Short Term Decumulation Strategies for Underspending Retirees

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August 30, 2021

Abstract

There is growing empirical evidence that many retirees are decumulating their assets very slowly, if at all. This fact is in stark contrast to the usual lifecycle models of spending. It appears that these underspending retirees adjust their withdrawals to avoid reducing their assets. In order to appeal to this class of retirees, we use optimal stochastic control techniques which maximize a multi-objective risk-reward problem. The reward is the total of withdrawals (over a five year period), while risk is based on a left tail measure. Our controls for this problem are the withdrawal amount per quarter, and the stock-bond asset allocation. We allow flexible withdrawals (even zero). This added flexibility results in a high probability of (i) retaining 90% of real wealth at the end of five years, and (ii) significant total spending over the five years. We suggest that these types of strategies will be appealing this underspending group of retirees.

Keywords: optimal control, expected shortfall, decumulation, short term
JEL codes: G11, G22
AMS codes: 91G, 65N06, 65N12, 35Q93

1 Introduction

Defined Benefit (DB) plans are being phased out in favour of Defined Contribution (DC) plans, due to the reluctance of governments and corporations to take on the funding risk. When the DC plan member retires, she is faced with the problem of devising an asset allocation strategy and withdrawal schedule which minimizes the probability of running out of cash. Perhaps one of the first studies that addressed this decumulation problem was Bengen (1994). This gave rise to the ubiquitous *four per cent rule*, i.e. a 65 year old retire should withdraw 4% (real) of her initial assets each year. If the retirement portfolio was invested 50% in stocks and 50% in bonds, then this decumulation strategy would have never run out of cash over any rolling 30 year historical period.

Of course, since this 1994 paper, there have been numerous other studies and proposed decumulation policies. The current literature on decumulation strategies for DC plan holders is succinctly summarized in Bernhardt and Donnelly (2018).

If should be noted that it is often suggested (at least in the academic literature) that DC plan participants should buy annuities in order to generate cash flows in retirement and hedge longevity risk, but they rarely do so in practice (Peijnenburg et al., 2016). MacDonald et al. (2013) put forth several arguments to explain why this behavior may be entirely rational.

It is commonplace to assume that DC plan investors desire to withdraw assets during the their decumulation stage at a constant (real) rate in order to fund necessary expenses (Bengen, 1994).

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Some flexibility can be added to improve cash withdrawal efficiency, but usually with a minimum amount of withdrawal to meet these expenses (Pfau, 2015). A typical flexible withdrawal scenario is modelled in Forsyth (2021a). It is assumed that a 65 year old retiree plans to decumulate his savings over a 30 year time horizon, with minimum required cash flows each year. The objective is to determine the cash withdrawal policy, and asset allocation strategy, which minimizes the left tail risk (as measured by expected shortfall), assuming a 30 year decumulation. Since living to age 95 is well beyond the median life expectancy of a 65 year old, this is regarded as a conservative strategy. However, recent work calls into question this basic model of DC plan decumulation.

Browning et al. (2016) note that various studies (De Nardi et al., 2009; Smith et al., 2009; Love et al., 2009; Poterba et al., 2011; Browning et al., 2015) indicate that the value of many retirees' financial assets actually remain constant or even increase over time. More recently, surveys conducted jointly by the Employee Benefit Research Institute[1] and BlackRock (Ackerly et al., 2021) verify this unexpected result:

> *"This was not what we expected to find: on average across all wealth levels, most current retirees still had 80% of their pre-retirement savings after almost two decades of retirement according to research conducted jointly with the Employee Benefit Research Institute... One-third even grew their assets over the course of retirement."*

One possible explanation for this observation is that these surveys are focused on retirees who left the workforce 20 years ago. In this case, it is plausible to assume that many of these retirees used a combination of government benefits and Defined Benefit (DB) pensions (more commonplace two decades ago) to cover necessary expenses, regarding their financial assets as a source of discretionary spending. This is confirmed in Bannerje (2021), where surveys show that for most retirees, the ratio of non-discretionary income to guaranteed income drops sharply to one after retirement, and remains very close to unity after the age of 70. However, Bannerje (2021) notes that it appears that the retirees have adjusted their lifestyle (i.e. their fixed expenses) to match their guaranteed cash flows. In other words, it seems that retirees are more flexible about their spending then previously thought, and reduce it to avoid drawing down their assets.

This is also consistent with the older study in the Canadian context (Hamilton, 2001), where retirees appear to be *asset rich, income poor*, and seem to lead comfortable lives with spending at a level of 50% of pre-retirement income.

Browning et al. (2016) notes that the *retirement consumption gap* is particularly noticeable for those with financial assets at the median level and above. Browning et al. (2016) suggests that these assets are being held as a reserve against unexpected medical expenses, but notes that this forgone spending is much larger than average medical expenses actually observed for retirees. Of course, high expense, low probability medical expenses may require a large reserve, but, as pointed out in Browning et al. (2016), this would be more rationally hedged using some form of insurance product.

Since Canada has a comprehensive public health care system, Canadian retirees are shielded from ruinous health expenses. Yet Hamilton (2001) finds that senior Canadian couples 85 and older either save or give away about 25% of their income.

This lack of spending does not appear to be due to a desire to leave bequests. Taylor et al. (2018) cite surveys which show that 48% of retirees view maintaining a comfortable standard of living as their main financial goal, while only 3% view leaving an estate as their primary goal. Taylor et al. (2018) also posits behavioural biases as a possible explanation for the decumulation paradox (i.e. underspending). This is consistent with the behavioural lifecycle model (Shefrin and

[1] `https://www.ebri.org/`

Thaler, 1988; Thaler, 1990), whereby different asset classes are not fungible. A classic example is real estate. Even though retirees may own their own homes outright, which may have increased in value considerably, they are reluctant to regard this asset as a financial hedge, even if real estate can be monetized readily using a reverse mortgage.

Another study (Ventura and Hoiroka, 2020), finds that more than 40% of elderly Italians are continuing to accumulate (real) wealth, and that more than 80% are generating positive amounts of saving. In contrast to the studies in the US, it appears that bequest and precautionary saving are the primary motivations to continue to accumulate wealth.

To summarize, it appears that many current retirees underspend their retirement savings by a considerable margin, compared to what would usually be expected based on the standard lifecycle model. Basically, this appears to be due to the fact that retirees seem reluctant to spend down their financial assets, even though they may be increasing in (real) value. These retirees are not well served by the usual lifecycle spending rules advocated in the literature.

The objective of this article is to suggest a decumulation strategy which may be more appealing to these underspending retirees. As an example, we consider a retiree with a stream of government benefits, DB plan payments, and other annuity-like cash flows which pay for the minimum basic expenses. The retiree also owns real estate, which is regarded as a hedge against medical expenses (i.e. long term care). In the event that this medical expense does not materialize, then the real estate is used as a bequest, or as a hedge against extreme left tail investment risk. In an act of *mental accounting*, the retiree regards their other financial assets as a source of discretionary spending. We recognize that these retirees are in a somewhat fortunate position. However, based on the surveys quoted above, it would appear that this situation (at least for current retirees) is not that unusual.

It has been argued previously (Forsyth, 2021b) that using a multi-stage approach to devising decumulation strategies may be more appealing to retirees. In Forsyth (2021b) it is suggested that applying an optimal strategy for the first 15 years of retirement (i.e. to the age of 80), allows retirees the re-evaluate their financial position after 15 years. If investment returns are good, then the now 80 year-old can continue to manage his investments. If returns are not so good, annuities become more attractive for 80-year old retirees, due to the mortality credits earned, and hence purchase of an annuity at this point may be a viable strategy to hedge longevity risk.

Continuing with this idea, in this paper, we consider even shorter time horizons, to make our decumulation strategy more attractive to reluctant spenders. We consider relatively short investment horizons, in this case five years. At the end of five years, the retiree can reevaluate her priorities, and start the decumulation strategy over again. This is clearly a sub-optimal strategy for long periods, but may be more acceptable for our target underspenders. Indeed, a major advantage of this short time horizon is that retirees can target higher spending during the early years of retirement, while being confident that their financial assets are little diminished in real terms.

Withdrawals from the retirement portfolio are assumed to occur quarterly. We place maximum and minimum bounds on each withdrawal. We consider a measure of reward to be the total withdrawals over the five year time horizon. Note that all quantities are real, and we do not discount the total withdrawals, since real interest rates at present are approximately zero. We set the minimum withdrawal bound to be zero, which then allows spending to be delayed during poor market conditions. This ameliorates sequence of return risk. Our thinking here is that this retiree group has guaranteed cash flows from other sources (government benefits, DB plans) which covers the required minimum cash flows. This financial *bucket* is used for discretionary spending, and these retirees are flexible in the timing of withdrawals, during the five year time horizon.

It is interesting to observe that a flexible spending strategy in retirement has been advocated previously,

> *"If we have a good year, we take a trip to China,...if we have a bad year, we stay home and play canasta."* Retired Mathematics professor Peter Ponzo, discussing his DC plan investment and withdrawal strategy. [2]

As we shall see, these *canasta-type* strategies are in fact optimal.

As a measure of risk we consider two possibilities. The first choice is Expected Shortfall (ES), which is simply the mean of the worst five per cent of the terminal wealth values, measured at five years. It could be argued that ES is unduly pessimistic, hence we also investigate an alternate choice of risk, Linear Shortfall (LS) with respect to a fixed target terminal wealth. LS is perhaps a bit more intuitive than ES, and a bit more aggressive in terms of the median terminal wealth. These measures of left tail risk should be appealing to those retirees who are focused on wealth preservation.

We consider this decumulation problem to be a problem in optimal stochastic control. We model the investment portfolio in terms of a stock index and a constant maturity bond index. It is assumed that the real (i.e. inflation adjusted) stock and bond indexes follow a jump diffusion process. The parameters for the jump processes are determined by fitting to 95 years of market data. We term the market where prices are determined by the parametric jump diffusion processes as the *synthetic* market. The controls for this problem are the withdrawal amount each quarter, and the allocation to the stock and bond indexes. We formulate this multi-objective optimization problem in terms of the risk and reward measures discussed above. We use a scalarization technique, coupled with a numerical dynamic programming approach, to determine optimal withdrawal (each quarter) as well as the (dynamic) asset allocation strategy.

We compute and store the optimal controls in the synthetic market. We then test these controls by using stationary block bootstrap resampling (Politis and Romano, 1994; Dichtl et al., 2016; Forsyth and Vetzal, 2019) of the actual historical data. We term the market which is driven by bootstrap resampled data to be the *historical* market. The efficient frontiers for the Expected Withdrawals (EW), linear shortfall (LS) are robust, in the sense that the efficient EW-LS frontiers in the synthetic and historical market are very similar. In the Expected Withdrawal, Expected Shortfall (ES) case, the EW-ES frontiers are slightly worse in the historical market compared to the synthetic market.

We believe that these strategies will be appealing to our underspending retiree. The risk of ending up (after five years) with a significantly smaller investment portfolio (in real terms) is very small. Yet the probability of a significant amount of total withdrawals over the five years is large.

2 Formulation

We assume that the investor has access to two funds: a broad market stock index fund and a constant maturity bond index fund.

The investment horizon is T. Let S_t and B_t respectively denote the real (inflation adjusted) *amounts* invested in the stock index and the bond index respectively. In general, these amounts will depend on the investor's strategy over time, as well as changes in the real unit prices of the assets. In the absence of an investor determined control (i.e. cash withdrawals or rebalancing), all changes in S_t and B_t result from changes in asset prices. We model the stock index as following a jump diffusion.

[2] `https://www.theglobeandmail.com/report-on-business/math-prof-tests-investing-formulas-strategies/article22397218/` Ponzo took half his DC plan assets and purchased an annuity, and put the other half into a discretionary spending DC account. However, Ponzo retired in 1993, when annuity rates were much higher than now. Perhaps the equivalent strategy today would utilize a tontine account to harvest mortality credits.

In addition, we follow the usual practitioner approach and directly model the returns of the constant maturity bond index as a stochastic process, see for example Lin et al. (2015); MacMinn et al. (2014). As in MacMinn et al. (2014), we assume that the constant maturity bond index follows a jump diffusion process as well.

Let $S_{t^-} = S(t-\epsilon), \epsilon \to 0^+$, i.e. t^- is the instant of time before t, and let ξ^s be a random number representing a jump multiplier. When a jump occurs, $S_t = \xi^s S_{t^-}$. Allowing for jumps permits modelling of non-normal asset returns. We assume that $\log(\xi^s)$ follows a double exponential distribution (Kou, 2002; Kou and Wang, 2004). If a jump occurs, u^s is the probability of an upward jump, while $1-u^s$ is the chance of a downward jump. The density function for $y = \log(\xi^s)$ is

$$f^s(y) = u^s \eta_1^s e^{-\eta_1^s y} \mathbf{1}_{y \geq 0} + (1-u^s)\eta_2^s e^{\eta_2^s y} \mathbf{1}_{y<0} \ . \tag{2.1}$$

We also define

$$\gamma_\xi^s \ = \ E[\xi^s - 1] = \frac{u^s \eta_1^s}{\eta_1^s - 1} + \frac{(1-u^s)\eta_2^s}{\eta_2^s + 1} - 1 \ . \tag{2.2}$$

In the absence of control, S_t evolves according to

$$\frac{dS_t}{S_{t^-}} \ = \ \left(\mu^s - \lambda_\xi^s \gamma_\xi^s\right) \, dt + \sigma^s \, dZ^s + d\left(\sum_{i=1}^{\pi_t^s} (\xi_i^s - 1) \right) , \tag{2.3}$$

where μ^s is the (uncompensated) drift rate, σ^s is the volatility, dZ^s is the increment of a Wiener process, π_t^s is a Poisson process with positive intensity parameter λ_ξ^s, and ξ_i^s are i.i.d. positive random variables having distribution (2.1). Moreover, ξ_i^s, π_t^s, and Z^s are assumed to all be mutually independent.

Similarly, let the amount in the bond index be $B_{t^-} = B(t-\epsilon), \epsilon \to 0^+$. In the absence of control, B_t evolves as

$$\frac{dB_t}{B_{t^-}} \ = \ \left(\mu^b - \lambda_\xi^b \gamma_\xi^b + \mu_c^b \mathbf{1}_{\{B_{t^-} < 0\}}\right) \, dt + \sigma^b \, dZ^b + d\left(\sum_{i=1}^{\pi_t^b} (\xi_i^b - 1) \right) , \tag{2.4}$$

where the terms in equation (2.4) are defined analogously to equation (2.3). In particular, π_t^b is a Poisson process with positive intensity parameter λ_ξ^b, and ξ_i^b has distribution

$$f^b(y = \log \xi^b) = u^b \eta_1^b e^{-\eta_1^b y} \mathbf{1}_{y \geq 0} + (1-u^b)\eta_2^b e^{\eta_2^b y} \mathbf{1}_{y<0} \ , \tag{2.5}$$

and $\gamma_\xi^b = E[\xi^b - 1]$. ξ_i^b, π_t^b, and Z^b are assumed to all be mutually independent. The term $\mu_c^b \mathbf{1}_{\{B_{t^-} < 0\}}$ in equation (2.4) represents the extra cost of borrowing (the spread).

The diffusion processes are correlated, i.e. $dZ^s \cdot dZ^b = \rho_{sb} dt$. The stock and bond jump processes are assumed mutually independent. See Forsyth (2020b) for justification of the assumption of stock-bond jump independence.

We define the investor's total wealth at time t as

$$\text{Total wealth } \equiv W_t = S_t + B_t. \tag{2.6}$$

We impose the constraints that (assuming solvency) shorting stock and using leverage (i.e. borrowing) are not permitted. In the event of insolvency (due to withdrawals), the portfolio is liquidated, trading ceases and debt accumulates at the borrowing rate. Due to the short time horizon and maximum withdrawal constraint, insolvency is improbable.

3 Notational conventions

Consider a set of discrete withdrawal/rebalancing times $\mathcal{T}$

$$\mathcal{T} = \{t_0 = 0 < t_1 < t_2 < \ldots < t_M = T\} \tag{3.1}$$

where we assume that $t_i - t_{i-1} = \Delta t = T/M$ is constant for simplicity. To avoid subscript clutter, in the following, we will occasionally use the notation $S_t \equiv S(t), B_t \equiv B(t)$ and $W_t \equiv W(t)$. Let the inception time of the investment be $t_0 = 0$. We let $\mathcal{T}$ be the set of withdrawal/rebalancing times, as defined in equation (3.1). At each rebalancing time t_i, $i = 0, 1, \ldots, M-1$, the investor (i) withdraws an amount of cash q_i from the portfolio, and then (ii) rebalances the portfolio. At $t_M = T$, the portfolio is liquidated. In the following, given a time dependent function $f(t)$, then we will use the shorthand notation

$$f(t_i^+) \equiv \lim_{\epsilon \to 0^+} f(t_i + \epsilon) \quad ; \quad f(t_i^-) \equiv \lim_{\epsilon \to 0^+} f(t_i - \epsilon) \ . \tag{3.2}$$

We assume that there are no taxes or other transaction costs, so that the condition

$$W(t_i^+) = W(t_i^-) - q_i \quad ; \quad t_i \in \mathcal{T} \tag{3.3}$$

holds. Typically, DC plan savings are held in a tax advantaged account, with no taxes triggered by rebalancing. With infrequent (e.g. yearly) rebalancing, we also expect transaction costs to be small, and hence can be ignored. It is possible to include transaction costs, but at the expense of increased computational cost (van Staden et al., 2018).

We denote by $X(t) = (S(t), B(t))$, $t \in [0,T]$, the multi-dimensional controlled underlying process, and by $x = (s, b)$ the realized state of the system. Let the rebalancing control $p_i(\cdot)$ be the fraction invested in the stock index at the rebalancing date t_i, i.e.

$$p_i\left(X(t_i^-)\right) = p\left(X(t_i^-), t_i\right) \quad = \quad \frac{S(t_i^+)}{S(t_i^+) + B(t_i^+)} \ . \tag{3.4}$$

Let the withdrawal control $q_i(\cdot)$ be the amount withdrawn at time t_i, i.e. $q_i\left(X(t_i^-)\right) = q\left(X(t_i^-), t_i\right)$. Formally, the controls depend on the state of the investment portfolio, before the rebalancing occurs, i.e. $p_i(\cdot) = p\left(X(t_i^-), t_i)\right) = p\left(X_i^-, t_i\right)$, and $q_i(\cdot) = q\left(X(t_i^-), t_i)\right) = q\left(X_i^-, t_i\right)$, $t_i \in \mathcal{T}$, where $\mathcal{T}$ is the set of rebalancing times.

However, it will be convenient to note that in our case, we find the optimal control $p_i(\cdot)$ amongst all strategies with constant wealth (after withdrawal of cash). Hence, with some abuse of notation, we will now consider $p_i(\cdot)$ to be function of wealth after withdrawal of cash

$$\begin{aligned}
p_i(\cdot) &= p(W(t_i^+), t_i) \\
W(t_i^+) &= S(t_i^-) + B(t_i^-) - q_i(\cdot) \\
S(t_i^+) &= S_i^+ = p_i(W_i^+)\ W_i^+ \\
B(t_i^+) &= B_i^+ = (1 - p_i(W_i^+))\ W_i^+ \ .
\end{aligned} \tag{3.5}$$

Remark 3.1 (Control depends on wealth only)**.** *Note that the control for $p_i(\cdot)$ depends only W_i^+. Since $p_i(\cdot) = p_i(W_i^- - q_i)$, then it follows that*

$$q_i(\cdot) = q_i(W_i^-) \tag{3.6}$$

which is proven in Forsyth (2021a).

A control at time t_i, is then given by the pair $(q_i(\cdot), p_i(\cdot))$ where the notation $(\cdot)$ denotes that the control is a function of the state.

Let $\mathcal{Z}$ represent the set of admissible values of the controls $(q_i(\cdot), p_i(\cdot))$. We impose no-shorting, no-leverage constraints (assuming solvency). We also impose maximum and minimum values for the withdrawals. We apply the constraint that in the event of insolvency due to withdrawals $(W(t_i^+) < 0)$, trading ceases and debt (negative wealth) accumulates at the appropriate bond rate of return (including a spread). We also specify that the stock assets are liquidated at $t = t_M$.

More precisely, let W_i^+ be the wealth after withdrawal of cash, then define

$$\mathcal{Z}_q = \begin{cases} [q_{\min}, q_{\max}] & t \in \mathcal{T} \; ; \; t \neq t_M \\ \{0\} & t = t_M \end{cases} , \tag{3.7}$$

$$\mathcal{Z}_p(W_i^+, t_i) = \begin{cases} [0,1] & W_i^+ > 0 \; ; \; t_i \in \mathcal{T} \; ; \; t_i \neq t_M \\ \{0\} & W_i^+ \leq 0 \; ; \; t_i \in \mathcal{T} \; ; \; t_i \neq t_M \\ \{0\} & t_i = t_M \end{cases} . \tag{3.8}$$

(3.9)

The set of admissible values for $(q_i, p_i), t_i \in \mathcal{T}$, can then be written a

$$(q_i, p_i) \in \mathcal{Z}(W_i^+, t_i) = \mathcal{Z}_q \times \mathcal{Z}_p(W_i^+, t_i) . \tag{3.10}$$

For implementation purposes, we have written equation (3.10) in terms of the wealth after withdrawal of cash. However, we remind the reader that since $W_i^+ = W_i^- - q$, the controls are formally a function of the state $X(t_i^-)$ before the control is applied.

The admissible control set $\mathcal{A}$ can then be written as

$$\mathcal{A} = \left\{ (q_i, p_i)_{0 \leq i \leq M} : (p_i, q_i) \in \mathcal{Z}(W_i^+, t_i) \right\} \tag{3.11}$$

An admissible control $\mathcal{P} \in \mathcal{A}$, where $\mathcal{A}$ is the admissible control set, can be written as,

$$\mathcal{P} = \{(q_i(\cdot), p_i(\cdot)) \; : \; i = 0, \ldots, M\} . \tag{3.12}$$

We also define $\mathcal{P}_n \equiv \mathcal{P}_{t_n} \subset \mathcal{P}$ as the tail of the set of controls in $[t_n, t_{n+1}, \ldots, t_M]$, i.e.

$$\mathcal{P}_n = \{(q_n(\cdot), p_n(\cdot)) \ldots, (p_M(\cdot), q_M(\cdot))\} . \tag{3.13}$$

For notational completeness, we also define the tail of the admissible control set $\mathcal{A}_n$ as

$$\mathcal{A}_n = \left\{ (q_i, p_i)_{n \leq i \leq M} : (q_i, p_i) \in \mathcal{Z}(W_i^+, t_i) \right\} \tag{3.14}$$

so that $\mathcal{P}_n \in \mathcal{A}_n$.

4 Risk and reward

4.1 Risk: definition of expected shortfall (ES)

Let $g(W_T)$ be the probability density function of wealth W_T at $t = T$. Suppose

$$\int_{-\infty}^{W_\alpha^*} g(W_T) \, dW_T = \alpha, \tag{4.1}$$

i.e. $Pr[W_T > W^*_\alpha] = 1 - \alpha$. We can interpret W^*_α as the Value at Risk (VAR) at level α[3]. The Expected Shortfall (ES) at level α is then

$$\text{ES}_\alpha = \frac{\int_{-\infty}^{W^*_\alpha} W_T \; g(W_T) \; dW_T}{\alpha}, \tag{4.2}$$

which is the mean of the worst α fraction of outcomes. Typically $\alpha \in \{.01, .05\}$. The definition of ES in equation (4.2) uses the probability density of the final wealth distribution, not the density of *loss*. Hence, in our case, a larger value of ES (i.e. a larger value of average worst case terminal wealth) is desired. The negative of ES is commonly referred to as Conditional Value at Risk (CVAR).

Define $X_0^+ = X(t_0^+), X_0^- = X(t_0^-)$. Given an expectation under control $\mathcal{P}$, $E_{\mathcal{P}}[\cdot]$, as noted by Rockafellar and Uryasev (2000), ES_α can be alternatively written as

$$\text{ES}_\alpha(X_0^-, t_0^-) \;\; = \;\; \sup_{W^*} E_{\mathcal{P}_0}^{X_0^+, t_0^+}\left[W^* + \frac{1}{\alpha} \min(W_T - W^*, 0) \right] . \tag{4.3}$$

The admissible set for W^* in equation (4.3) is over the set of possible values for W_T.

The notation $\text{ES}_\alpha(X_0^-, t_0^-)$ emphasizes that ES_α is as seen at (X_0^-, t_0^-). In other words, this is the pre-commitment ES_α. A strategy based purely on optimizing the pre-commitment value of ES_α at time zero is *time-inconsistent*, hence has been termed by many as *non-implementable*, since the investor has an incentive to deviate from the time zero pre-commitment strategy at $t > 0$. However, in the following, we will consider the pre-commitment strategy merely as a device to determine an appropriate level of W^* in equation (4.3). If we fix W^* $\forall t > 0$, then this strategy is the induced time consistent strategy (Strub et al., 2019), hence is implementable. We delay further discussion of this subtle point to later sections.

4.2 Risk: Linear Shortfall (LS)

Another possibility for a measure of risk is linear shortfall (LS) with shortfall target W^*,

$$\text{LS}_{W^*} \;\; = \;\; E[\min(W_T - W^*, 0)] \; . \tag{4.4}$$

Note that, if

$$E[\mathbf{1}_{W_T < W^*}] \;\; = \;\; \alpha \; , \tag{4.5}$$

then

$$\text{ES}_\alpha \;\; = \;\; W^* + \frac{\text{LS}^*_W}{\alpha} \; . \tag{4.6}$$

4.3 A measure of reward: expected total withdrawals (EW)

We will use expected total withdrawals as a measure of reward in the following. More precisely, we define EW (expected withdrawals) as

$$\text{EW}(X_0^-, t_0^-) = E_{\mathcal{P}_0}^{X_0^+, t_0^+}\left[\sum_{i=0}^{M} q_i \right] . \tag{4.7}$$

Note that there is no discounting term in equation (4.7), since all quantities are real, and the current real short term rate is approximately zero (or even negative).

[3] In practice, the negative of W^*_α is often the reported VAR.

5 Problem EW-ES

Since expected withdrawals (EW) and expected shortfall (ES) are conflicting measures, we use a scalarization technique to find the Pareto points for this multi-objective optimization problem. Informally, for a given scalarization parameter $\kappa > 0$, we seek to find the control $\mathcal{P}_0$ that maximizes

$$\text{EW}(X_0^-, t_0^-) + \kappa \ \text{ES}_\alpha(X_0^-, t_0^-) \ . \tag{5.1}$$

More precisely, we define the pre-commitment EW-ES problem ($PCES_{t_0}(\kappa)$) problem in terms of the value function $J(s,b,t_0^-)$

$$(PCES_{t_0}(\kappa)): \qquad J\left(s,b,t_0^-\right) = \sup_{\mathcal{P}_0 \in \mathcal{A}} \sup_{W^*} \Bigg\{ E_{\mathcal{P}_0}^{X_0^+,t_0^+} \Bigg[\sum_{i=0}^{M} q_i \ + \ \kappa \bigg(W^* + \frac{1}{\alpha} \min(W_T - W^*, 0) \bigg) \bigg| X(t_0^-) = (s,b) \Bigg] \Bigg\} \tag{5.2}$$

$$\text{subject to} \quad \begin{cases} (S_t, B_t) \text{ follow processes (2.3) and (2.4)}; \ \ t \notin \mathcal{T} \\ W_\ell^+ = S_\ell^- + B_\ell^- - q_\ell \,; \ X_\ell^+ = (S_\ell^+, B_\ell^+) \\ S_\ell^+ = p_\ell(\cdot) W_\ell^+ \,; \ B_\ell^+ = (1 - p_\ell(\cdot)) W_\ell^+ \\ (q_\ell(\cdot), p_\ell(\cdot)) \in \mathcal{Z}(W_\ell^+, t_\ell) \\ \ell = 0, \ldots, M \ ; \ t_\ell \in \mathcal{T} \end{cases} . \tag{5.3}$$

Interchange the sup sup$(\cdot)$ in equation (5.2), so that value function $J\left(s,b,t_0^-\right)$ can be written as

$$J\left(s,b,t_0^-\right) = \sup_{W^*} \sup_{\mathcal{P}_0 \in \mathcal{A}} \Bigg\{ E_{\mathcal{P}_0}^{X_0^+,t_0^+} \Bigg[\sum_{i=0}^{M} q_i \ + \ \kappa \bigg(W^* + \frac{1}{\alpha} \min(W_T - W^*, 0) \bigg) \bigg| X(t_0^-) = (s,b) \Bigg] \Bigg\} . \tag{5.4}$$

Noting that the inner supremum in equation (5.4) is a continuous function of W^*, and noting that the optimal value of W^* in equation (5.4) is bounded[4], then define

$$\mathcal{W}^*(s,b) = \operatorname*{arg\,max}_{W^*} \Bigg\{ \sup_{\mathcal{P}_0 \in \mathcal{A}} \Bigg\{ E_{\mathcal{P}_0}^{X_0^+,t_0^+} \Bigg[\sum_{i=0}^{M} q_i \ + \ \kappa \bigg(W^* + \frac{1}{\alpha} \min(W_T - W^*, 0) \bigg) \bigg| X(t_0^-) = (s,b) \Bigg] \Bigg\} . \tag{5.5}$$

We refer the reader to Forsyth (2020a) for an extensive discussion concerning pre-commitment and time consistent ES strategies. We summarize the relevant results from that research here. Denote the investor's initial wealth at t_0 by W_0^-. Then we have the following result.

Proposition 5.1 (Pre-commitment strategy equivalence to a time consistent policy for an alternative objective function)**.** *The pre-commitment EW-ES strategy $\mathcal{P}^*$ determined by solving $J(0, W_0, t_0^-)$ (with $\mathcal{W}^*(0, W_0^-)$ from equation (5.5)) is the time consistent strategy for the equivalent problem*

[4] This is the same as noting that a finite value at risk exists. This easily shown, assuming $0 < \alpha < 1$, since our investment strategy uses no leverage and no-shorting.

TCEQ (with fixed $\mathcal{W}^*(0,W_0^-)$*), with value function* $\tilde{J}(s,b,t)$ *defined by*

$$(TCEQ_{t_n}(\kappa/\alpha)) : \qquad \tilde{J}\left(s,b,t_n^-\right) = \sup_{\mathcal{P}_n \in \mathcal{A}} \Bigg\{ E_{\mathcal{P}_n}^{X_n^+,t_n^+} \Bigg[\sum_{i=n}^{M} q_i \ + \ \frac{\kappa}{\alpha} \min(W_T - \mathcal{W}^*(0,W_0^-),0) \Bigg| X(t_n^-) = (s,b) \Bigg] \Bigg\} . \tag{5.6}$$

Proof. This follows similar steps as in Forsyth (2020a), proof of Proposition 6.2, with the exception that the reward in Forsyth (2020a) is expected terminal wealth, while here the reward is total withdrawals. □

Remark 5.1 (An Implementable Strategy)**.** *Given an initial level of wealth* W_0^- *at* t_0*, then the optimal control*[5] *for the pre-commitment problem (5.2) is the same optimal control for the time consistent problem*[6] $(TCEQ_{t_n}(\kappa/\alpha))$ *(5.6),* $\forall t > 0$*. Hence we can regard problem* $(TCEQ_{t_n}(\kappa/\alpha))$ *as the* EW-ES induced time consistent strategy*. Thus, the induced strategy is implementable, in the sense that the investor has no incentive to deviate from the strategy computed at time zero, at later times (Forsyth, 2020a).*

Remark 5.2 (EW-ES Induced Time Consistent Strategy)**.** *In the following, we will consider the actual strategy followed by the investor for any* $t > 0$ *as given by the induced time consistent strategy* $(TCEQ_{t_n}(\kappa/\alpha))$ *in equation (5.6), with a fixed value of* $\mathcal{W}^*(0,W_0^-)$*, which is identical to the EW-ES strategy at time zero. Hence, we will refer to this strategy in the following as the EW-ES strategy, with the understanding that this refers to strategy* $(TCEQ_{t_n}(\kappa/\alpha))$ *for any* $t > 0$*.*

6 Problem EW-LS

Now, using LS as the risk measure, we use a scalarization parameter $\hat{\kappa} > 0$ to determine the Pareto optimal points for the problem with objective function

$$\text{EW}(X_0^-,t_0^-) + \hat{\kappa}\ \text{LS}_{W^*}(X_0^-,t_0^-) \ . \tag{6.1}$$

We define the time-consistent EW-LS problem ($EWLS(\hat{\kappa})$) in terms of the value function $\hat{J}(s,b,t_0^-)$

$$(EWLS_{t_0}(\hat{\kappa})) : \qquad J\left(s,b,t_0^-\right) = \sup_{\mathcal{P}_0 \in \mathcal{A}} \Bigg\{ E_{\mathcal{P}_0}^{X_0^+,t_0^+} \Bigg[\sum_{i=0}^{M} q_i \ + \ \hat{\kappa}\ \min(W_T - W^*, 0) \Bigg| X(t_0^-) = (s,b) \Bigg] \Bigg\} , \tag{6.2}$$

with the same constraints as in equation (5.3).

[5]To be perfectly precise here, in the event that the control is non-unique, we impose a tie-breaking strategy to generate a unique control.

[6]Assuming that the same tie breaking strategy is used as for the pre-commitment problem.

7 Formulation as a Dynamic Program

7.1 Formulation for optimal expected-withdrawals expected-shortfall (EW-ES) strategy

We use the method in Forsyth (2020a) to solve problem (5.4). We write equation (5.4) as

$$J(s,b,t_0^-) \quad = \quad \sup_{W^*} V(s,b,0^-) \ , \tag{7.1}$$

where the auxiliary function $V(s,b,W^*,t)$ is defined as

$$V(s,b,W^*,t_n^-) \quad = \quad \sup_{\mathcal{P}_n \in \mathcal{A}_n} \left\{ E_{\mathcal{P}_n}^{\hat{X}_n^+, t_n^+} \left[\sum_{i=n}^{M} q_i + \kappa \left(W^* + \frac{1}{\alpha} \min((W_T - W^*),0) \right) \middle| \hat{X}(t_n^-) = (s,b,W^*) \right] \right\} . \tag{7.2}$$

$$\text{subject to} \qquad \begin{cases} (S_t, B_t) \text{ follow processes (2.3) and (2.4)}; \quad t \notin \mathcal{T} \\ W_\ell^+ = S_\ell^- + B_\ell^- - q_\ell \, ; \ X_\ell^+ = (S_\ell^+, B_\ell^+) \\ S_\ell^+ = p_\ell(\cdot) W_\ell^+ \, ; \ B_\ell^+ = (1 - p_\ell(\cdot)) W_\ell^+ \\ (q_\ell(\cdot), p_\ell(\cdot)) \in \mathcal{Z}(W_\ell^+, t_\ell) \\ \ell = n, \ldots, M \ ; \ t_\ell \in \mathcal{T} \end{cases} . \tag{7.3}$$

We have now decomposed the original problem (5.4) into two steps

- For given initial cash W_0, and a fixed value of W^*, solve problem (7.2) using dynamic programming (see Appendix A and Forsyth (2021a) for details) to determine $V(0,W_0,W^*,0^-)$.
- Solve problem (5.4) by maximizing over W^*

$$J(0,W_0,0^-) \quad = \quad \sup_{W^*} V(0,W_0,W^*,0^-) \ . \tag{7.4}$$

7.2 Formulation for optimal expected-withdrawals linear-shortfall (EW-LS) strategy

Problem ($EWLS_{t_0}(\hat{\kappa})$) is essentially a special case of Problem ($PCES_{t_0}(\kappa)$), with W^* fixed. Define

$$\hat{V}(s,b,t_n^-) \quad = \quad \sup_{\mathcal{P}_n \in \mathcal{A}_n} \left\{ E_{\mathcal{P}_n}^{\hat{X}_n^+, t_n^+} \left[\sum_{i=n}^{M} q_i + \hat{\kappa} \left(\min((W_T - W^*),0) \right) \middle| \hat{X}(t_n^-) = (s,b,W^*) \right] \right\} , \tag{7.5}$$

with constraints (7.3). We solve for $\hat{V}(s,b,t)$ using dynamic programming, as in Section 7.1, noting the trivial identity

$$\hat{J}(0,W_0,0^-) \quad = \quad \hat{V}(0,W_0,0^-) \ . \tag{7.6}$$

7.3 Controls for EW-ES and EW-LS

From the definitions (5.2) and (6.2) and Proposition 5.1 we have the following result

Proposition 7.1 (Condition for identical controls)**.** *Suppose we solve problem (5.2) with given values of* (κ, α)*, and we solve problem (6.2) with given values of* $(\hat{\kappa}, W^*)$*. If the solution to problem (6.2) is such that*

$$E[\mathbf{1}_{W_T < W^*}] = \alpha \ ; \ \hat{\kappa} = \frac{\kappa}{\alpha} \ , \tag{7.7}$$

and the controls for problem (5.2) are unique, then the controls for problems (5.2) and (6.2) are identical.[7]

8 Continuous withdrawal/rebalancing limit

In order to develop some intuition about the nature of the optimal controls, we will examine the limit as the rebalancing interval becomes vanishingly small.

Proposition 8.1 (Bang-bang withdrawal control in the continuous withdrawal limit)**.** *Assume that*

- *the stock and bond processes follow (2.3) and (2.4),*
- *the portfolio is continuously rebalanced, and withdrawals occur at a continuous (finite) rate* $\hat{q} \in [\hat{q}_{\min}, \hat{q}_{\max}]$*,*
- *the HJB equation for the EW-ES and the EW-LS problem in the continuous rebalancing limit has bounded derivatives w.r.t. total wealth,*
- *in the event of ties for the control* $\hat{q}$*, the smallest withdrawal is selected,*

then the optimal withdrawal control $\hat{q}^*(\cdot)$ *for the EW-ES problem* $(PCES_{t_0}(\kappa))$ *and for the EW-LS problem* $(EWLS_{t_0}(\hat{\kappa}))$ *is bang-bang,* $\hat{q}^* \in \{\hat{q}_{\min}, \hat{q}_{\max}\}$*.*

Proof. This follows the same steps as in Forsyth (2021a). □

Remark 8.1 (Bang-bang control for discrete rebalancing/withdrawals)**.** *Proposition 8.1 suggests that, for sufficiently small rebalancing intervals, we can expect the optimal q control (finite withdrawal amount) to be bang-bang, i.e. it is only optimal to withdraw either the maximum amount* $q_{\max}$ *or the minimum amount* $q_{\min}$*. However, it is not clear that this will continue to be true for the case of quarterly rebalancing (which we specify in our numerical examples), and finite amount controls q. In fact, we do observe that the finite amount control q is very close to bang-bang in our numerical experiments, even for quarterly rebalancing. We term this control to be* quasi-bang-bang.

9 Numerical algorithms and stabilization

A brief overview of the numerical algorithms is described in Appendix A, along with a numerical convergence verification.

[7] A unique control can always be defined by specifying a tie-breaking strategy.

9.1 Stabilization

If $W_t \gg W^*$, and $t \to T$, then $Pr[W_T < W^*] \simeq 0$ (recall that W^* is fixed for problem $(TCEQ_{t_n}(\kappa/\alpha))$ (5.6) . In addition, for large values of W_t, the withdrawal will be capped at $q_{\max}$. In this case, the control only weakly effects the objective function. To avoid this ill-posedness for the controls, we changed the objective function (5.2) to

$$J\left(s,b,t_0^-\right) \;=\; \sup_{\mathcal{P}_0\in\mathcal{A}} \sup_{W^*} \Bigg\{ E_{\mathcal{P}_0}^{X_0^+,t_0^+} \Bigg[\sum_{i=0}^{M} q_i \;+\; \kappa \Bigg(W^* + \frac{1}{\alpha}\min(W_T - W^*, 0) \Bigg) \overbrace{+\epsilon W_T}^{stabilization} \Bigg| X(t_0^-) = (s,b) \Bigg] \Bigg\} . \tag{9.1}$$

We used the value $\epsilon = +10^{-6}$ in the following test cases. Using a positive value for ϵ has the effect of forcing the strategy to invest in stocks when W_t is very large, and $t \to T$, when the control problem is ill-posed. In other words, when the probability that W_T is less than W^* is very small, then the ES risk is practically zero, hence the investor might as well invest in risky assets. There is little to lose, and much to gain. Using this small value of $\epsilon = 10^{-6}$ gave the same results as $\epsilon = 0$ for the summary statistics, to four digits. This is simply because the states with very large wealth have low probability. However, this stabilization procedure produced smoother heat maps for large wealth values, without altering the summary statistics appreciably.

Similarly, we changed the objective function for Problem EW-LS problem ($EWLS(\hat{\kappa})$) to

$$J\left(s,b,t_0^-\right) \;=\; \sup_{\mathcal{P}_0\in\mathcal{A}} \Bigg\{ E_{\mathcal{P}_0}^{X_0^+,t_0^+} \Bigg[\sum_{i=0}^{M} q_i \;+\; \hat{\kappa}\; \min(W_T - W^*, 0) \overbrace{+\epsilon W_T}^{stabilization} \Bigg| X(t_0^-) = (s,b) \Bigg] \Bigg\} , \tag{9.2}$$

10 Data

We use data from the Center for Research in Security Prices (CRSP) on a monthly basis over the 1926:1-2020:12 period.[8] Our base case tests use the CRSP US 30 day T-bill for the bond asset and the CRSP value-weighted total return index for the stock asset. This latter index includes all distributions for all domestic stocks trading on major U.S. exchanges. All of these various indexes are in nominal terms, so we adjust them for inflation by using the U.S. CPI index, also supplied by CRSP. We use real indexes since investors funding retirement spending should be focused on real (not nominal) wealth goals.

We use the threshold technique (Mancini, 2009; Cont and Mancini, 2011; Dang and Forsyth, 2016) to estimate the parameters for the parametric stochastic process models. The data is inflation adjusted, so that all parameters reflect real returns. Table 10.1 shows the results of calibrating the models to the historical data. The correlation ρ_{sb} is computed by removing any returns which occur at times corresponding to jumps in either series, and then using the sample covariance. Further discussion of the validity of assuming that the stock and bond jumps are independent is given in Forsyth (2020b).

[8]More specifically, results presented here were calculated based on data from Historical Indexes, ©2020 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services (WRDS) was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third-party suppliers.

CRSP	μ^s	σ^s	λ^s	u^s	η_1^s	η_2^s	ρ_{sb}
	0.089124	0.14685	0.3263157	0.22580	4.36258	5.5335	0.08420
30-day T-bill	μ^b	σ^b	λ^b	u^b	η_1^b	η_2^b	ρ_{sb}
	0.0046	0.0130	0.5053	0.3958	65.8012	57.7929	0.08420

TABLE 10.1: *Estimated annualized parameters for double exponential jump diffusion model. Value-weighted CRSP index, 30-day T-bill index deflated by the CPI. Sample period 1926:1 to 2020:12.*

11 Investment scenario

Table 11.1 shows our base case investment scenario. We will use thousands as our units of wealth in the following. For example, a withdrawal of 40 per year corresponds to $40,000 per year (all values are real, i.e. inflation adjusted), with an initial wealth of 1000 ($1,000,000). Thus, a withdrawal of 40 per year would correspond to the use of the four per cent rule (Bengen, 1994).

We assume that the retiree has other benefits (or other DC plans) which are enough to provide for basic living expenses. We also assume that the retiree has discretionary DC plan holdings at retirement of $1,000,000.

For the EW-LS case, we use the fixed value of $W^* = 900$. In other words, we are targeting a real total wealth decumulation rate (over the five year horizon) of about 2% per year.

Investment horizon T (years)	5.0
Equity market index	CRSP Cap-weighted index (real)
Bond index	30-day T-bill (US) (real)
Initial portfolio value W_0	1000
Cash withdrawal/rebalancing times	$t = 0, 0.25, 0.50, \ldots, 4.75$
Maximum withdrawal (per quarter)	$q_{\max} = 25$
Minimum withdrawal (per quarter)	$q_{\min} = 0.0$
Equity fraction range	[0,1]
Borrowing spread μ_c^b	0.02
Rebalancing interval (years)	0.25
Fixed W^* (EW-LS)	900
α (EW-ES)	.05
Market parameters	See Table 10.1

TABLE 11.1: *Input data for examples. Monetary units: thousands of dollars.*

11.1 Synthetic market

We fit the parameters for the parametric stock and bond processes (2.3 - 2.4) as described in Section 10. We then compute and store the optimal controls based on the parametric market model. Finally, we compute various statistical quantities by using the stored control, and then carrying out Monte Carlo simulations, based on processes (2.3 - 2.4).

Data series	Optimal expected block size $\hat{b}$ (months)
Real 10-year Treasury index	50.6
Real CRSP value-weighted index	3.42

TABLE 11.2: *Optimal expected blocksize $\hat{b} = 1/v$ when the blocksize follows a geometric distribution $Pr(b = k) = (1 - v)^{k-1}v$. The algorithm in Patton et al. (2009) is used to determine $\hat{b}$. Historical data range 1926:1-2020:12.*

11.2 Historical market

We compute and store the optimal controls based on the parametric model (2.3-2.4) as for the synthetic market case. However, we compute statistical quantities using the stored controls, but using bootstrapped historical return data directly. We remind the reader that all returns are inflation adjusted. We use the stationary block bootstrap method (Politis and Romano, 1994; Politis and White, 2004; Patton et al., 2009; Dichtl et al., 2016). A crucial parameter is the expected blocksize. Sampling the data in blocks accounts for serial correlation in the data series. We use the algorithm in Patton et al. (2009) to determine the optimal blocksize for the bond and stock returns separately, see Table 11.2. We use a paired sampling approach to simultaneously draw returns from both time series. In this case, a reasonable estimate for the blocksize for the paired resampling algorithm would be about 2.0 years. We will give results for a range of blocksizes as a check on the robustness of the bootstrap results. Detailed pseudo-code for block bootstrap resampling is given in Forsyth and Vetzal (2019).

12 Constant weight, constant withdrawals

As a preliminary example, we consider a strategy whereby, each quarter, the investor (i) withdraws at a constant annual rate (fixed amount per quarter) and (ii) rebalances the portfolio to a constant equity weight. Table 12.1 shows the results for constant weight, constant withdrawal case, in the synthetic market. The constant withdrawal rate is 10 per quarter, which is annualized as 40 per year (consistent with the advice in Bengen (1994)). The largest value of ES (least risky) is for $p = 0.10$. This value of $ES = 737$, which is not particularly good, given that the initial investment is 1000.

Table 12.2 gives similar results for constant weight, constant withdrawals scenarios, based on the historical bootstrapped market. The historical results give the best value of $ES = 651$ for $p = 0.20$, which is significantly worse than the best result for the synthetic market ES.

Equity fraction p	$E[\sum_i q_i]/T$	ES (5%)	$Median[W_T]$
0.0	40	726.50	821.45
.10	40	736.90	857.95
.20	40	706.76	894.31
.30	40	664.76	930.87
.40	40	619.43	967.15
.50	40	572.98	1003.00
.60	40	526.17	1038.18
.70	40	479.29	1072.44
.80	40	432.84	1105.93
.90	40	385.62	1138.22
1.0	40	338.70	1169.38

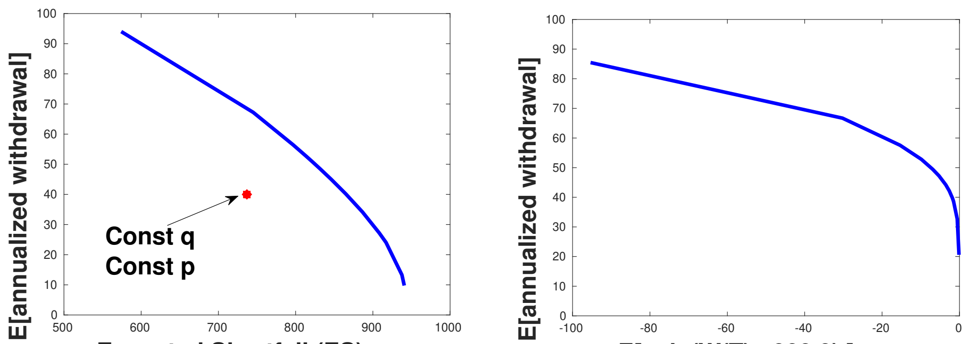
TABLE 12.1: *Constant weight, constant withdrawals, synthetic market results. Constant withdrawals are 10 per quarter (40 per year). Stock index: real capitalization weighted CRSP stocks; bond index: 30-day T-bills. Parameters from Table 10.1. Scenario in Table 11.1. Units: thousands of dollars. Statistics based on* 2.56×10^6 *Monte Carlo simulation runs.* $T = 5$ *years*

Equity fraction p	$E[\sum_i q_i]/T$	ES (5%)	$Median[W_T]$
0.0	40	608.37	819.64
0.1	40	636.25	856.93
0.2	40	651.38	896.60
0.3	40	647.72	935.27
0.4	40	625.88	974.05
0.5	40	592.80	1012.60
0.6	40	553.51	1051.13
0.7	40	511.02	1089.76
0.8	40	466.69	1127.30
0.9	40	421.40	1165.00
1.0	40	375.64	1201.42

TABLE 12.2: *Constant weight, constant withdrawals, historical market. Historical data range 1926:1-2020:12. Constant withdrawals are 10 per quarter. Stock index: real capitalization weighted CRSP stocks; bond index: 30-day T-bills. Scenario in Table 11.1. Units: thousands of dollars. Statistics based on* 10^6 *bootstrap simulation runs. Blocksize = 2 years.* $T = 5$ *years*

13 Efficient frontiers: synthetic market

Figure 13.1 shows the efficient frontiers, in the synthetic market, for both the EW-ES objective function and the EW-LS objective function. The frontiers are generated by varying κ for Problems 5.2 and $\hat{\kappa}$ for Problem 6.2. For ease of comparison, we show the expected withdrawals (EW) as average annualized withdrawals. For example, an average withdrawal of 10 per quarter (over the entire five year period) would be 40 per year (annualized). The detailed efficient frontier results are given in Appendix B. The red dot in Figure 13.1(a) shows the best result (in terms of EW-ES efficiency) for the constant weight, constant withdrawal case from Table 12.1.



(a) *EW-ES frontier* (b) *EW-LS frontier*

FIGURE 13.1: *Comparison of synthetic frontiers, and frontier generated from the synthetic market. Parameters based on real CRSP index, real 30-day T-bills (see Table 10.1). The Const q, Const p case has* $q = 40$, $p = 0.10$, *which is the best result from Table 12.1. Units: thousands of dollars.*

Recall from Proposition 7.1 that if there is a point on the EW-LS frontier (i.e. a value of $\hat{\kappa}$) such that

$$E[\mathbf{1}_{W_T < W^*}] = \alpha \tag{13.1}$$

then this point corresponds to a point on the EW-ES frontier (with $\kappa = \hat{\kappa}\alpha$), and that both these points have the same optimal controls. This can be seen clearly from Figure 13.2, where we plot the frontiers obtained from the EW-ES problem, and the EW-LS problem, but in terms of the ES risk measure. This is obviously an unfair comparison, since the EW-LS controls are not designed to be optimal in the EW-ES sense. In particular, the EW-LS controls use a fixed value of $W^* = 900$ (see equation (6.2)). However, we can see that both frontiers overlap near ES= 880. In this case, from Proposition 7.1, we can deduce that $W^* = 900$ corresponds to a VAR of 900.

14 Historical market frontiers

In this section, we examine the efficient frontiers by first computing and storing the optimal controls in the synthetic market. Then, these controls are tested using block bootstrap resampling of the historical data (see Section 11.2).

14.1 EW-LS Bootstrap Frontiers

Figure 14.1(a) shows the efficient frontier, for the EW-LS controls, tested in the historical market. Recall from Section 11.2, that the block bootstrap resampling method requires us to specify an

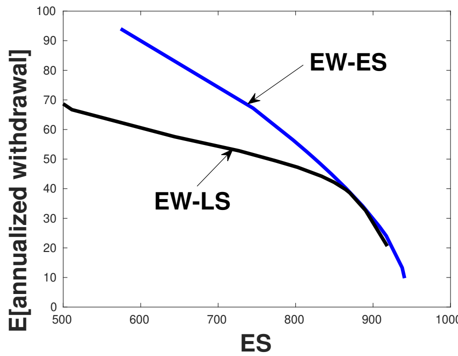


FIGURE 13.2: *Comparison of synthetic frontiers, and frontier generated from the synthetic market. Parameters based on real CRSP index, real 30-day T-bills (see Table 10.1). Units: thousands of dollars. Note that this is an unfair comparison: the EW-LS frontier is not designed to be EW-ES optimal.*

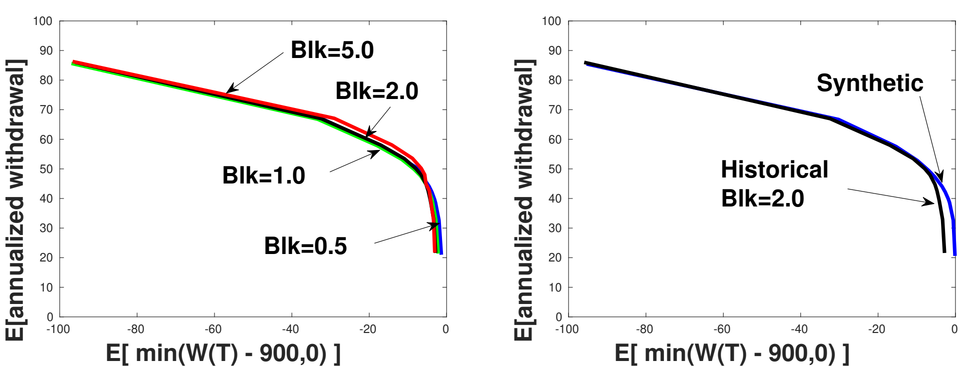
expected blocksize. The estimated blocksizes from Table 11.2 are quite different for the bond and stock time series. In Figure 14.1(a), we can see that the efficient frontiers are insensitive to the choice of expected blocksize, for blocksizes ranging from $0.5 - 5.0$ years.

In Figure 14.1(b) we show

- The efficient EW-LS frontier, controls computed in the synthetic market, and tested in the synthetic market.

- The efficient EW-LS frontier, controls computed in the synthetic market, and tested in the historical market (expected blocksize 2.0 years).

These two curves essentially overlap, except near the LS = 0.0 boundary. This suggests that the EW-LS controls are very robust in terms of parametric model misspecification.



(a) *EW-LS frontier: effect of blocksize*

(b) *EW-LS frontier: synthetic vs. historical*

FIGURE 14.1: *EW-LS frontiers, historical market. Bootstrap simulations, 10^6 samples. Optimal control generated in the synthetic market. Parameters based on real CRSP index, real 30-day T-bills (see Table 10.1). Historical data in range 1926:1-2020:12. Units: thousands of dollars.*

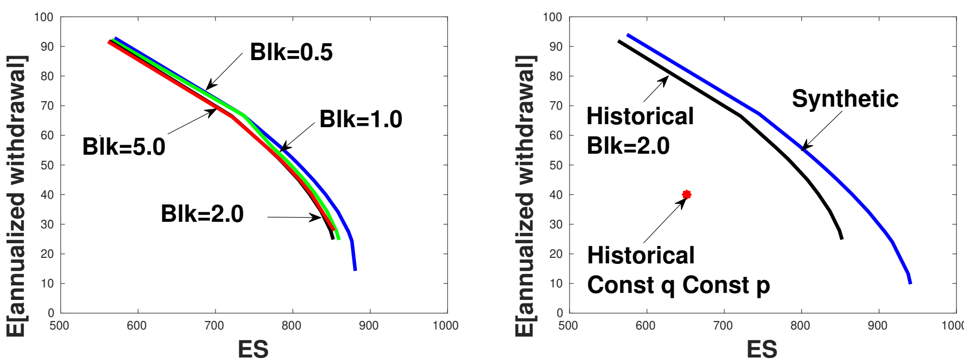
14.2 EW-ES Bootstrap Frontiers

Figure 14.2 shows the efficient frontier, for the EW-ES controls, tested in the historical market. In Figure 14.1(a), we can see that the efficient frontiers are slightly more sensitive to the choice of expected blocksize, compared to the EW-LS case.

In Figure 14.2(b) we show

- The efficient EW-ES frontier, controls computed in the synthetic market, and tested in the synthetic market.
- The efficient EW-ES frontier, controls computed in the synthetic market, and tested in the historical market (expected blocksize 2.0 years).
- The best result for the constant withdrawal ($q = 40$ per year) and constant weight ($p = 0.2$) computed in the historical market.

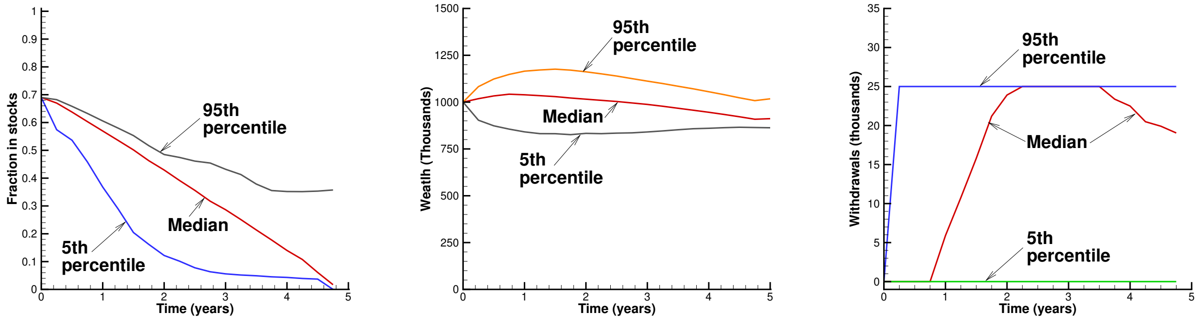
In this case, the synthetic market frontier is a bit above the historical market frontier, indicating that the ES risk measure is a slightly more sensitive to the market model, compared to the LS risk measure. However, note that the frontier tested in the historical market plots well above the constant weight, constant withdrawal point (tested in the historical market).



(a) *EW-ES frontier: effect of blocksize*

(b) *EW-ES frontier: synthetic vs. historical*

FIGURE 14.2: *EW-ES frontiers, historical market. Bootstrap simulations,* 10^6 *samples. Optimal control generated in the synthetic market. Parameters based on real CRSP index, real 30-day T-bills (see Table 10.1). Historical data in range 1926:1-2020:12. Units: thousands of dollars.*



(a) *Percentiles fraction in stocks* (b) *Percentiles wealth* (c) *Percentiles withdrawals*

FIGURE 15.1: *Scenario in Table 11.1. Mean-LS control. Optimal control computed from problem EW-LS Problem (6.2). Parameters based on the real CRSP index, and real 30-day T-bills (see Table 10.1). Control computed and stored from the Problem (6.2) in the synthetic market. Control used in the historical market,* 10^6 *bootstrap samples.* $q_{min} = 0, q_{\max} = 25$ *(per quarter),* $\hat{\kappa} = 6.25$. $W^* = 900$. *Units: thousands of dollars.*

15 Detailed Historical market results: EW-LS controls

We now present some representative results from testing the optimal EW-LS controls in the historical (bootstrapped) market. The summary statistics for various values of $\hat{\kappa}$ are given in Appendix C.

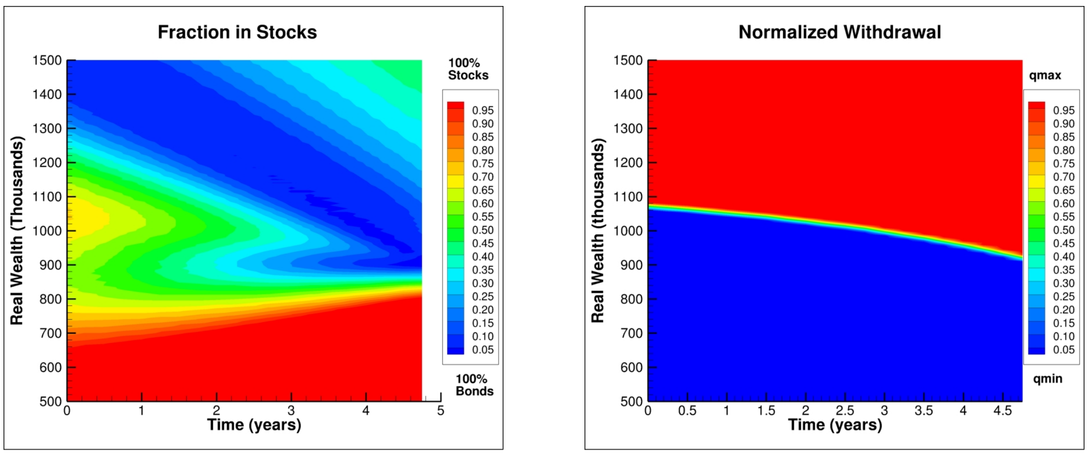
15.1 EW-LS $\hat{\kappa} = 6.25$

For the case of $\hat{\kappa} = 6.25$ (Problem 6.2) the average annualized withdrawal is EW= 50.10, with an ES= 766.72 and $Median[W_T] = 911.65$. Although this strategy is optimal in terms of the LS risk measure, it outperforms the constant weight, constant withdrawal strategy (using ES as a risk measure, in the historical market), where the best result is $(EW,ES) = (40, 651)$ with $Median[W_T] = 897$. However, unlike the constant withdrawal case, there is some probability of withdrawing less (on average) than 40 per year.

Figure 15.1 shows the percentiles of fraction in stocks, wealth, and withdrawals versus time. Figure 15.1(a) indicates that the initial fraction in stocks is about 0.7. The median value of stocks drops steadily over time, reaching zero at 4.75 years. Figure 15.1(b) shows that the total wealth in the portfolio is tightly constrained about the median value, which is a desirable feature. However, this comes at a cost of variable withdrawals, as seen in Figure 15.1(c). This figure shows the percentiles of the quarterly withdrawals, with $(q_{\min}, q_{\max}) = (0.0, 25)$. The median withdrawal is zero until the end of year one. The median withdrawal then increases rapidly reaching the maximum value at the end of year two, and dropping somewhat near five years.

Figure 15.2 shows the heat maps of the optimal fraction in stocks and the optimal withdrawals. The optimal fraction in stocks starts out around 0.7, and then adjusts to the observed performance of the portfolio. Initially, if the returns are good, then the fraction in stocks is decreased. Conversely, if returns are poor, then the fraction in stocks is again initially decreased (to reduce downside risk). However, if very poor returns are observed, then the fraction in stocks is increased, in order to attempt to increase the probability of gains in future periods. At $t \to T$, and if the wealth is above the target value 900, then the portfolio is de-risked completely. If wealth is significantly below 900, then the portfolio is switched to 100% stocks, in an attempt to recover. However, this is a fairly low probability event, since the bond floor (the blue region) is an attractor, i.e. once we are in the high fraction of bonds region, there is very little probability of leaving this region.

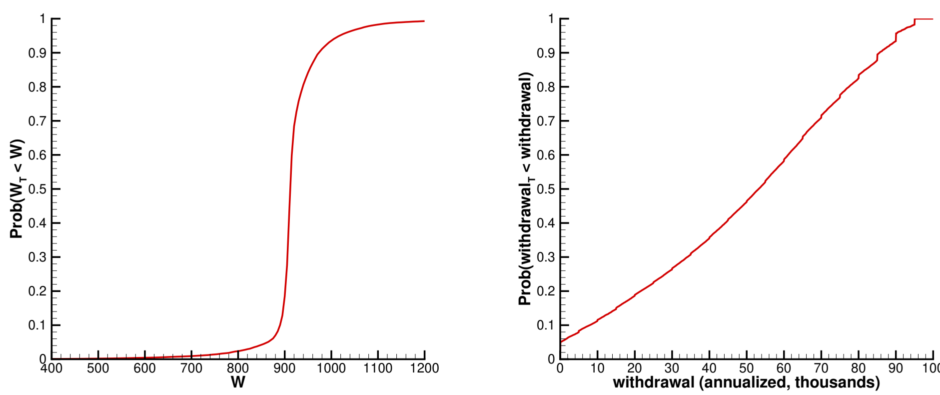
Figure 15.2(b) indicates that the optimal withdrawal controls are essentially bang-bang, i.e. withdraw at either the maximum or minimum withdrawal amounts. Note that we did not assume this to be true in our numerical algorithm. However, from Proposition 8.1, we learn that in the continuous rebalancing, continuous withdrawal limit, the withdrawal control is bang-bang. It is interesting to see that this result appears to hold (with a very small transition zone) in the case of discrete rebalancing and withdrawals.



(a) *Fraction in stocks* (b) *Withdrawals*

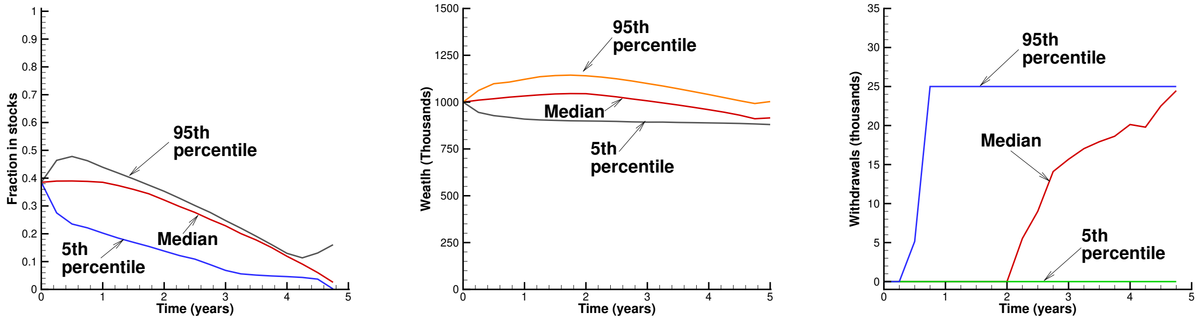
FIGURE 15.2: *Optimal EW-LS. Heat map of controls: fraction in stocks and withdrawals, computed from Problem EW-LS (6.2). cap-weighted real CRSP, real 30-day T-bills. Scenario given in Table 11.1. Control computed and stored from the Problem 6.2 in the synthetic market.* $q_{min} = 0, q_{\max} = 25$ *(quarterly).* $\hat{\kappa} = 6.25$. $W^* = 900$. $\epsilon = 10^{-6}$. *Normalized withdrawal* $(q - q_{\min})/(q_{\max} - q_{\min})$. *Units: thousands of dollars.*

Figure 15.3 shows the CDFs of the terminal wealth and annualized withdrawals. The final wealth CDF in Figure 15.3(a) is tightly clustered around $W = 900$, which is consistent with the objective function (6.2), where risk is measured in terms of the shortfall relative to $W = 900$. There is, of course, no free lunch here, as we can see in the CDF of the annualized withdrawals in Figure 15.3(b). There is about a 5% chance that the total withdrawals over the five year period will be zero (in order to preserve final wealth). On the other hand, the median annualized withdrawal is about 55 per year, which is quite impressive.



(a) *CDF of terminal wealth.* (b) *CDF of total withdrawals.*

FIGURE 15.3: *Scenario in Table 11.1. CDFs of terminal wealth and total withdrawals (annualized). EW-LS control. Optimal control computed from problem Problem 6.2. Parameters based on the real CRSP index, and real 30-day T-bills (see Table 10.1). Control computed and stored from the EW-LS Problem (6.2) in the synthetic market. Control used in the historical market, 10^6 bootstrap samples. $q_{min} = 0, q_{\max} = 25$ (per quarter), $\hat{\kappa} = 6.25$. $W^* = 900$. Units: thousands of dollars.*



(a) *Percentiles fraction in stocks*

(b) *Percentiles wealth*

(c) *Percentiles withdrawals*

FIGURE 15.4: *Scenario in Table 11.1. EW-LS control. Optimal control computed from problem Problem (6.2). Parameters based on the real CRSP index, and real 30-day T-bills (see Table 10.1). Control computed and stored from the Problem (6.2) in the synthetic market. Control used in the historical market,* 10^6 *bootstrap samples.* $q_{min} = 0, q_{\max} = 25$ *(per quarter),* $\hat{\kappa} = 17.0$. $W^* = 900$. $ES = 821$. *Units: thousands of dollars.*

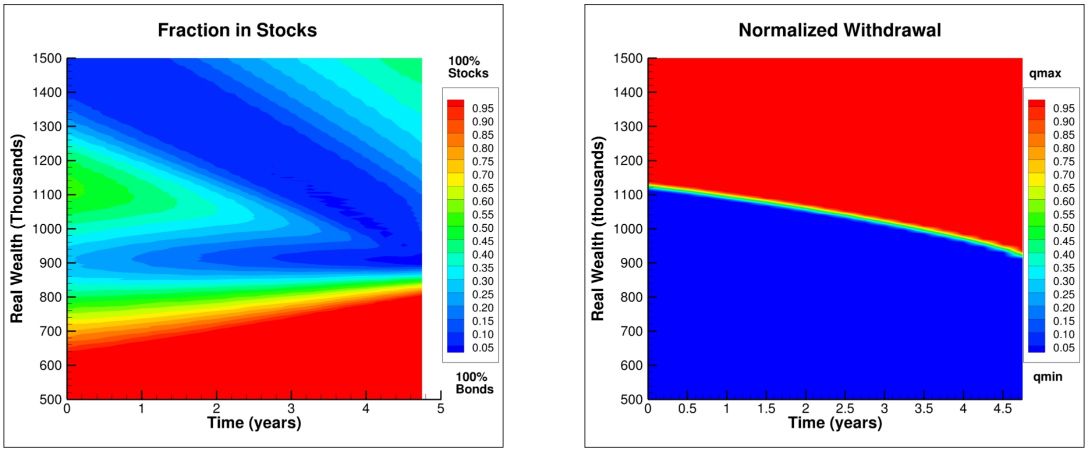
15.2 EW-LS $\kappa = 17.0$

For this case, we solve Problem 6.2, with $\hat{\kappa} = 17$, putting increased emphasis on LS risk term, which gives an average annualized withdrawal of EW= 40.06, LS $= -4.32$, ES $=$ 824 and $Median[W_T] =$ 916.

Figure 15.4 shows the percentiles of fraction in stocks, wealth, and withdrawals. Note that in this case, the initial fraction in stocks is about 0.4, indicating a preference for less risk. Figure 15.4(b) shows an even tighter distribution of wealth about the median, compared to Figure 15.1(b). However, the price to be paid for this tighter wealth distribution is evident from Figure 15.4(c), where the median withdrawals are zero until the end of year two, in contrast to Figure 15.1(c).

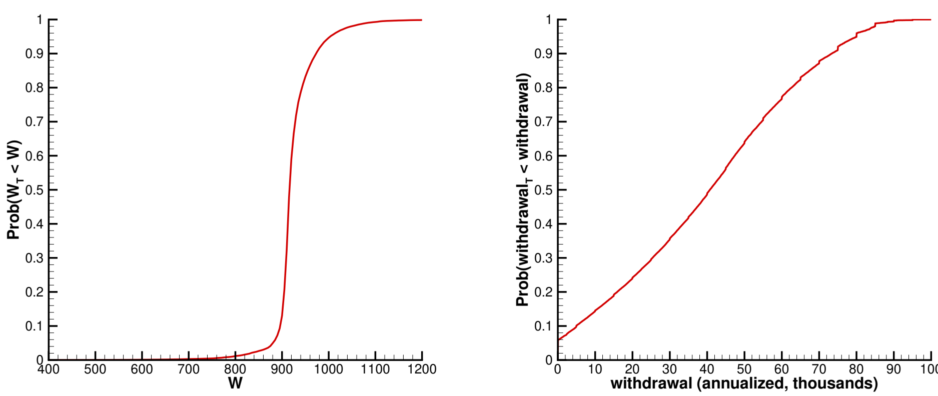
Figure 15.5(a) illustrates the heat map of the equity allocation controls. Note that in this case, if the total wealth decreases initially, then the optimal strategy is go heavily into bonds. This protects the downside risk. Similarly, if wealth increases, the investor also de-risks. The effect of this is to cause the tight distribution about the median. Figure 15.5(b) shows the optimal withdrawal controls are, for all practical purposes, bang-bang.

The CDFs of the terminal wealth distributions and annualized withdrawals are given in Figure 15.6. These plots are qualitatively similar to Figure 15.3.



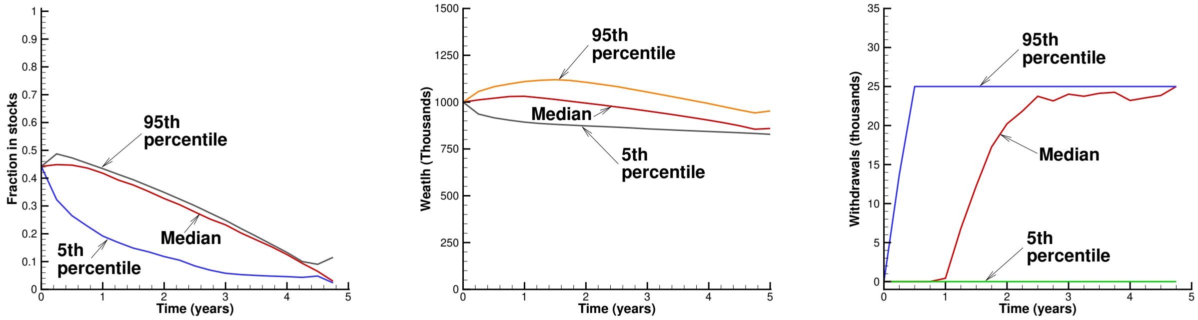
(a) *Fraction in stocks* (b) *Withdrawals*

FIGURE 15.5: *Optimal EW-LS. Heat map of controls: fraction in stocks and withdrawals, computed from Problem 6.2, cap-weighted real CRSP, real 30-day T-bills. Scenario given in Table 11.1. Control computed and stored from the Problem 6.2 in the synthetic market.* $q_{min} = 0, q_{\max} = 25$ *(quarterly).* $\hat{\kappa} = 17.0$. $W^* = 900$. $\epsilon = 10^{-6}$. *Normalized withdrawal* $(q - q_{\min})/(q_{\max} - q_{\min})$. *Units: thousands of dollars.*



(a) *CDF of terminal wealth.* (b) *CDF of total withdrawals.*

FIGURE 15.6: *Scenario in Table 11.1. CDFs of terminal wealth and total withdrawals (annualized). Mean-LS control. Optimal control computed from problem Problem 6.2. Parameters based on the real CRSP index, and real 30-day T-bills (see Table 10.1). Control computed and stored from the Problem 6.2 in the synthetic market. Control used in the historical market,* 10^6 *bootstrap samples.* $q_{min} = 0, q_{\max} = 25$ *(per quarter),* $\kappa = 17.0$. $W^* = 900$. *Units: thousands of dollars.*



(a) *Percentiles fraction in stocks*

(b) *Percentiles wealth*

(c) *Percentiles withdrawals*

FIGURE 16.1: *Scenario in Table 11.1. EW-ES control. Optimal control computed from problem Problem 5.2. Parameters based on the real CRSP index, and real 30-day T-bills (see Table 10.1). Control computed and stored from the Problem 5.2 in the synthetic market. Control used in the historical market,* 10^6 *bootstrap samples.* $q_{min} = 0, q_{\max} = 25$ *(per quarter),* $\kappa = 1.17$*.* $W^* = 840$*. Units: thousands of dollars. ES = 787.*

16 Detailed historical market results: EW-ES controls

In this section, we present some detailed results for the EW-ES controls (Problem 5.2), computed in the synthetic market, and tested in the historical market. The summary statistics for various values of κ are given in Appendix C.

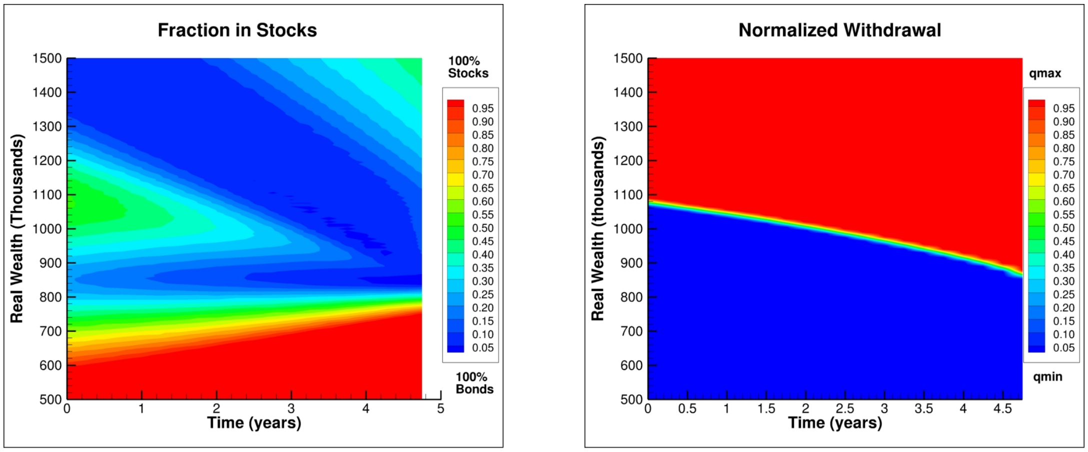
16.1 Bootstrap EW-ES $\kappa = 1.17$

For $\kappa = 1.17$ in Problem 5.2, the expected annualized withdrawals are EW=50.5, with ES=788 and $Median[W_T] = 859.33$. This can be compared with the $\hat{\kappa} = 6.25$ for the EW-LS controls. In this case, the EW values for both strategies are similar, the ES for the EW-LS control is a bit worse, while the $Median[W_T] = 912$ for the EW-LS control is better. Hence there is a tradeoff here for these two strategies, which depends on the investor's preferences (larger ES or larger $Median[W_T]$).

Figure 16.1 shows the percentiles of fraction in stocks, wealth, and withdrawals versus time. From Figure 16.1(a), we can see that in this case, the median fraction in stocks starts out at a conservative allocation of 45% and drops to zero over time. There is very little spread between the median and the 95^{th} percentile. The 5^{th} percentile shows a very rapid de-risking. Figure 16.1(b) shows a very tight range for the total wealth over the entire investment horizon. The cost for this very predictable wealth can be seen in Figure 16.1(c), where the median withdrawal is zero until well into the second year.

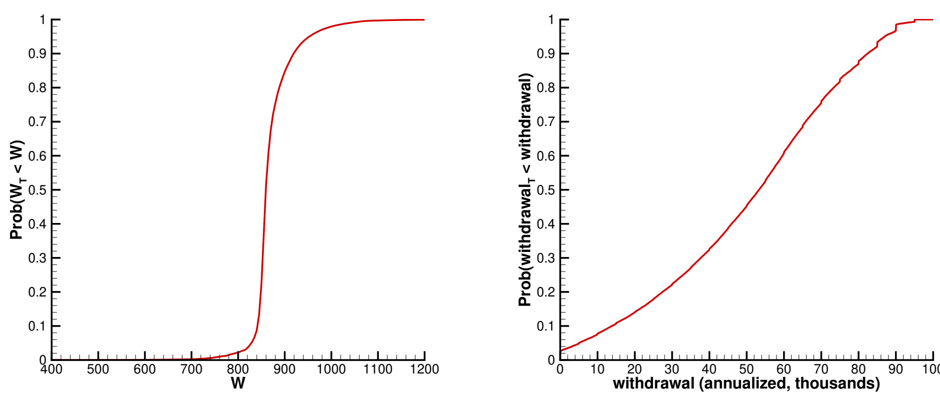
The heat maps of the optimal equity fraction and the optimal withdrawals are given in Figure 16.2. Figure 16.2(a) should be compared to Figure 15.2(a) (EW-LS control, approximately the same EW). The EW-ES control is more cautious, with a large bond fraction extending to the $t = 0$ boundary, below the initial wealth. The optimal withdrawals continue to be approximately bang-bang.

The CDFs of the terminal wealth and the total withdrawals are given in Figure 16.3. The probability of zero total withdrawals is considerably smaller than the EW-LS control (with similar total expected withdrawals, see Figure 16.6(b)).



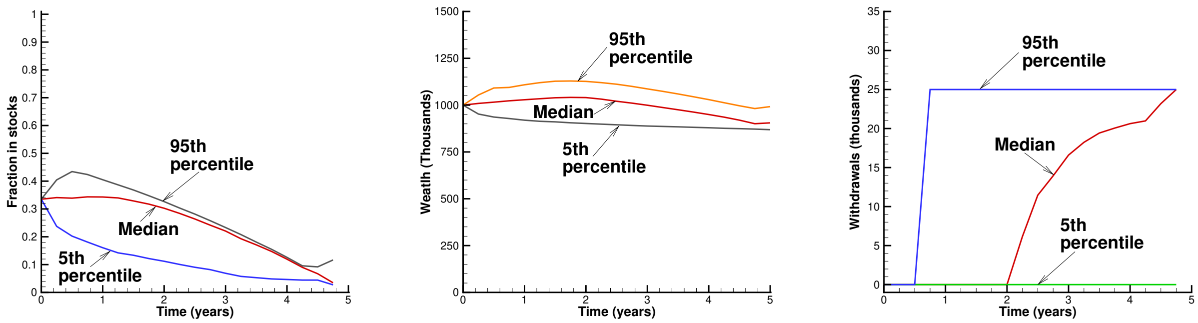
(a) *Fraction in stocks* (b) *Withdrawals*

FIGURE 16.2: *Optimal EW-ES heat map of controls: fraction in stocks and withdrawals, computed from Problem 5.2 cap-weighted real CRSP, real 30-day T-bills. Scenario given in Table 11.1. Control computed and stored from the Problem 6.2 in the synthetic market.* $q_{min} = 0, q_{\max} = 25$ *(quarterly).* $\kappa = 1.17$. $W^* = 840$. $\epsilon = 10^{-6}$. *Normalized withdrawal* $(q - q_{\min})/(q_{\max} - q_{\min})$. *Units: thousands of dollars.*



(a) *CDF of terminal wealth.* (b) *CDF of total withdrawals.*

FIGURE 16.3: *Scenario in Table 11.1. CDFs of terminal wealth and total withdrawals (annualized). EW-ES control. Optimal control computed from problem Problem 5.2. Parameters based on the real CRSP index, and real 30-day T-bills (see Table 10.1). Control computed and stored from the Problem 5.2 in the synthetic market. Control used in the historical market,* 10^6 *bootstrap samples.* $q_{min} = 0, q_{\max} = 25$ *(per quarter),* $\kappa = 1.17$. $W^* = 840$. *Units: thousands of dollars.* $ES = 787$.



(a) *Percentiles fraction in stocks*

(b) *Percentiles wealth*

(c) *Percentiles withdrawals*

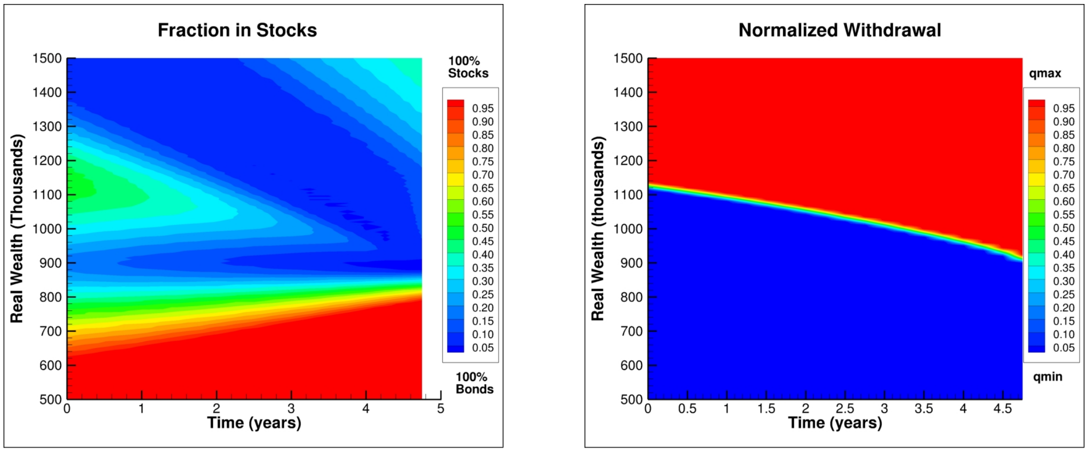
FIGURE 16.4: *Scenario in Table 11.1. EW-ES control. Optimal control computed from problem Problem 5.2. Parameters based on the real CRSP index, and real 30-day T-bills (see Table 10.1). Control computed and stored from the Problem 5.2 in the synthetic market. Control used in the historical market,* 10^6 *bootstrap samples.* $q_{min} = 0, q_{\max} = 25$ *(per quarter),* $\kappa = 1.37$. $W^* = 884$. *Units: thousands of dollars.*

16.2 Bootstrap EW-ES $\kappa = 1.37$

In this section, we give the detailed results for $\kappa = 1.37$ in Problem 5.2. This value of κ generates an annualized value of EW=40.26, and ES=821 and $Median[W_T] = 905.25$. This should compared with the EW-LS case with $\hat{\kappa} = 17$, since both strategies have approximately the same EW. Note that $W^* = 884$ for Problem 5.2, compared to $W^* = 900$ for Problem 6.2.

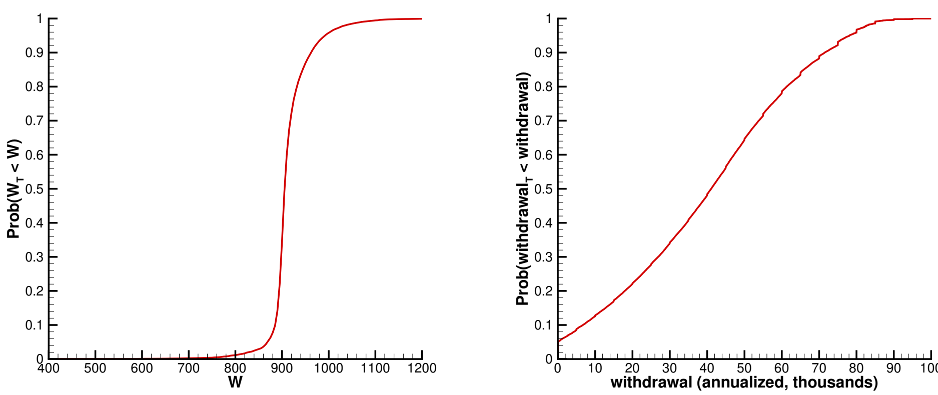
From Figure 16.4(a), we can see that the fraction in stocks begins at a very conservative level of 35%, and drops rapidly over time. The EW-ES strategy continues to have a very narrow spread (5^{th} to 95^{th} percentile) of the total wealth (see Figure 16.4(b)). However, Figure 16.4(c) shows that the median withdrawal remains at zero until well into the third year.

The heat maps of the controls for $\kappa = 1.37$ are given in Figure 16.5, and should be compared with Figure 15.5(a) (EW-LS control with approximately the same EW). The cumulative distribution functions for the terminal wealth and the total withdrawals are given in Figure 16.6(b).



(a) *Fraction in stocks* (b) *Withdrawals*

FIGURE 16.5: *Optimal EW-ES control hear maps: fraction in stocks and withdrawals, computed from Problem 5.2 cap-weighted real CRSP, real 30-day T-bills. Scenario given in Table 11.1. Control computed and stored from the Problem 6.2 in the synthetic market.* $q_{min} = 0, q_{\max} = 25$ *(quarterly).* $\kappa = 1.37$. $W^* = 884$. $\epsilon = 10^{-6}$. *Normalized withdrawal* $(q - q_{\min})/(q_{\max} - q_{\min})$. *Units: thousands of dollars.*



(a) *CDF of terminal wealth.* (b) *CDF of total withdrawals.*

FIGURE 16.6: *Scenario in Table 11.1. CDFs of terminal wealth and total withdrawals (annualized). EW-ES control. Optimal control computed from problem Problem 5.2. Parameters based on the real CRSP index, and real 30-day T-bills (see Table 10.1). Control computed and stored from the Problem 5.2 in the synthetic market. Control used in the historical market,* 10^6 *bootstrap samples.* $q_{min} = 0, q_{\max} = 25$ *(per quarter),* $\kappa = 1.37$. $W^* = 884$. *Units: thousands of dollars.*

17 Discussion

We remind the reader that the EW-ES problem (6.2) is formally of the pre-commitment type, and hence is time inconsistent. However, we follow the usual practice, and consider the policy followed for $t > 0$ to be the induced time consistent policy (Strub et al., 2019; Forsyth, 2020a). From Proposition 5.1, we learn that for any given point on the EW-ES frontier, there is a pair $(\hat{\kappa}, W^*)$ for the EW-LS problem (6.2) which has identical controls. This is illustrated in Figure 13.2, where we fix the value of W^* for the EW-LS problem.

Hence, in some sense, the difference between an EW-LS policy and an EW-ES policy might be deemed to be just a matter of interpretation. However, the target shortfall W^* in the EW-LS objective function is easily interpreted as a desired lower bound for the terminal wealth. In the EW-ES case, the effective target W^* is specified in terms of the mean of the worst α fraction of results, which is a bit more obscure.

Generally speaking, the EW-LS controls show a fairly tight distribution around the specified value of the target (which in our examples is 900, compared to the initial wealth of 1000), as can be seen from the cumulative distribution functions (Figures 15.3(a) and 15.6(a)).

The EW-ES wealth CDFs (Figures 16.3(a) and 16.6(a) are either comparable with the EW-LS CDFs (from Proposition 5.1) or have a lower median value of the terminal wealth, but with less tail risk.

It could be argued that the EW-ES control is too focused on extreme outcomes. On the other hand, the EW-LS control has an intuitive parameter W^*, which represents the desired lower bound on terminal wealth. The EW-LS efficient frontier is also very robust to parametric model mis-specification; the EW-LS efficient frontiers in both the synthetic and historical market virtually coincide.

Both strategies protect the desired wealth target by delaying withdrawals for $1 - 2$ years, and increasing the withdrawals thereafter if stocks do well. For larger values of expected total withdrawals (ES), the EW-LS strategy begins with a larger value in stocks than the EW-ES control. Both strategies rapidly de-risk into bonds as $t \to T$.

Consider the heat maps for the equity fraction controls for the EW-ES control, Figure 16.2(a) and for the EW-LS control, Figure 15.2(a). These controls have roughly the same EW, but different values of W^*, indicating that the strategies are different. Both strategies de-risk if the portfolio does well. The EW-ES control also moves rapidly into bonds if the portfolio does poorly (to protect the downside). The EW-LS control does this only for larger times, and tends to take on more risk early on.

The EW-ES control (where it is different from the EW-LS control) then has a larger probability of *cashing out* early on in the investment process. Once the investor has moved into a portfolio largely dominated by bonds, there is little chance of escaping from this basin of attraction. This may be an undesirable characteristic of this strategy.

As a final point of comparison, note that the best result for the constant weight, constant withdrawal policy in the historical market was $(EW, ES) = (40{,}651)$. The comparable EW-LS policy had (40,864). Of course, this greatly reduced tail risk came at the expense of only an expected annualized withdrawal $EW = 40$ for the EW-LS policy, compared with the guaranteed yearly withdrawal of 40 in Table 12.2. On the other hand, there is about a 55% probability that the average annualized withdrawals will be larger than 40 for the EW-LS strategy.

In summary, it would seem that the EW-LS strategy is to be preferred, since the parameter W^* has an easy intuitive interpretation as a minimum final wealth target, while at the same time producing impressive expected total withdrawals (EW). The EW-LS strategy is also formally time consistent, without having to consider an induced time consistent strategy (as required for the

EW-ES strategy).

18 Conclusion

In view of the empirical fact that many retirees are decumulating their total wealth very slowly, or even accumulating, we propose a strategy to make it more palatable for these retirees to withdraw significant sums during early years of retirement.

We consider two closely related strategies in this paper: expected withdrawal-expected shortfall (EW-ES) and expected withdrawal-linear shortfall (EW-LS). The controls are the withdrawal amount per quarter, and the allocation to stocks and bonds. The optimal controls are determined using dynamic programming, based on a parametric stochastic model of historical stock and bond returns. These strategies are tested on bootstrapped resampled historical data.

We use a short time horizon (five years) to facilitate withdrawals during the early years of retirement. The optimal strategies for both EW-ES and EW-LS objective functions have the following characteristics

- The median optimal withdrawal policy is to withdraw zero amounts for first 1-2 years, and then to increase withdrawals rapidly.
- The optimal allocation to stocks is high at the start, then declines rapidly, in order to protect total portfolio wealth.
- With a suitable choice of parameters, there is a high probability of preserving at least 90% of the initial wealth in real terms (after five years).
- There is a high probability that the annualized average withdrawals will exceed 4% real of the initial wealth.

Note that the excellent wealth preservation aspect of this strategy is due to both the allocation strategy, and the ability to delay spending for the first few years of the five year cycle, if necessary. Hence, since it appears that the *reluctant spenders* desire to preserve wealth, and are flexible in their spending needs, these types of strategies should be appealing to this class of retirees.

19 Acknowledgements

P. A. Forsyth's work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) grant RGPIN-2017-03760.

Appendix

A Numerical techniques

We solve problems (5.2) and (6.2) using the techniques described in detail in Forsyth and Labahn (2019); Forsyth (2020a; 2021a). We give only a brief overview here.

We localize the infinite domain to $(s,b) \in [s_{\min}, s_{\max}] \times [b_{\min}, b_{\max}]$, and discretize $[b_{\min},b_{\max}]$ using an equally spaced $\log b$ grid, with n_b nodes. Similarly, we discretize $[s_{\min}, s_{\max}]$ on an equally spaced $\log s$ grid, with n_s nodes. Localization errors are minimized using the domain extension method in (Forsyth and Labahn, 2019).

At rebalancing dates, we solve the local optimization problem by discretizing $(q(\cdot), p(\cdot))$ and using exhaustive search. Between rebalancing dates, we solve a two dimensional partial integro-differential equation (PIDE) using Fourier methods (Forsyth and Labahn, 2019; Forsyth, 2021a).

Finally, the optimization problem (7.4) is solved using a one-dimensional optimization technique (this final step is only required for the EW-ES case).

We compute and store the optimal controls from solving Problem 5.2 or Problem (6.2) using the parametric model of the stock and bond processes. We then use the stored controls in Monte Carlo simulations to generate statistical results. As a robustness check, we also use the stored controls and simulate results using bootstrap resampling of historical data.

A.1 Convergence Test

Table A.1 shows a detailed convergence test for the base case problem given in Table 11.1, for the EW-LS problem. The results are given for a sequence of grid size, for the dynamic programming algorithm (7.1). The dynamic programming algorithm appears to converge at roughly a second order rate. The optimal control computed using dynamic programming is stored, and then used in Monte Carlo computations. The MC results are in good agreement with the dynamic programming solution.

For all the numerical examples, we will use the 2048×2048 grid, since this seems to be accurate enough for our purposes.

Algorithm in Section 7.1				Monte Carlo	
Grid	$E[(W_T - W^*)]$	$E[\sum_i q_i]/T$	Value Function	$E[(W_T - W^*)]$	$E[\sum_i q_i]/T$
512×512	-5.3355	42.382	158.55	-3.3374	42.71 (.16)
1024×1024	-3.9939	43.301	176.56	-3.3373	43.48 (.18)
2048×2048	-3.6996	44.154	183.77	-3.5384	44.20 (.16)
4096×4096	-3.6442	44.368	185.39	-3.5982	44.38 (.19)

TABLE A.1: *Convergence test, optimal EW-LS strategy, real stock index: deflated real capitalization weighted CRSP, real bond index: deflated 30-day T-bills. Scenario in Table 11.1. Parameters in Table 10.1. The Monte Carlo method used* 2.56×10^6 *simulations.* $\kappa = 10$, $W^* = 900$. *Grid refers to the grid used in the Algorithm in Section 7.1:* $n_x \times n_b$, *where* n_x *is the number of nodes in the* $\log s$ *direction, and* n_b *is the number of nodes in the* $\log b$ *direction. Units: thousands of dollars (real).* $T = 5$ *years.* $q_{\min} = 0.0$. $q_{\max} = 100$ *(quarterly). Algorithm in Section 7.1. The numbers in brackets are the standard errors at the 99% confidence level.*

B Detailed efficient frontiers: synthetic market

$\hat{\kappa}$	$E[\min(W - W^*,0)]$	$E[\sum_i q_i]/T$	ES (5%)	$Median[W_T]$
1.0	-95.35	85.46	404.38	885.79
2.5	-30.23	66.70	511.22	904.93
3.75	-15.35	57.56	643.97	907.91
5.0	-9.85	52.82	726.77	910.05
6.25	-6.91	49.44	773.95	912.42
7.5	-5.29	47.24	802.57	912.66
10.0	-3.54	44.20	833.64	913.67
12.5	-2.63	42.17	849.68	914.88
17.0	-1.83	39.78	864.33	916.41
20.0	-1.52	38.60	869.95	917.89
50	-0.57	32.65	890.32	929.37
500	-0.12	20.60	918.33	964.61

TABLE B.1: *EW-LS synthetic market results for optimal strategies, assuming the scenario given in Table 11.1. Stock index: real capitalization weighted CRSP stocks; bond index: 30-day T-bills. Parameters from Table 10.1. Units: thousands of dollars. Statistics based on* 2.56×10^6 *Monte Carlo simulation runs. Control is computed using the Algorithm in Section 7.1, stored, and then used in the Monte Carlo simulations.* $q_{\min} = 0.0$, $q_{\max} = 100$ *(quarterly).* $T = 5$ *years* $W^* = 900$. $\epsilon = 10^{-6}$.

κ	ES (5%)	$E[\sum_i q_i]/T$	W^*	$Median[W_T]$
0.5	574.19	94.03	582.28	601.11
1.0	745.03	67.23	762.63	779.26
1.1	796.52	56.50	814.50	833.79
1.13	813.50	52.71	832.81	851.68
1.17	821.70	50.83	840.80	860.24
1.20	831.63	48.50	850.57	870.67
1.25	846.18	44.97	865.59	886.36
1.37	864.87	40.19	884.00	905.94
1.5	886.48	34.18	906.07	929.31
1.75	908.19	27.23	929.98	953.36
2.0	917.04	24.05	938.04	963.92
5.0	937.61	13.29	960.33	999.09
10.0	940.64	9.77	964.67	1013.23

TABLE B.2: *EW-ES synthetic market results for optimal strategies, assuming the scenario given in Table 11.1. Stock index: real capitalization weighted CRSP stocks; bond index: 30-day T-bills. Parameters from Table 10.1. Units: thousands of dollars. Statistics based on* 2.56×10^6 *Monte Carlo simulation runs. Control is computed using the Algorithm in Section 7.1, stored, and then used in the Monte Carlo simulations.* $q_{\min} = 0.0$, $q_{\max} = 100$ *(quarterly).* $T = 5$ *years* $\epsilon = 10^{-6}$.

C Detailed efficient frontiers: historical market

$\hat{\kappa}$	$E[\min(W - W^*,0)]$	$E[\sum_i q_i]/T$	ES (5%)	$Median[W_T]$
1.0	-95.96	86.00	430.03	874.91
2.5	-32.40	66.98	528.98	902.76
3.75	-10.93	53.47	724.50	909.05
6.25	-7.96	50.10	766.72	911.65
7.5	-6.55	47.92	789.93	911.96
10.0	-5.31	44.80	810.14	912.90
12.5	-4.77	42.61	817.97	914.09
17.0	-4.32	40.06	824.06	915.77
20.0	-4.13	38.86	827.31	917.22
50.0	-3.32	32.81	839.73	928.60
500.0	-3.06	21.69	843.44	963.24

TABLE C.1: *EW-LS historical market results for optimal strategies, assuming the scenario given in Table 11.1. Stock index: real capitalization weighted CRSP stocks; bond index: 30-day T-bills. Parameters from Table 10.1. Units: thousands of dollars. Statistics based on* 10^6 *bootstrap simulation runs. Blocksize* $= 2$ *years. Control is computed using the Algorithm in Section 7.1, stored, and then used in the bootstrap simulations.* $q_{\min} = 0.0$, $q_{\max} = 100$ *(quarterly).* $T = 5$ *years* $W^* = 900$. $\epsilon = 10^{-6}$.

κ	ES (5%)	$E[\sum_i q_i]/T$	$Median[W_T]$
0.5	562.69	91.91	778.30
1.0	721.28	66.45	778.30
1.1	766.30	55.96	832.96
1.13	781.17	52.31	850.77
1.17	787.87	50.51	859.33
1.20	794.53	48.63	868.15
1.25	807.53	44.97	885.57
1.37	821.34	40.26	905.25
1.50	836.30	34.47	928.79
1.75	848.77	27.89	953.13
2.0	852.15	24.84	963.31
5.0	849.19	15.40	997.96
10.0	846.81	12.48	1011.58

TABLE C.2: *EW-ES historical market results for optimal strategies, assuming the scenario given in Table 11.1. Stock index: real capitalization weighted CRSP stocks; bond index: 30-day T-bills. Parameters from Table 10.1. Units: thousands of dollars. Statistics based on* 10^6 *bootstrap simulation runs. Blocksize* $= 2$ *years. Control is computed using the Algorithm in Section 7.1, stored, and then used in the bootstrap simulations.* $q_{\min} = 0.0$, $q_{\max} = 100$ *(quarterly).* $T = 5$ *years* $\epsilon = 10^{-6}$.

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