frac{c^2 i^2}{2} + O(i)$, yielding a total of at most $2ci^2 + c^2 i^2 + O(i)$ charges received by v from vertices on γ_1. Similarly, there are at most $2ci^2 + c^2 i^2 + O(1)$ charges from vertices on γ_2. □

By the above lemma, the total number of charges is at least $2ci(t_i^{\rm mo} - O(|\Lambda_i|))$ (there are $O(|\Lambda_i))$ exceptional vertices) and is at most $(4(c+1)i + O(1))t_i^{\rm bi} + (2c(c+2)i^2 + O(i))(\Delta t_i^{\rm mo} + \Delta t_i^{\rm bi} + O(n_i))$. Dividing by $2ci$, we get

$$t_i^{\rm mo} \;\le\; \left(2 + \frac{2}{c} + O(1/i)\right) t_i^{\rm bi} + ((c+2)i + O(1))(\Delta t_i^{\rm mo} + \Delta t_i^{\rm bi}) + O(n_i i + |\Lambda_i|).$$

We have thus obtained an inequality of the form stated in Theorem 2.5. There is just one (major!) problem: the coefficient c_2 of the $t_i^{\rm bi}$ term here is greater than 2, regardless of the choice of c, but in order for Corollaries 2.3 and 2.6 to yield any nontrivial bound for levels in 3-d, we need the coefficient to be strictly less than 2.

3.2 Helpers

Despite the apparent problem, we will not abandon the above charging scheme but will amend it by looking for places for improvement. Specifically, the following kinds of configurations (see Figure 4 (left)) help reduce our bound on the number of charges received:

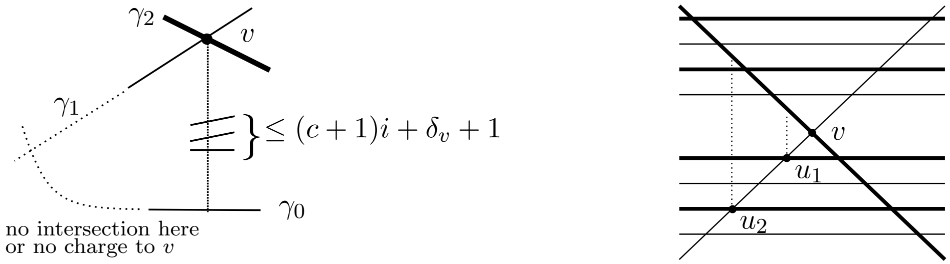

Figure 4: (Left) A helper. (Right) An example where there are no helpers at v, but there are helpers at u_1 and u_2.

Definition 3.5 Suppose v is a vertex defined by curves γ_1 and γ_2 and lies on the vertical line ℓ. Let γ_0 be a curve with the same color as γ_1.

- For a bichromatic interior vertex v, we say that (v, γ_0) is a *helper (at v)* if
  - (B1) $m(\gamma_0, \gamma_1, \ell) \leq (c+1)i + \chi_{v,\gamma_0}\delta_v + 1$, and
  - (B2) no intersection of γ_0 and γ_1 sends a charge to v.
- For a (monochromatic or bichromatic) boundary vertex v, we say that (v, γ_0) forms $ci - j(\gamma_0, \gamma_1, \ell)$ *helpers (at v)* if the same conditions (B1) and (B2) hold.

By inspecting the proofs of Lemma 3.4(ii) and (iii), we immediately see that

Lemma 3.6

(i) *Each interior bichromatic vertex v receives at most* $4(c+1)i-$*[the number of helpers at v]*$+O(1)$ *charges.*

(ii) *Each boundary (monochromatic or bichromatic) vertex v receives at most* $2c(c+2)i^2 -$ *[the number of helpers at v]* $+ O(i)$ *charges.*

Is it always possible to find many helpers in an arrangement? A "canonical" example where there are no helpers at a vertex v is shown in Figure 4 (right), but in this example one can find helpers at nearby vertices. This suggests hope of a positive answer. . .

We now classify helpers into a few specific types. The list of definitions below is somewhat elaborate, because of the desire to obtain better constants c_1, c_2 in our proof and to handle general non-pseudoline curves.

Definition 3.7 Suppose u is an ordinary (not necessarily interior) monochromatic vertex defined by curves γ_0 and γ_1, and v is a vertex defined by curves γ_1 and γ_2 and lies on the vertical line ℓ.

- For an interior bichromatic vertex v, we say that (v, γ_0) is a *strong helper (from u)* if
  - (C1) $(\gamma_0, \gamma_1, \ell')$ is within range for all vertical lines ℓ' between u and v, and
  - (C2) v is backward w.r.t. u.
- For an interior bichromatic vertex v, we say that (v, γ_0) is a *moderate helper (from u)* if

(D1) $(\gamma_0, \gamma_1, \ell)$ is within range, and

(D2) $(\gamma_0, \gamma_1, \ell')$ is out of range for some vertical line ℓ' between u and v, and

(D3) $m(\gamma_0, \gamma_1, \ell'), j(\gamma_0, \gamma_1, \ell') \leq c_3 i$ for all vertical lines ℓ' between u and v.

A moderate helper is further classified as a *moderate forward* or *moderate backward* helper depending on whether v is forward or backward w.r.t. u.

- For an interior bichromatic vertex v, any helper (v, γ_0) that is not strong or moderate are classified as a *weak helper*.
- For a boundary vertex v, we say that (v, γ_0) forms $ci - j(\gamma_0, \gamma_1, \ell)$ *moderate helpers (from u)* if (D1)–(D3) hold. These moderate helpers are classified as *moderate forward* helpers (regardless of whether v is actually forward or backward w.r.t. u).

Remarks: Note that if (v, γ_0) is a strong or moderate helper from u, then γ_0 cannot cross γ_1 between u and v, because otherwise u would define a lens that is $(c_3 i)$-light (by condition (C1) or (D3)) and would thus be exceptional for a sufficiently large c_3 (like in Figure 3 (left), with "$(c+2)i+1$" replaced by "$c_3 i$"). For a similar reason, (v, γ_0) cannot be a strong or moderate helper from both a vertex u left of v and another vertex u' right of v (like in Figure 3 (right)). Nor can we have both a strong or moderate helper (v, γ_0) from a vertex u left of v, and v receiving a charge from another vertex u' right of v.

It can then be checked that each helper (v, γ_0) can indeed be only one of the classified types: strong, moderate forward, moderate backward, and weak; and furthermore, a strong or moderate helper is indeed a helper (because (B1) is implied by (C1) or (D1), and (B2) is implied by (C2) or (D2)).

The following lemma is useful in further improving constants. It turns out that strong helpers can not only reduce the number of charges received but also boost the number of charges sent. Furthermore, an abundance of moderate backward helpers automatically imply an abundance of moderate forward helpers.

Lemma 3.8

(i) *Each ordinary interior monochromatic vertex u sends at least $2ci$ + [the number of strong helpers from u] charges.*

(ii) *From each ordinary (not necessarily interior) monochromatic vertex u, the number of moderate forward helpers is at least the number of moderate backward helpers.*

Proof:

(i) This follows by inspecting the proof of Lemma 3.4(i). During the sweep for vertex u, when ℓ passes through a bichromatic backward vertex v on $\gamma_0 \cup \gamma_1$, we get a strong helper at v from u and the counter $j(\gamma_0, \gamma_1, \ell)$ decreases by 1, allowing for one more subsequent increment (and thus a charge to one more interior vertex) or a lower final value for $j(\gamma_0, \gamma_1, \ell)$ (and thus an extra charge to a boundary vertex).

(ii) Perform the same left-to-right sweep starting at u, except this time the sweep is terminated only when $j(\gamma_0, \gamma_1, \ell)$ or $m(\gamma_0, \gamma_1, \ell)$ exceeds $c_3 i$. The regions swept by ℓ during which $(\gamma_0, \gamma_1, \ell)$ is within range form a union of disjoint "windows". Take one such window, excluding the initial window containing u (there is no initial window if u is not an interior vertex). When ℓ passes through a bichromatic forward (resp. backward) vertex v on $\gamma_0 \cup \gamma_1$ within this window, we get a moderate forward (resp. backward) helper at v from u and the counter $j(\gamma_0, \gamma_1, \ell)$ increases (resp. decreases) by 1. At the right boundary of the window, $j(\gamma_0, \gamma_1, \ell)$ reaches ci or ℓ passes through a boundary vertex v on $\gamma_0 \cup \gamma_1$. In the latter case, we get $ci - j(\gamma_0, \gamma_1, \ell)$ moderate forward helper at v. Thus, there are at least as many moderate forward helpers as moderate backward helpers within each window. A similar argument holds for a right-to-left sweep. □

Let H_{strong}, H_{for}, H_{back}, and H_{weak} denote the number of strong, moderate forward, moderate backward, and weak helpers respectively.

By Lemmas 3.8(i) and 3.6, the total number of charges is at least $2ci(t_i^{\text{mo}} - O(|\Lambda_i|)) + H_{\text{strong}}$ and is at most $(4(c+1)i + O(1))t_i^{\text{bi}} + (2c(c+2)i^2 + O(i))(\Delta t_i^{\text{mo}} + \Delta t_i^{\text{bi}} + O(n_i)) - H_{\text{strong}} - H_{\text{for}} - H_{\text{back}} - H_{\text{weak}}$. Dividing by $2ci$ and using the fact that $H_{\text{for}} - H_{\text{back}} \geq 0$ by Lemma 3.8(ii), we get

$$\begin{aligned} t_i^{\text{mo}} \;\; \leq \;\; & \left(2 + \frac{2}{c} + O(1/i)\right) t_i^{\text{bi}} + ((c+2)i + O(1))(\Delta t_i^{\text{mo}} + \Delta t_i^{\text{bi}}) + O(n_i i + |\Lambda_i|) \\ & - \frac{H_{\text{strong}} + (0.5 - \lambda)H_{\text{for}} + (0.5 + \lambda)H_{\text{back}} + 0.5 H_{\text{weak}}}{ci} \qquad \forall \lambda \geq 0. \end{aligned} \tag{2}$$

3.3 A More Sophisticated Charging Scheme

We now prove the abundance of helpers by devising a second charging scheme, this time, with charges sent from interior bichromatic vertices to helpers. Let $\alpha > 2/c$ and $\beta \in (\alpha, 1-\alpha)$ be constants, to be set later.

Sweeping. Take an ordinary interior bichromatic vertex v, defined by red curve γ_1 and blue curve γ_2. W.l.o.g., say γ_1 is below γ_2 slightly to the left of v (the other case is symmetric). Let h be the number of helpers of the form (v, γ) with γ red and below v, or blue and above v.

Move a vertical sweep line ℓ from right to left, starting at v. Maintain the following counters (see Figure 5 (left)):

- Let j_1 (resp. j_2) be the number of blue (resp. red) curves between γ_1 and γ_2.
- Let m_σ (resp. m'_σ) be the number of bichromatic forward (resp. backward) vertices on γ_σ between ℓ and v ($\sigma \in \{1, 2\}$). Let $m = m_1 + m_2$.
- Let p_σ (resp. p'_σ) be the number of monochromatic forward (resp. backward) vertices on γ_σ between ℓ and v.

As ℓ passes through a bichromatic forward vertex on $\gamma_1 \cup \gamma_2$, the counter m increases by 1. As soon as ℓ hits a boundary vertex on $\gamma_1 \cup \gamma_2$ or m reaches $(1-\alpha)ci - h - 1$, terminate the sweep.

Observation 3.9 $j_1 + m'_1,\ j_2 + m'_2 \leq m + i + \delta_v + 1 < ci.$

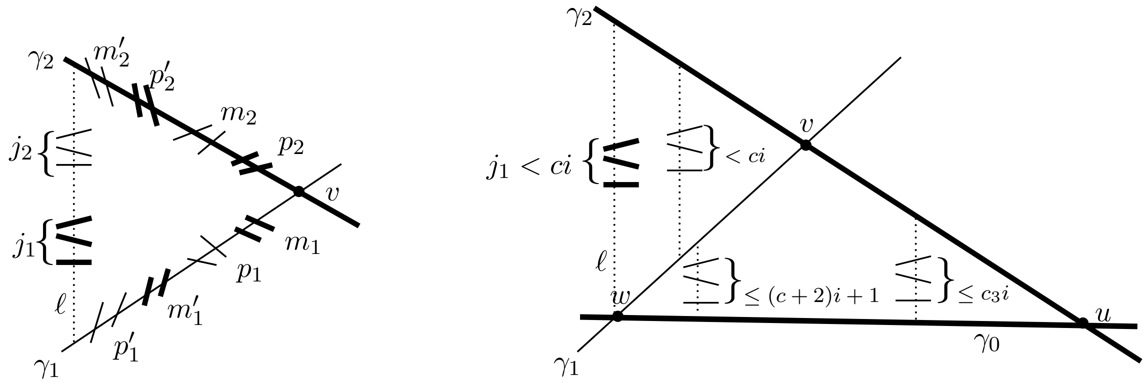

Figure 5: The more sophisticated charging scheme. (Left) Counters $j_\sigma, m_\sigma, m'_\sigma, p_\sigma, p'_\sigma$. (Right) Case 1: a strong or moderate backward helper (w, γ_2) from u.

Proof: By the definition of levels, $m_2 + p'_2 - m'_2 - p_2 \geq \delta_{\gamma_2 \cap \ell} - \delta_v - 1$. Thus, $j_1 = m_1 + p_2 - m'_1 - p'_2 \leq m_1 + m_2 - m'_1 - m'_2 + \delta_v - \delta_{\gamma_2 \cap \ell} + 1$, Since $\delta_v, \delta_{\gamma_2 \cap \ell} \in [-i, +i]$, we have $j_1 + m'_1 \leq m + i + \delta_v + 1 \leq (1 - \alpha)ci + 2i < ci$. The other inequality for $j_2 + m'_2$ is similar. □

Remark: Note that during the right-to-left sweep, γ_1 cannot cross γ_2, because otherwise v would define a lens that is (ci)-light (by Observation 3.9) and would thus be exceptional.

Actually, in the above sweep, ℓ cannot hit a boundary vertex on $\gamma_1 \cup \gamma_2$. To see this, suppose ℓ reaches a boundary vertex w on γ_1 (the other case is similar). For at least $(c + 1)i + \delta_v + 1 - h$ red curves γ below v, some intersection y_γ of γ and γ_1 sends a charge to v (by the definition of helpers and the number h); and y_γ must be to the right of w (in order for y_γ to send a charge to v). Now, γ must cross the vertical line segment between w and $\gamma_2 \cap \ell$, or cross γ_2 between $\gamma_2 \cap \ell$ and v, because otherwise γ would cross γ_1 between w and y_γ, and y_γ would define a lens that is (ci)-light (since j_1 and j_2 always stay below ci by Observation 3.9) and would thus be exceptional. It follows that $j_2 + m'_2$ reaches at least $(c + 1)i + \delta_v + 1 - h$, and by Observation 3.9, m reaches at least $ci - h > (1 - \alpha)ci - h - 1$: a contradiction.

Charging. For each of the h helpers of the form (v, γ) with γ red and below v, or blue and above v, as initialization, v sends 1 unit of charge to the helper.

Suppose ℓ passes through a forward interior bichromatic vertex w on $\gamma_1 \cup \gamma_2$. W.l.o.g., say w is defined by γ_1 and a blue curve γ_0 (the other possibility is similar). The heart of the proof lies in the following case analysis (see Figures 5 and 6):

Lemma 3.10 *At least one of the following cases must hold:*

- Case 0*: w is an exceptional vertex.*
- Case 1*: (w, γ_2) is a strong or moderate backward helper.*
- Case 2*: (v, γ_0) is a strong, moderate backward, or weak helper.*
- Case 3*: At some vertical line ℓ' between w and v, we have γ_0 below γ_1, and there are more than ci blue curves or more than $(c + 2)i + 1$ red curves between γ_0 and γ_1.*

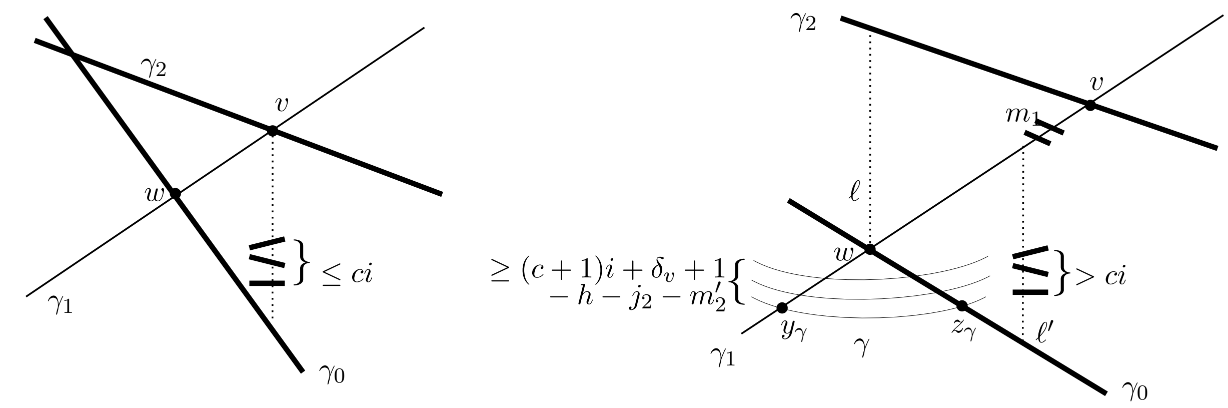

Figure 6: The more sophisticated charging scheme, continued. (Left) Case 2: a strong, moderate backward, or weak helper (v, γ_0). (Right) Case 3: strong helpers (z_γ, γ_1) from y_γ for many γ's.

Proof: Suppose that Case 3 does not hold. Then at every vertical line ℓ' between w and v, if γ_0 is below γ_1, there are at most ci blue curves and at most $(c+2)i+1$ red curves between γ_0 and γ_1. In particular, if γ_0 is below γ_1 at ℓ, there are at most $ci \leq (c+1)i + \delta_v + 1$ blue curves between γ_0 and γ_1 at ℓ.

We may assume that γ_0 and γ_1 do not cross between w and v (and thus γ_0 is indeed below γ_1 at ℓ), because otherwise w would define a lens that is $((c+2)i+1)$-light and we would be in Case 0.

If v receives a charge from a vertex u on γ_0, or if (v, γ_0) is a moderate forward helper from u, then (w, γ_2) is a strong or moderate backward helper from u and we are in Case 1 ((D1) and (D3) hold since j_1 and j_2 stay below ci by Observation 3.9, as one can see from Figure 5 (right) for a sufficiently large $c_3 > 2c+2$). On the other hand, if (v, γ_0) is a strong, moderate backward, or weak helper, then we are in Case 2. □

Our charging scheme is as follows:

- In Case 0, v sends 1 unit of charge to the vertex w itself.
- In Case 1, v sends 1 unit of charge to the helper (w, γ_2).
- In Case 2, v sends 1 unit of charge to the helper (v, γ_0).
- In Case 3, note that for at least $(c+1)i + \delta_v + 1 - h$ red curves γ below v, some intersection y_γ of γ and γ_1 sends a charge to v (by definition of helpers and the number h). If y_γ is to the right of w, then γ must cross the vertical line segment between w and $\gamma_2 \cap \ell$, or cross γ_2 between $\gamma_2 \cap \ell$ and v, because otherwise γ would cross γ_1 between w and y_γ, and y_γ would define a lens that is (ci)-light (since j_1 and j_2 always stay below ci by Observation 3.9) and would thus be exceptional. Thus, by excluding at most $j_2 + m_2'$ candidates for γ, we can ensure that y_γ is to the left of w.

  Since y_γ sends a charge to v, there are at most ci blue curves and at most $(c+2)i+1$ red curves between γ and γ_1 at ℓ' (by (A1), with Observation 3.2). In particular, γ must be above γ_0 at ℓ', due to the condition stated in Case 3. Thus, some intersection z_γ of γ and γ_0 must be backward w.r.t. y_γ, and so (z_γ, γ_1) is a strong helper from y_γ (see Figure 6 (right); condition

(C1) holds for this helper because y_γ sends a charge to v). The number of candidates for γ is at least $(c+1)i + \delta_v + 1 - h - j_2 - m'_2 \geq ci - h - m$ by Observation 3.9. We make v send $\frac{1}{\beta ci}$ units of charge to each such strong helper.

This completes the description of the right-to-left sweeping and charging process for v. We perform a similar process for v, this time, sweeping from left to right.

Analysis. Each ordinary interior bichromatic vertex v sends at least the following number of charges in the right-to-left sweep (including the h initial charges):

$$h + \sum_{m=0}^{\lfloor (1-\alpha)ci - h - 1 \rfloor} \min\left\{1,\, (ci - h - m)\frac{1}{\beta ci}\right\}$$

$$\geq \begin{cases} h + [(1-\beta)ci - h] + \frac{\lfloor \beta ci \rfloor + \cdots + \lfloor \alpha ci \rfloor}{\beta ci} - O(1) & \text{if } h \leq (1-\beta)ci \\ h + \frac{(ci-h) + \cdots + \lfloor \alpha ci \rfloor}{\beta ci} - O(1) & \text{if } (1-\beta)ci < h \leq (1-\alpha)ci \\ h & \text{if } h > (1-\alpha)ci \end{cases}$$

$$\geq \begin{cases} (1-\beta)ci + \frac{(\beta^2 - \alpha^2)c^2 i^2}{2\beta ci} - O(1) & \text{if } h \leq (1-\beta)ci \\ h + \frac{(ci-h)^2 - \alpha^2 c^2 i^2}{2\beta ci} - O(1) & \text{if } (1-\beta)ci < h \leq (1-\alpha)ci \\ h & \text{if } h > (1-\alpha)ci \end{cases}$$

It can be checked that this last expression is monotone increasing in h (ignoring the $O(1)$ terms), and so the number of charges is lower-bounded by the value in the first case, i.e., $\left(1 - \frac{\beta}{2} - \frac{\alpha^2}{2\beta}\right) ci - O(1)$. Therefore, v sends a total of at least $(2 - \beta - \alpha^2/\beta)ci - O(1)$ charges in both sweeps.

In regards to the number of charges received:

- Each exceptional vertex w receives $O(i)$ charges due to Case 0, since $m_1, m_2 \leq (1-\alpha)ci$ implies that given w, there are $O(i)$ candidates for γ_2, and thus for v.

- Each strong helper receives 1 charge initially, at most 1 charge due to Case 1, at most 1 charge due to Case 2, and at most $(1-\alpha)ci\frac{1}{\beta ci} = (1-\alpha)/\beta$ charges due to Case 3, since $m_1, m_2 \leq (1-\alpha)ci$ implies that given (z_γ, γ_1), there are at most $(1-\alpha)ci$ candidates for γ_2, and thus for v.

- Each moderate forward helper receives 1 charge during initialization and no more afterwards.

- Each moderate backward helper receives 1 charge during initialization, at most 1 charge due to Case 1, and at most 1 charge due to Case 2.

- Each weak helper receives 1 charge during initialization, and at most 1 charge due to Case 2.

To summarize, we have shown that the total number of charges in the above scheme is at least $((2 - \beta - \alpha^2/\beta)ci - O(1))(t_i^{\text{bi}} - O(|\Lambda_i|))$ and is at most $(3 + (1-\alpha)/\beta)H_{\text{strong}} + H_{\text{for}} + 3H_{\text{back}} + 2H_{\text{weak}} + O(i|\Lambda_i|)$. Since $3 + (1-\alpha)/\beta > 4$,

$$H_{\text{strong}} + 0.25H_{\text{for}} + 0.75H_{\text{back}} + 0.5H_{\text{weak}} \geq \left(\frac{2 - \beta - \alpha^2/\beta}{3 + (1-\alpha)/\beta} ci - O(1)\right)(t_i^{\text{bi}} - O(|\Lambda_i|)) - O(i|\Lambda_i|).$$

The coefficient $\frac{2-\beta-\alpha^2/\beta}{3+(1-\alpha)/\beta}$ exceeds 0.310102 by setting $\alpha = 0.15505$ and $\beta = 0.53485$. (Note that indeed $\beta \in (\alpha, 1-\alpha)$.)

So finally, inequality (2) with $\lambda = 0.25$ yields

$$t_i^{\rm mo} \;\le\; \left(2 + \frac{2}{c} - 0.310102 + O(1/i)\right) t_i^{\rm bi} + ((c+2)i + O(1))(\Delta t_i^{\rm mo} + \Delta t_i^{\rm bi}) + O(n_i i + |\Lambda_i|).$$

The coefficient of the $t_i^{\rm bi}$ term here is strictly less than 2 for a sufficiently large choice of the parameter c. The proof of Theorem 2.5 is now complete. For Corollaries 2.3 and 2.6, we have $c_0 = \frac{3(c+2)}{0.310102-2/c}$, which is below 286.97 by setting $c = 13.8312$, with $c_1 = 15.8312$ and $c_2 < 1.834499$. (Note that indeed $\alpha > 2/c$.) □

4 Final Remarks

Just like in earlier work on levels in arrangements of curves, we have shown how a nontrivial cutting number bound translates to a nontrivial level bound for any family of surfaces in 3-d. In light of this result, we reiterate the following open problem [7]: do general fixed-degree algebraic curves in the plane have a subquadratic cutting number? If so, a subcubic level bound would immediately follow for graphs of fixed-degree bivariate polynomial functions in 3-d.

Perhaps it might be possible to obtain slight improvements on the constants in the 2-d main inequality by lengthening the proof with an even more detailed case analysis, but it would be more desirable to find a simpler yet smarter charging argument that could yield more drastic improvements. An intriguing question is to determine what is the smallest value c_2 attainable in the 2-d inequality. Alternatively, can one prove the 3-d inequality directly?

It is doubtful that our approach could improve known upper bounds for levels for planes (i.e., the k-set problem) in 3-d. A more intriguing direction to pursue would be the case of hyperplanes in higher dimensions, where the previous upper bounds are very weak. Unfortunately, our reduction to 2-d fails to yield $o(n^d)$ bounds as soon as the dimension d reaches 4, because there could be as many bichromatic vertices ($t_i^{\rm bi}$) as monochromatic vertices ($t_i^{\rm mo}$) on average in the resulting 2-d subarrangements, and so we need the coefficient c_2 to be strictly less than 1—an impossible demand (as the example in Figure 1 indicates). Still, it is possible to adapt the approach of this paper to obtain new k-sensitive upper bounds for hyperplanes in 4-d for a certain range of k values, as the author has shown in a recent paper [11]. New k-sensitive upper bounds might also be possible in higher dimensions if we could somehow get c_2 closer to 1. In 2-d and 3-d, our approach has been shown [11] to lead to new results for a bichromatic version of the k-set problem.

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