Continuous Time Mean Variance Asset Allocation: A Time-consistent Strategy *

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Abstract

We develop a numerical scheme for determining the optimal asset allocation strategy for time-consistent, continuous time, mean variance optimization. Any type of constraint can be applied to the investment policy. The optimal policies for time-consistent and pre-commitment strategies are compared. When realistic constraints are applied, the efficient frontiers for the pre-commitment and time-consistent strategies are similar, but the optimal investment strategies are quite different.

Keywords: time-consistent mean variance asset allocation, piecewise constant policy timestepping, constrained policies

AMS Classification 65N06, 93C20

1 Introduction

Recently, there has been considerable interest in continuous time mean variance asset allocation [21, 14, 18, 13, 3, 6, 20, 9, 10, 19]. The optimal strategy in these papers was based on the *pre-commitment* strategy [2]. The pre-commitment strategy, for time $t + \Delta t$, computed at time t will not necessarily agree with the strategy for time $t + \Delta t$, computed at time $t + \Delta t$.

On the other hand, it can be argued that there are many economic reasons for requiring that the investment strategy be *time-consistent*. The time-consistent strategy chooses, at each instant in time, the best possible mean variance strategy, assuming optimal mean variance strategies are selected at each later instant in time [4, 2].

In [2], the time-consistent and pre-commitment mean variance policies were compared based on an analytic solution. However, this analytic solution assumes that the investment policy is unconstrained. This allows for infinite borrowing and shorting, and permits trading to continue even if the investor is insolvent. From [19], we learn that the the optimal policies for the pre-commitment strategy behave quite differently when realistic constraints (e.g. no bankruptcy, finite

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borrowing, no shorting) are applied to the investment policy. It is therefore of interest to compare the pre-commitment and time-consistent strategies under practical policy constraints.

Since we can view the time-consistent mean variance strategy as the pre-commitment strategy with a constraint forcing time consistency, it is immediately obvious that the efficient frontier for the time-consistent strategy can never be above the efficient frontier for the pre-commitment strategy. In addition, the time-consistent formulation must have a dynamic programming principle. This contrasts with the pre-commitment strategy, where there is no natural dynamic programming principle. However, the pre-commitment problem can be recast into an equivalent convex optimization problem [16, 3, 8]. In [19], a general numerical scheme was developed for determining the optimal mean variance strategy with realistic constraints.

The main results of this paper are

- We develop a fully numerical scheme for determining the optimal time-consistent mean variance strategy. Any type of constraint can be applied to the optimal policy.
- The method is based on the piecewise constant policy technique in [12]. In our case, since the time-consistent problem can be formulated as a system of Hamilton-Jacobi-Bellman differential algebraic equations, this falls outside the viscosity solution theory in [12]. Hence we have no formal proof of convergence of our method. Nevertheless, our technique does converge to analytic solutions where available.
- The efficient frontier for the time-consistent strategy is never above the efficient frontier for the pre-commitment strategy. If realistic constraints are applied to both strategies (e.g. no bankruptcy, no shorting), the efficient frontiers for both strategies become very close. However, the optimal investment polices are quite different for each strategy.

Consequently, if realistic constraints are applied, the time-consistent and pre-commitment strategies are not easily distinguished in terms of their efficient frontiers. Rather, is necessary to examine the optimal policies in each case, which are qualitatively different.

2 Pre-commitment Policy vs Time-consistent Policy

In this paper, we consider the problem of determining the mean variance efficient strategy for a pension plan. It is common to write the efficient frontier in terms of the investor's final wealth. We will refer to this problem in the following as the *wealth* case.

Suppose there are two assets in the market: one is risk free (e.g. a government bond) and the other is risky (e.g. a stock index). The risky asset S follows the stochastic process

$$dS = (r + \xi_1\sigma_1)S\ dt + \sigma_1 S\ dZ_1\ , \tag{2.1}$$

where dZ_1 is the increment of a Wiener process, σ_1 is volatility, r is the interest rate, ξ_1 is the market price of risk (or Sharpe ratio) and the stock drift rate can then be defined as $\mu_S = r + \xi_1\sigma_1$. Suppose that the plan member continuously pays into the pension plan at a constant contribution rate π in the unit time. Let $W(t)$ denote the wealth accumulated in the pension plan at time t, let p denote the proportion of this wealth invested in the risky asset S, and let $(1-p)$ denote the fraction of wealth invested in the risk free asset. Then,

$$\begin{aligned} dW &= [(r + p\xi_1\sigma_1)W + \pi]dt + p\sigma_1 W dZ_1\ , \\ W(t=0) &= \hat{w}_0 \geq 0\ . \end{aligned} \tag{2.2}$$

Define,

$$\begin{aligned}
E[\cdot] \quad &: \quad \text{expectation operator,} \\
Var[\cdot] \quad &: \quad \text{variance operator,} \\
\text{Std}[\cdot] \quad &: \quad \text{standard deviation operator,} \\
E_{t,w}[\cdot],\ Var_{t,w}[\cdot] \text{ or } \text{Std}_{t,w}[\cdot] \quad &: \quad E[\cdot|W(t)=w],\ Var[\cdot|W(t)=w] \text{ or } \text{Std}[\cdot|W(t)=w] \text{ when sitting} \\
&\quad\ \ \text{at time } t, \\
E^q_{t,w}[\cdot],\ Var^q_{t,w}[\cdot] \text{ or } \text{Std}^q_{t,w}[\cdot] \quad &: \quad E_{t,w}[\cdot],\ Var_{t,w}[\cdot] \text{ or } \text{Std}_{t,w}[\cdot],\text{ with } q(s,W(s)),\ s \geq t, \text{ being the} \\
&\quad\ \ \text{policy along path } W(t) \text{ from stochastic process (2.2), where } q \text{ can} \\
&\quad\ \ \text{be } p \text{ (the proportion of the total wealth invested in the risky asset),} \\
&\quad\ \ \text{or } pw \text{ (the monetary amount invested in the risky asset)} \ . \qquad (2.3)
\end{aligned}$$

2.1 Pre-commitment Policy

We seek the optimal policy which solves the following optimization problem,

$$J(w,t) = \sup_{q(s\geq t, W(s))} \{E^q_{t,w}[W_T] - \lambda Var^q_{t,w}[W_T]\} \ , \tag{2.4}$$

subject to stochastic process (2.2), and where $\lambda > 0$ is a given Lagrange multiplier and $q(t, W(t))$ is the investment strategy. In this paper, the strategy q can be p (the proportion of the total wealth invested in the risky asset), or pw (the monetary amount invested in the risky asset). We will discuss this in detail in later Sections. The multiplier λ can be interpreted as a coefficient of risk aversion. The optimal policy for (2.4) is called a pre-commitment policy [2].

Let $q^*_t(s,w)$, $s \geq t$, be the optimal policy for problem (2.4). Then, $q^*_{t+\Delta t}(s,w)$, $s \geq t+\Delta t$, is the optimal policy for

$$J(W(t+\Delta t), t+\Delta t) = \sup_{q(s\geq t+\Delta t, W(s))} \{E^q_{t+\Delta t, W(t+\Delta t)}[W_T] - \lambda Var^q_{t+\Delta t, W(t+\Delta t)}[W_T]\} \ . \tag{2.5}$$

However, in general

$$q^*_t(s, W(s)) \neq q^*_{t+\Delta t}(s, W(s)) \ ; \ s \geq t + \Delta t \ , \tag{2.6}$$

i.e. the solution of problem (2.4) is time inconsistent [2, 4]. Therefore, the dynamic programming principle cannot be directly applied to solve this problem. However, problem (2.4) can be embedded into a class of auxiliary stochastic Linear-Quadratic (LQ) problems using the method in [21, 14]. The optimal strategy $q^*_t(s,w)$ can be determined by solving those LQ problems with dynamic programming principle. We have discussed the pre-commitment policy in detail in [19].

2.2 Time-consistent Policy

In this paper, we will focus on the so called time-consistent policy. We can determine the time-consistent policy by solving problem (2.4) with an additional constraint,

$$q^*_t(s,w) = q^*_{t'}(s,w) \ ; \ s \geq t', \ t' \in [t, T] \ . \tag{2.7}$$

In other words, we optimize problem (2.4) at time t, given that we follow the optimal policy at time t' in the future, which is determined by solving (2.4) at each future instant. Obviously, dynamic programming can be applied to the time-consistent problem.

3 Time-consistent Mean Variance Policy: Wealth Case

Let,

$$
\begin{aligned}
\mathbb{D} &:= \text{the set of all admissible wealth } W(t), \text{ for } 0 \le t \le T; \\
\mathbb{Q} &:= \text{the set of all admissible controls } q(t,w), \text{ for } 0 \le t \le T \text{ and } w \in \mathbb{D}.
\end{aligned} \tag{3.1}
$$

As discussed in the previous section, we want to find the optimal policy for the problem

$$
\begin{aligned}
J(w,t) &= \sup_{\substack{q(s\ge t, W(s))\\ q\in\mathbb{Q}}} \left\{ E^q_{t,w}[W_T] - \lambda Var^q_{t,w}[W_T] \right\} , \\
\text{s.t. } q^*_t(s,w) &= q^*_{t'}(s,w) \ ; \ s \ge t', \ t' \in [t,T] ,
\end{aligned} \tag{3.2}
$$

subject to stochastic process (2.2), and where q^* denotes the optimal control. Varying $\lambda \in (0,\infty)$ allows us to draw an efficient frontier.

We can now drop the subscript "t" from $q^*_t(s,w)$, since we will impose constraint (2.7) and the optimal policy at time $t+\Delta t$ does not depend on the policy at time t. Note we are following here time consistency as defined in [2].

Define,

$$
U(w,t) = E^{q^*(s\ge t, W(s))}_{t,w}[W_T] , \tag{3.3}
$$

$$
V(w,t) = E^{q^*(s\ge t, W(s))}_{t,w}[W_T^2] , \tag{3.4}
$$

with terminal condition

$$
\begin{aligned}
U(w,t=T) &= w, \\
V(w,t=T) &= w^2 .
\end{aligned} \tag{3.5}
$$

Note that

$$
U(w,t) = E^{q^*(t\le s\le t+\Delta t, W(s))}_{t,w}[U(W(t+\Delta t), t+\Delta t)] , \tag{3.6}
$$

$$
V(w,t) = E^{q^*(t\le s\le t+\Delta t, W(s))}_{t,w}[V(W(t+\Delta t), t+\Delta t)] . \tag{3.7}
$$

Then, $J(w,t)$ can be rewritten as

$$
\begin{aligned}
J(w,t) &= \sup_{\substack{q(s\ge t, W(s))\\ q\in\mathbb{Q}}} \left\{ E^{q(s\ge t,W(s))}_{t,w}[W_T] - \lambda\{E^{q(s\ge t,W(s))}_{t,w}[W_T^2] - (E^{q(s\ge t,W(s))}_{t,w}[W_T])^2\} \right\} \\
&= \sup_{\substack{q(t\le s\le t+\Delta t), W(s))\\ q\in\mathbb{Q}}} \left\{ E^{q(t\le s\le t+\Delta t, W(s))}_{t,w}[E^{q^*(s\ge t+\Delta t, W(s))}_{t+\Delta t, W(t+\Delta t)}(W_T)] \right. \\
&\quad -\lambda E^{q(t\le s\le t+\Delta t, W(s))}_{t,w}[E^{q^*(s\ge t+\Delta t, W(s))}_{t+\Delta t, W(t+\Delta t)}(W_T^2)] \\
&\quad \left. +\lambda(E^{q(t\le s\le t+\Delta t, W(s))}_{t,w}[E^{q^*(s\ge t+\Delta t, W(s))}_{t+\Delta t, W(t+\Delta t)}(W_T)])^2 \right\} \\
&= \sup_{\substack{q(t\le s\le t+\Delta t, W(s))\\ q\in\mathbb{Q}}} \left\{ E^q_{t,w}[U(W(t+\Delta t), t+\Delta t)] \right. \\
&\quad \left. -\lambda(E^q_{t,w}[V(W(t+\Delta t), t+\Delta t)] - \{E^q_{t,w}[U(W(t+\Delta t), t+\Delta t)]\}^2) \right\} .
\end{aligned} \tag{3.8}
$$

Assume that the set of all controls $\mathbb{Q}$ is compact, and that $E^q_{t,w}[\cdot]$ is a bounded, upper semi-continuous function of the control q. Given $q^*(s \geq t + \Delta t, W(s))$, suppose we can determine $(U(W(t + \Delta t), t + \Delta t), V(W(t + \Delta t), t + \Delta t))$. Then,

$$
\begin{aligned}
q^*(t \leq s \leq t + \Delta t, W(s)) \in \underset{\substack{q(t \leq s \leq t+\Delta t, W(s)) \\ q \in \mathbb{Q}}}{\arg\max} & \Big\{ E^q_{t,w}[U(W(t + \Delta t), t + \Delta t)] \\
& - \lambda (E^q_{t,w}[V(W(t + \Delta t), t + \Delta t)] - \{E^q_{t,w}[U(W(t + \Delta t), t + \Delta t)]\}^2) \Big\} .
\end{aligned} \tag{3.9}
$$

Equations (3.6-3.9) can be used as the basis for a recursive algorithm to determine $V(w, t), U(w, t)$ for any t (see later in Algorithm (5.10)). Assuming $V(w, t = 0), U(w, t = 0)$ are known, then for a given λ, we can compute the pair $(Var^{q^*}_{t=0,w}[W_T], E^{q^*}_{t=0,w}[W_T])$ from $Var^{q^*}_{t=0,w}[W_T] = V(w, t = 0) - [U(w, t = 0)]^2$.

Remark 3.1 *The classic multi-period portfolio selection problem can be stated as the following: given some investment choices (assets) in the market, an investor seeks an optimal asset allocation strategy over a period T with an initial wealth $\hat{w}_0$. This problem has been widely studied in terms of a pre-commitment strategy [17, 21, 14, 16, 3, 15]. If we use the mean variance approach with a time-consistent strategy to solve this problem, then the best strategy $q^*(w, t)$ can be defined as a solution of problem (3.2). We still assume there is one risk free bond and one risky asset in the market. In this case,*

$$
\begin{aligned}
dW &= (r + p\xi_1\sigma_1)W dt + p\sigma_1 W dZ_1 , \\
W(t = 0) &= \hat{w}_0 > 0 .
\end{aligned} \tag{3.10}
$$

Clearly, the pension plan problem we introduced previously can be reduced to the classic multi-period portfolio selection problem by simply setting the contribution rate $\pi = 0$. All equations and terminal conditions stay the same.

4 Wealth-to-income Ratio Case

In the previous section, we considered the expected value and variance of the terminal wealth in order to construct an efficient frontier. Many studies have shown that a desirable feature of a pension plan is that the holder's wealth W is large compared to her annual salary Y the year before she retires. In this section, instead of the terminal wealth, we determine the mean variance efficient strategy in terms of the terminal wealth-to-income ratio $X = \frac{W}{Y}$. In the following, we give a brief overview of the model developed in [5]. We still assume there are two underlying assets in the pension plan: one is risk free and the other is risky. Recall from equation (2.1) that the risky asset S follows the Geometric Brownian Motion,

$$
dS = (r + \xi_1\sigma_1)S \ dt + \sigma_1 S \ dZ_1 . \tag{4.1}
$$

Suppose that the plan member continuously pays into the pension plan at a fraction π of her yearly salary Y, which follows the process

$$
dY = (r + \mu_Y)Y \ dt + \sigma_{Y_0} Y \ dZ_0 + \sigma_{Y_1} Y \ dZ_1 , \tag{4.2}
$$

where μ_Y, σ_{Y_0} and σ_{Y_1} are constants, and dZ_0 is another increment of a Wiener process, which is independent of dZ_1. Let p denote the proportion of this wealth invested in the risky asset S, and let $1-p$ denote the fraction of wealth invested in the risk-free asset. Then

$$\begin{aligned} dW &= (r + p\xi_1\sigma_1)W\ dt + p\sigma_1 W dZ_1 + \pi Y dt\ , \\ W(t=0) &= \hat{w}_0 \geq 0\ . \end{aligned} \tag{4.3}$$

Define a new state variable $X(t) = W(t)/Y(t)$, then by Ito's Lemma, we obtain

$$\begin{aligned} dX &= [\pi + X(-\mu_Y + p\sigma_1(\xi_1 - \sigma_{Y_1}) + \sigma_{Y_0}^2 + \sigma_{Y_1}^2)]dt \\ &\quad -\sigma_{Y_0} X dZ_0 + X(p\sigma_1 - \sigma_{Y_1})dZ_1\ , \\ X(t=0) &= \hat{x}_0 \geq 0\ . \end{aligned} \tag{4.4}$$

We can write this problem in the form of problem (3.2), if we let the control $q = p$ or $q = pw$. The time-consistent control problem is then to determine the strategy $q(t, X(t) = x)$ such that $q(t,x)$ maximizes

$$\begin{aligned} J(x,t) &= \sup_{\substack{q(s\geq t, X(s))\\ q\in\mathbb{Q}}} \left\{ E_{t,x}^q[X_T] - \lambda Var_{t,x}^q[X_T] \right\}\ , \\ \text{s.t. } q_t^*(s,x) &= q_{t'}^*(s,x)\ ;\ s \geq t',\ t' \in [t,T]\ . \end{aligned} \tag{4.5}$$

subject to stochastic process (4.4). Similar to the wealth case, let

$$\begin{aligned} U(x,t) &= E_{t,x}^{q^*(s\geq t, X(s))}[X_T]\ , \\ V(x,t) &= E_{t,x}^{q^*(s\geq t, X(s))}[X_T^2]\ , \end{aligned} \tag{4.6}$$

with,

$$\begin{aligned} q^*(t \leq s \leq t+\Delta t, X(s)) \in & \operatorname*{arg\,max}_{\substack{q(t\leq s\leq t+\Delta t, X(s))\\ q\in\mathbb{Q}}} \Big\{ E_{t,x}^q[U(X(t+\Delta t), t+\Delta t)] \\ & -\lambda(E_{t,x}^q[V(X(t+\Delta t), t+\Delta t)] - \{E_{t,x}^q[U(X(t+\Delta t), t+\Delta t)]\}^2) \Big\}\ . \end{aligned} \tag{4.7}$$

Then, $J(x,t)$ can be rewritten as

$$\begin{aligned} J(x,t) = & \sup_{\substack{q(t\leq s\leq t+\Delta t, X(s))\\ q\in\mathbb{Q}}} \Big\{ E_{t,x}^q[U(X(t+\Delta t), t+\Delta t)] \\ & -\lambda(E_{t,x}^q[V(X(t+\Delta t), t+\Delta t)] - \{E_{t,x}^q[U(X(t+\Delta t), t+\Delta t)]\}^2) \Big\}\ . \end{aligned} \tag{4.8}$$

Remark 4.1 *The problem described in Section 3 can be seen as a special case of the problem described in this section. We can simply set the salary Y to be a constant (let $\sigma_{Y_0} = \sigma_{Y_1} = 0$ and $\mu_y = -r$), then $X(t)$ is reduced to $W(t)$ and problem (4.5) is reduced to problem (3.2).*

5 Discretization

In this section, we develop a discretization scheme to solve the mean variance time-consistent problem numerically. Let $z = w$ for the wealth case, and $z = x$ for the wealth-to-income ratio case. The optimal control problem in both cases is then

$$
\begin{aligned}
U(z,t) &= E_{t,z}^{q^*(t \le s \le t+\Delta t, Z(s))}[U(Z(t+\Delta t), t+\Delta t)] \ , && (5.1)\\
V(z,t) &= E_{t,z}^{q^*(t \le s \le t+\Delta t, Z(s))}[V(Z(t+\Delta t), t+\Delta t)] \ , && (5.2)\\
q^*(t \le s \le t+\Delta t, Z(s)) &\in \underset{\substack{q(t \le s \le t+\Delta t, Z(s))\\ q \in \mathbb{Q}}}{\arg\max} \Big\{ E_{t,z}^{q}[U(Z(t+\Delta t), t+\Delta t)] && (5.3)\\
&\quad -\lambda(E_{t,z}^{q}[V(Z(t+\Delta t), t+\Delta t)] - \{E_{t,z}^{q}[U(Z(t+\Delta t), t+\Delta t)]\}^2) \Big\} \ ,
\end{aligned}
$$

with terminal condition

$$
\begin{aligned}
U(z, t=T) &= z,\\
V(z, t=T) &= z^2 \ . && (5.4)
\end{aligned}
$$

The form of constraints applied to the control will dictate a choice of $q = p$ or $q = pz$. This will be discussed in later Sections.

5.1 Piecewise Constant Timestepping

For general constraints, we cannot find an analytic solution for the time-consistent strategy. Therefore, the control has to be to determined numerically. One possible approach for solution of problem (5.1-5.3) is to use piecewise constant policy timestepping [12].

We can replace the set of admissible controls $\mathbb{Q}$ by an approximation $\hat{\mathbb{Q}}$. Define

$$
\begin{aligned}
\hat{\mathbb{Q}} = [q_0, q_1, ..., q_m] \quad &, \quad \text{with } q_0 = q_{\min} \ ; \ q_m = q_{\max} \ ,\\
\max_{0 \le j \le m-1} (q_{j+1} - q_j) \ &= \ C_1 h \ , && (5.5)
\end{aligned}
$$

where C_1 is a positive constant. Let $\Delta t = \frac{T}{N}$. Define a set of discrete times,

$$
\begin{aligned}
\{t_n \quad &| \quad t_n = n\Delta t \ , \ 0 \le n \le N\} \ ,\\
\Delta t \ &= \ C_2 h \ , && (5.6)
\end{aligned}
$$

where C_2 is a positive constant. We assume the control is a constant over the period $[t_n, t_{n+1}]$. Set

$$
\begin{aligned}
q^n(w) &= q(t_n, w) \ ; \ U^n(w) = U(t_n, w) \ ; \ V^n(w) = V(t_n, w) && (5.7)\\
U_j^n(w) &= E_{t_n,w}^{q_j}[U^{n+1}(W(t_{n+1}))] \ , && (5.8)\\
V_j^n(w) &= E_{t_n,w}^{q_j}[V^{n+1}(W(t_{n+1}))] \ , && (5.9)
\end{aligned}
$$

We compute the solutions of equations (5.8) and (5.9) for each control q_j, $0 \le j \le m$, then find the optimal control q_{j^*} according to the objective function, and update the values for U^n and V^n. This gives us the following algorithm.

Piecewise Constant Timestepping Algorithm

$U_j^N(w) = w \ , \ V_j^N(w) = w^2 \ , \ \text{for all } 0 \le j \le m$
For timestep $n = N-1, \ldots, 0$
  For $j = 0, \ldots, m$
    $U_j^n(w) = E_{t_n,w}^{q_j}[U^{n+1}(W(t+\Delta t))]$
    $V_j^n(w) = E_{t_n,w}^{q_j}[V^{n+1}(W(t+\Delta t))]$
  EndFor
  $j^* \in \underset{0\le j\le m}{\arg\max}\{U_j^n(w) - \lambda(V_j^n(w) - (U_j^n(w))^2)\}$
  $(q^n(w))^* = q_{j^*} \ ; \ U_j^n(w) = U_{j^*}^n(w) \ ; \ V_j^n(w) = V_{j^*}^n(w) \ , \ \text{for all } 0 \le j \le m$
EndFor

(5.10)

Remark 5.1 *In [12], the authors applied the piecewise constant timestepping to a scalar Hamilton-Jacobi-Bellman (HJB) equation, and proved that the solution given by the piecewise constant timestepping method converges to the viscosity solution. However, the problem we study in this paper is more complex, since we solve a system set of expectations and a nonlinear algebraic equation. We have no proof that Algorithm (5.10) converges to the solutions of equations (3.6-3.9), although we will see in Section 7 that our numerical solutions converge to the analytic solutions where available.*

5.2 Computing the Expectations

Algorithm (5.10) gives a piecewise constant timestepping method for solution of the optimal stochastic control problem. However, it is not clear how we can compute $U_j^n(w)$ and $V_j^n(w)$. Recall that

$$\begin{aligned} U_j^n(w) &= E_{t_n,w}^{q_j}[U^{n+1}(W(t_{n+1}))] \ , \\ V_j^n(w) &= E_{t_n,w}^{q_j}[V^{n+1}(W(t_{n+1}))] \ . \end{aligned}$$

According to [12], given a constant control q_j, we can determine $U_j^n(w)$ and $V_j^n(w)$ by solving

$$-U_t = \mu_z^q U_z + \frac{1}{2}(\sigma_z^q)^2 U_{zz} \quad ; \quad z \in \mathbb{D} \ , \tag{5.11}$$

$$-V_t = \mu_z^q V_z + \frac{1}{2}(\sigma_z^q)^2 V_{zz} \quad ; \quad w \in \mathbb{D}, \tag{5.12}$$

over the interval $[t_{n+1}, t_n]$ (we solve backward in time) with $U(z, t = t_{n+1})$, $V(z, t = t_{n+1})$ computed from the previous step of Algorithm (5.10), and at $t = t_N$ $(t = T)$,

$$\begin{aligned} U(z, t = T) &= z, \\ V(z, t = T) &= z^2 \ , \end{aligned} \tag{5.13}$$

and where

$$\begin{aligned}\mu_z^q &= \mu_w^q = \pi + w(r + p\sigma_1\xi_1) \\ (\sigma_z^q)^2 &= (\sigma_w^q)^2 = (p\sigma_1 w)^2 \ .\end{aligned} \tag{5.14}$$

for the wealth case introduced in Section 3; and

$$\begin{aligned}\mu_z^q &= \mu_x^q = \pi + x(-\mu_Y + p\sigma_1(\xi_1 - \sigma_{Y_1}) + \sigma_{Y_0}^2 + \sigma_{Y_1}^2) \\ (\sigma_z^q)^2 &= (\sigma_x^q)^2 = x^2(\sigma_{Y_0}^2 + (p\sigma_1 - \sigma_{Y_1})^2) \ .\end{aligned} \tag{5.15}$$

for the wealth-to-income ratio case introduced in Section 4. Note that in equations (5.14) and (5.15), we set $q = p$ (use p as the control). If we want to use pw (the monetary amount invested in the risky asset) as the control, we can set $q = pw$ (px) and replace pw (px) by q in equation (5.14) ((5.15)).

Since we solve PDEs (5.11) and (5.12) backward in time, in order to derive the discretization of the PDEs using conventional notations, let $\tau = T - t$. Then, $\tau_n = T - t_{N-n}$ for $0 \leq n \leq N$. We define

$$\hat{U}(\tau, z) = U(T - t, z) \ , \tag{5.16}$$

$$\hat{V}(\tau, z) = U(T - t, z) \ . \tag{5.17}$$

Then equations (5.11) and (5.12) become to

$$\hat{U}_\tau = \mu_z^q \hat{U}_z + \frac{1}{2}(\sigma_z^q)^2 \hat{U}_{zz} \quad ; \quad z \in \mathbb{D} \ , \tag{5.18}$$

$$\hat{V}_\tau = \mu_z^q \hat{V}_z + \frac{1}{2}(\sigma_z^q)^2 \hat{V}_{zz} \quad ; \quad w \in \mathbb{D}, \tag{5.19}$$

with terminal condition

$$\begin{aligned}\hat{U}(z, \tau = 0) &= z \ , \\ \hat{V}(z, \tau = 0) &= z^2 \ .\end{aligned} \tag{5.20}$$

We then can find the values for $\hat{U}_j^n(w)$ and $\hat{V}_j^n(w)$ by solving PDEs (5.18) and (5.19), over the interval $[\tau_n, \tau_{n+1}]$ in Algorithm (5.10).

5.3 Localization

Let,

$$\begin{aligned}\hat{\mathbb{D}} &:= \text{a finite computational domain which approximates the set } \mathbb{D}. \\ \hat{\mathbb{Q}} &:= \text{a finite computational set which approximates the set } \mathbb{Q}.\end{aligned} \tag{5.21}$$

In order to solve PDEs (5.18) and (5.19) we need to use a finite computational domain, $\hat{\mathbb{D}} = [z_{\min}, z_{\max}]$. When $z \to \pm\infty$, we assume that

$$\begin{aligned}\hat{U}(z \to \pm\infty, \tau) &\simeq A_1(\tau) z \ , \\ \hat{V}(z \to \pm\infty, \tau) &\simeq B_1(\tau) z^2 \ .\end{aligned} \tag{5.22}$$

Then, taking into account the initial conditions (3.5),

$$\begin{aligned} \hat{U}(z \to \pm\infty, \tau) &\simeq e^{k_1 \tau} z \ , \\ \hat{V}(z \to \pm\infty, \tau) &\simeq e^{(2k_1+k_2)\tau} z^2 \ , \end{aligned} \tag{5.23}$$

If $q = p$ (use p as the control), then $k_1 = r + q\sigma_1\xi_1$ and $k_2 = (q\sigma_1)^2$ for the wealth case; $k_1 = -\mu_Y + q\sigma_1(\xi_1 - \sigma_{Y_1}) + \sigma_{Y_0}^2 + \sigma_{Y_1}^2$ and $k_2 = \sigma_{Y_0}^2 + (q\sigma_1 - \sigma_{Y_1})^2$ for the wealth-to-income ratio case. If $q = pz$ (use pw or px as the control), then $k_1 = r + \frac{q}{w}\sigma_1\xi_1$ and $k_2 = \frac{(q\sigma_1)^2}{w^2}$ for the wealth case; $k_1 = -\mu_Y + \frac{q}{x}\sigma_1(\xi_1 - \sigma_{Y_1}) + \sigma_{Y_0}^2 + \sigma_{Y_1}^2$ and $k_2 = \sigma_{Y_0}^2 + (\frac{q}{x}\sigma_1 - \sigma_{Y_1})^2$ for the wealth-to-income ratio case.

Since in Algorithm (5.10), we update the values for U and V at the end of each timestep according to the optimal strategy, it is more appropriate to compute $\hat{U}$ and $\hat{V}$ at $z \to \pm\infty$ by using the updated values. Rewriting equation (5.23) gives

$$\begin{aligned} \hat{U}(z \to \pm\infty, \tau + \Delta\tau) &\simeq e^{k_1 \Delta\tau} \hat{U}(z \to \pm\infty, \tau) \ , \\ \hat{V}(z \to \pm\infty, \tau + \Delta\tau) &\simeq e^{(2k_1+k_2)\Delta\tau} \hat{V}(z \to \pm\infty, \tau) \ . \end{aligned} \tag{5.24}$$

More discussion of these boundary condition is given in Section 6.

5.4 Discretization of PDEs

In this section, we give a brief overview of method used to solve PDEs (5.18) and (5.19). See [19] for more detail. Given a control q, define operator $\mathcal{L}^q$ as

$$\mathcal{L}^q \hat{V} \equiv a(z,q)\hat{V}_{zz} + b(z,q)\hat{V}_z \ , \tag{5.25}$$

where

$$\begin{aligned} a(z,q) &= \frac{1}{2}(\sigma_z^q)^2 \ , \\ b(z,q) &= \mu_z^q \ . \end{aligned} \tag{5.26}$$

Then,

$$\hat{U}_\tau = \mathcal{L}^q \hat{U} \ , \tag{5.27}$$

$$\hat{V}_\tau = \mathcal{L}^q \hat{V} \ . \tag{5.28}$$

Define a grid $\{z_0, z_1, \ldots, z_l\}$ with $z_0 = z_{\min}$, $z_l = z_{\max}$. Given a control q_j, let $\hat{V}_{i,j}^n$ be a discrete approximation to $\hat{V}(z_i, \tau^n)$ with control q_j. Let $\hat{V}_j^n = [\hat{V}_{0,j}^n, \ldots, \hat{V}_{l,j}^n]'$, and let $(\mathcal{L}_h^{q_j} \hat{V}_j^n)_i$ denote the discrete form of the differential operator (5.25) at node (z_i, τ^n) with a control q_j. The operator (5.25) can be discretized using forward, backward or central differencing in the z direction to give

$$(\mathcal{L}_h^{q_j} \hat{V}_j^{n+1})_i = \alpha_{i,j}^{n+1} \hat{V}_{i-1,j}^{n+1} + \beta_{i,j}^{n+1} \hat{V}_{i+1,j}^{n+1} - (\alpha_{i,j}^{n+1} + \beta_{i,j}^{n+1}) \hat{V}_{i,j}^{n+1} \ . \tag{5.29}$$

Here $\alpha_{i,j}$, $\beta_{i,j}$ are defined in Appendix A.

Equations (5.28) can now be approximated by

$$\frac{\hat{V}_{i,j}^{n+1} - \hat{V}_{i,j}^n}{\Delta\tau} = (\mathcal{L}_h^{q_j} \hat{V}_j^{n+1})_i \ , \tag{5.30}$$

Similarly equation (5.27) can be discretized as

$$\frac{\hat{U}_{i,j}^{n+1} - \hat{U}_{i,j}^n}{\Delta\tau} = (\mathcal{L}_h^{q_j} \hat{U}_j^{n+1})_i \ . \tag{5.31}$$

5.5 Algorithm for Construction of the Efficient Frontier

Given a positive value for λ, by solving PDEs (5.18) and (5.19) over each period $[\tau_n, \tau_{n+1}]$, we can compute the numerical solutions of equations (5.1), (5.2) and (5.3). For the convenience of the reader, we rewrite Algorithm (5.10) in terms of $\tau = T - t$, where the expectations are given by solving equations (5.30) and (5.31). The algorithm is given below.

Algorithm for the Time-consistent Policy

$\hat{U}^0_{i,j} = z_i$, $\hat{V}^0_{i,j} = z_i^2$, for all $0 \le i \le l$ and $0 \le j \le m$

For timestep $n = 0, \ldots, N-1$

  For $j = 0, \ldots, m$

    Solve equations (5.31) and (5.30)

  EndFor

  For $i = 0, \ldots, l$ (5.32)

    $j^* \in \arg\max_{0 \le j \le m} \{\hat{U}^{n+1}_{i,j} - \lambda(\hat{V}^{n+1}_{i,j} - (\hat{U}^{n+1}_{i,j})^2)\}$

    $(q_i^{n+1})^* = q_{j^*}$; $\hat{U}^{n+1}_{i,k} = \hat{U}^{n+1}_{i,j^*}$; $\hat{V}^{n+1}_{i,k} = \hat{V}^{n+1}_{i,j^*}$, for all $0 \le k \le m$,

  EndFor

EndFor

Given an initial value $\hat{z}_0$, Algorithm (5.33) is used to obtain the efficient frontier. Since the Z grid is discretized over the interval $[z_{min}, z_{max}]$, we can use Algorithm (5.33) to obtain the efficient frontier for any initial wealth $\hat{z}_0 \in [z_{min}, z_{max}]$ by interpolation. Of course, if we choose $\hat{z}_0$ to be a node in the discretized Z grid, then there is no interpolation error.

Algorithm for Constructing the Efficient Frontier

For $\lambda = \lambda_{min}, \lambda_1, \ldots, \lambda_{max}$

  Compute solutions of equations (5.1), (5.2) and (5.3) by Algorithm (5.32)

  Given the initial $\hat{z}_0$, use interpolation to get the numerical values of

$$(\hat{U}(\hat{z}_0, t=0), \hat{V}(\hat{z}_0, t=0))_\lambda \text{ at } Z(t=0) = \hat{z}_0$$

  Then $E^{q^*}_{t=0,\hat{z}_0}[Z_T] = \hat{U}(\hat{z}_0, t=0)$

  and $\text{Std}^{q^*}_{t=0,\hat{z}_0}[Z_T] = \sqrt{\hat{V}(\hat{z}_0, t=0) - [\hat{U}(\hat{z}_0, t=0)]^2}$

EndFor

Construct the efficient frontiers from the points

$$(\text{Std}^{q^*}_{t=0,\hat{z}_0}[Z_T], E^{q^*}_{t=0,\hat{z}_0}[Z_T])_\lambda, \lambda \in [\lambda_{\min}, \lambda_{\max}]$$

(5.33)

6 Various Constraints

In this section, we apply various constraints to the control policy q. We consider three cases: allowing bankruptcy, no bankruptcy (no shorting stocks) and bounded control. We will see later that these constraints have different effects on boundary conditions and dramatically change the properties of the efficient frontiers.

We summarize the various cases in Table 1 below.

Case	Control q	Original Domain: $\mathbb{D}$, $\mathbb{Q}$	Localized Domain: $\hat{\mathbb{D}}$, $\hat{\mathbb{Q}}$
Bankruptcy	pz	$(-\infty, +\infty)$, $(-\infty, +\infty)$	$[z_{\min}, z_{\max}]$, $[q_{\min}, q_{\max}]$
No Bankruptcy	p or pz	$[0, +\infty)$, $[0, +\infty)$	$[0, z_{\max}]$, $[0, q_{\max}]$
Bounded Control	p	$[0, +\infty)$, $[0, q_{\max}]$	$[0, z_{\max}]$, $[0, q_{\max}]$

Table 1: *Summary of cases.*

6.1 Allowing Bankruptcy, Unbounded Controls

In this case, we assume there are no constraints on $Z(t)$ or on the control q, i.e., $\mathbb{D} = (-\infty, +\infty)$ and $\mathbb{Q} = (-\infty, +\infty)$. Since $Z(t) = z$ can be negative, bankruptcy is allowed. We call this case the *allowing bankruptcy* case. We solve this problem by using the monetary amount invested in the risky asset as the control ($q = pz$). Note that the amount invested in the risky asset was also used as the control in [3] to determine analytic solution for the pre-commitment policy.

Our numerical problem uses

$$\hat{\mathbb{D}} = [z_{\min}, z_{\max}] \ , \ \hat{\mathbb{Q}} = [q_{\min}, q_{\max}] \ , \tag{6.1}$$

where $\hat{\mathbb{D}} = [z_{\min}, z_{\max}]$ and $\hat{\mathbb{Q}} = [q_{\min}, q_{\max}]$ are approximations to the original set $\mathbb{D} = (-\infty, +\infty)$ and $\mathbb{Q} = (-\infty, +\infty)$. At $z = z_{\min}, z_{\max}$ we apply the Dirichlet conditions (5.24).

These artificial boundary conditions will cause some error. However, we can make these errors small by choosing large values for $(|z_{\min}|, z_{\max})$ and $(|q_{\min}|, q_{\max})$. The error will be small if $(|z_{\min}|, z_{\max})$ and $(|q_{\min}|, q_{\max})$ are sufficiently large [1]. We will verify this in some subsequent numerical tests.

An analytic solution exists for the wealth case [2]. The efficient frontier solution is

$$\begin{cases} Var_{t=0,\hat{w}_0}[W_T] = \frac{\xi_1^2}{4\lambda^2} T \\ E_{t=0,\hat{w}_0}[W_T] = \hat{w}_0 e^{rT} + \pi \frac{e^{rT}-1}{r} + \xi\sqrt{T}\text{Std}(W_T) \ , \end{cases} \tag{6.2}$$

and the optimal control ($q = pw$) at any time $t \in [0, T]$ is

$$q^*(t, w) = \frac{\xi_1}{2\lambda\sigma_1} e^{-r(T-t)} \ . \tag{6.3}$$

We can then see directly from the SDE (2.2), that $W(t)$ can be negative in this case. Hence, $\mathbb{D} = (-\infty, +\infty)$. From equation (6.3), given a time t, the optimal monetary amount $q^* = p^* w$ invested in the risky asset is a positive constant. Hence the investor is always long stock.

The efficient frontier $(\text{Std}_{q^*}^{t=0}[W_T], E_{q^*}^{t=0}[W_T])$ in this case is a straight line. We will use this analytic result to check our numerical solution.

Remark 6.1 *For the wealth case, from equation (6.3), we can see that if we use* p *as the control, then*

$$p^*(t, w) = \frac{\xi_1}{2\lambda\sigma_1 w} e^{-r(T-t)} \ . \tag{6.4}$$

Clearly, this will cause some difficulties near $w = 0$*, as discussed in [19]. We can avoid these problem in this case by using the control* $q = pw$*, which is always finite from equation (6.3).*

6.2 No Bankruptcy, No Short Sales

In this case, we assume that bankruptcy is prohibited and the investor cannot short the stock index, i.e., $\mathbb{D} = [0, +\infty)$ and $\mathbb{Q} = [0, +\infty)$. We call this case the *no bankruptcy* (or *bankruptcy prohibition*) case. We can solve this problem by either using the proportion p as the control ($q = p$) or using the monetary amount pw as the control ($q = pw$).

Our numerical problem uses,

$$\hat{\mathbb{D}} \ = \ [0, z_{\max}] \ , \ \hat{\mathbb{Q}} = [0, q_{\max}] \ . \tag{6.5}$$

We prohibit the possibility of bankruptcy ($Z(t) < 0$) by requiring that (see Remark 6.2 below) the optimal monetary amount $\lim_{z\to 0}(p^* z) = 0$, so that PDEs (5.18) and (5.19) reduce to (at $z = 0$)

$$\begin{aligned} \hat{U}_\tau(0, \tau) &= \pi \hat{U}_z \ , \\ \hat{V}_\tau(0, \tau) &= \pi \hat{V}_z \ . \end{aligned} \tag{6.6}$$

Remark 6.2 *It is important to know the behavior of the optimal monetary amount* $p^* z$ *as* $z \to 0$*, since it helps us determine whether negative wealth is admissible or not. Negative wealth is admissible for the case of allowing bankruptcy. In the case of no bankruptcy, although* $p \in \mathbb{P} = [0, +\infty)$*, we must have* $\lim_{z\to 0}(p^* z) = 0$ *so that* $Z(t) \geq 0$ *for all* $0 \leq t \leq T$*. In particular, we need to make sure that the optimal strategy never generates negative wealth, i.e., Probability*$(Z(t) < 0 | p^*) = 0$ *for all* $0 \leq t \leq T$*. We will see from the numerical solutions that boundary condition (6.6) does in fact result in* $\lim_{z\to 0}(p^* z) = 0$*. Hence, negative wealth is not admissible under the optimal strategy. More discussion of this issue is given in Section 7. For the bounded control case, the control is finite, thus* $\lim_{z\to 0}(p^* z) = 0$ *and negative wealth is not admissible.*

6.3 No Bankruptcy, Bounded Control

This is a realistic case, in which we assume that bankruptcy is prohibited and infinite borrowing is not allowed. As a result, $\mathbb{D} = [0, +\infty)$ and $\mathbb{Q} = [0, q_{\max}]$. We call this case the *bounded control* case. Since the borrowing upper bound $q_{\max}$ is usually based on the investor's total wealth (e.g, the investor can borrow at most 50% of her total wealth), we use the proportion of the total wealth invested in the risky asset as the control ($q = p$) for this case.

Our numerical problem uses,

$$\hat{\mathbb{D}} \ = \ [0, z_{\max}] \ , \ \hat{\mathbb{Q}} = \mathbb{Q} = [0, q_{\max}] \ . \tag{6.7}$$

where $z_{\max}$ is an approximation to the infinity boundary. In this case we also specify that $q \geq 0$ (no shorting the risky asset). Other assumptions and the boundary conditions for V and U are the same as those of no bankruptcy case introduced in Section 6.2.

7 Numerical Results

In this section, we carry out numerical tests for the defined contribution pension plan problem. We examine both the wealth case (addressed in Section 3) and the wealth-to-income ratio case (addressed in Section 4).

7.1 Wealth Case

r	0.03	ξ_1	0.33
σ_1	0.15	π	0.1
T	20 years	$W(t=0)$	1

TABLE 2: *Parameters used in the pension plan examples.*

We first consider the wealth case introduced in Section 3. When bankruptcy is allowed, analytic solutions exist. We use the monetary amount pw as the control. Table 3 and 4 show the numerical results. Table 3 reports the value of $E^{q^*}_{t=0,w}[W_T^2]$, which is the solution of equation (5.9). Table 4 reports the value of $E^{q^*}_{t=0,w}[W_T]$, which is the solution of equation (5.8). Given $E^{q^*}_{t=0,w}[W_T^2]$ and $E^{q^*}_{t=0,w}[W_T]$, the standard deviation is can be easily computed, which is also reported in Table 4. The results show that the numerical solutions of $E^{q^*}_{t=0,w}[W_T^2]$ and $E^{q^*}_{t=0,w}[W_T]$ converge to the analytic values at a first order rate as mesh and timestep size tends to zero.

Nodes ($W \times Q$)	Timesteps	Normalized CPU Time	$E^{q^*}_{t=0,w}[W_T^2]$	Ratio
180 × 105	40	1	43.0211	
360 × 209	80	7.24	40.3870	
720 × 417	160	56.16	41.4764	-2.418
1440 × 833	320	437.04	42.0794	1.807
2880 × 1665	640	3445.49	42.3825	1.989
5760 × 3329	1280	31277.09	42.5347	1.991

TABLE 3: *Convergence study of the wealth case, allowing bankruptcy. The monetary amount invested in the risky asset is used as the control ($q = pw$). Fully implicit timestepping is applied, using constant timesteps. Parameters are given in Table 2, with $\lambda = 0.6$. Values of $E^{q^*}_{t=0,w}[W_T^2]$ are reported at ($W = 1, t = 0$). Ratio is the ratio of successive changes in the computed values for decreasing values of the discretization parameter h. Analytic solution is $E^{q^*}_{t=0,w}[W_T^2] = 42.6873$. CPU time is normalized. We take the CPU time used for the first test in this table as one unit of CPU time, which uses 180×105 nodes for $W \times Q$ grid and 40 timesteps.*

We also solve the problem for the no bankruptcy case and the bounded control case. Analytic solutions do not exist for these cases. The efficient frontiers are shown in Figure 1, with parameters given in Table 2. The straight line is the efficient frontier for the allowing bankruptcy case. This result agrees with the analytic solution (equations (6.2)). The curve for the case of no bankruptcy is actually two overlapping curves. One is from the solutions obtained by using the monetary amount invested in the risky asset as the control, and the other is from the solutions using proportion as the control. The lower curve is for the bounded control case. Clearly, the strategy given by the

Nodes $(W \times Q)$	Timesteps	$\text{Std}_{t=0,w}^{q^*}[W_T]$	$E_{t=0,w}^{q^*}[W_T]$	Ratio for $\text{Std}_{t=0,w}^{q^*}[W_T]$	Ratio for $E[W_T]$
180 × 105	40	1.74390	6.32297		
360 × 209	80	1.32762	6.21486		
720 × 417	160	1.28790	6.31013	10.480	-1.135
1440 × 833	320	1.26536	6.36226	1.762	1.828
2880 × 1665	640	1.25392	6.38828	1.970	2.003
5760 × 3329	1280	1.24812	6.40132	1.972	1.995

TABLE 4: *Convergence study of the wealth case, allowing bankruptcy. The monetary amount invested in the risky asset is used as the control ($q = pw$). Fully implicit timestepping is applied, using constant timesteps. Parameters are given in Table 2, with $\lambda = 0.6$. Values of $Std_{t=0,w}^{q^*}[W_T]$ and $E_{t=0,w}^{q^*}[W_T]$ are reported at $(W = 1, t = 0)$. Ratio is the ratio of successive changes in the computed values for decreasing values of the discretization parameter h. Analytic solution is $(Std_{t=0,w}^{q^*}[W_T], E_{t=0,w}^{q^*}[W_T]) = (1.24226, 6.41437)$.*

allowing bankruptcy case is the most efficient, and the strategy given by the bounded control case is the least efficient.

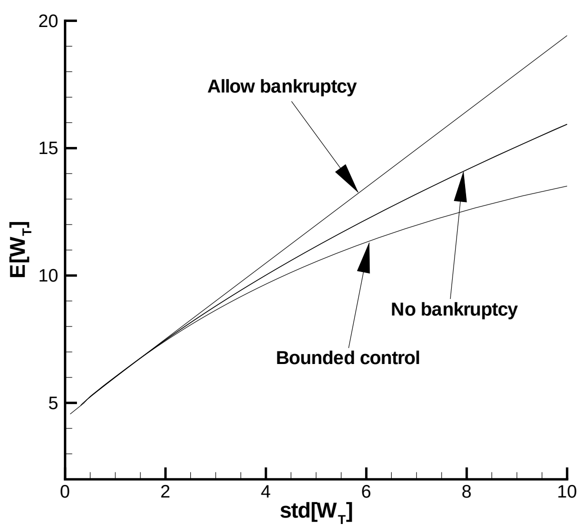


FIGURE 1: *Time-consistent efficient frontiers (wealth case) for allowing bankruptcy ($\mathbb{D} = (-\infty, +\infty)$ and $\mathbb{Q} = (-\infty, +\infty)$), no bankruptcy ($\mathbb{D} = [0, +\infty)$ and $\mathbb{Q} = [0, +\infty)$) and bounded control ($\mathbb{D} = [0, +\infty)$ and $\mathbb{Q} = [0, 1.5]$) cases. Parameters are given in Table 2. Values are reported at $(W = 1, t = 0)$.*

As mentioned in Section 6.1, some error is introduced using the artificial boundaries. However, we can make these errors small by choosing large values for $(|w_{\min}|, w_{\max})$ and $(|q_{\min}|, q_{\max})$. Table 5 shows the values of $E_{t=0,w}^{q^*}[W_T^2]$ and $E_{t=0,w}^{q^*}[W_T]$ for different large boundaries. We can see that once $(|w_{\min}|, w_{\max})$ and $(|u_{\min}|, u_{\max})$ are large enough, the values of $E^{q^*}_{t=0,w}[W_T^2]$ and $E^{q^*}_{t=0,w}[W_T]$ are insensitive to the location of these large boundaries.

$(w_{\min}, w_{\max})$	$(q_{\min}, q_{\max})$	$E^{q^*}_{t=0,w}[W_T^2]$	$E^{q^*}_{t=0,w}[W_T]$
(-1000, 1000)	(-1000, 1000)	42.5347	6.40132
(-2000, 2000)	(-2000, 2000)	42.5347	6.40132
(-5000, 5000)	(-5000, 5000)	42.5347	6.40132
(-10000, 10000)	(-10000, 10000)	42.5347	6.40132

TABLE 5: *Effect of finite boundary, wealth case, allowing bankruptcy. The monetary amount invested in the risky asset is used as the control ($q = pw$). Parameters are given in Table 2, with $\lambda = 0.6$. There are 1280 timesteps for each test. Recall that $q = pw$, which is the monetary amount invested in the risky asset.*

As discussed in Remark 3.1, the wealth case can be reduced to the classic multi-period portfolio selection problem. The efficient frontier solutions of a particular multi-period portfolio selection problem are shown in Figure 2. As for the wealth case, the efficient frontier for the bankruptcy case is a straight line. The curve for the case of no bankruptcy is actually two overlapping curves. One is from the solution obtained using the monetary amount invested in the risky asset as the control, and the other is from the solution computed using the proportion as the control. Again, the strategy given by the allowing bankruptcy case is the most efficient, and the strategy given by the bounded control case is the least efficient.

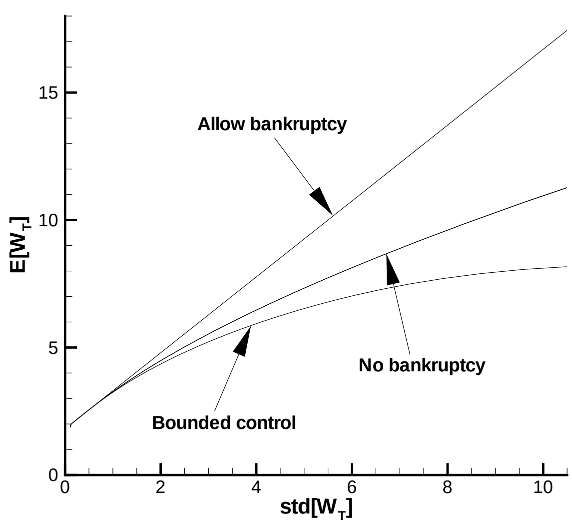


FIGURE 2: *Time-consistent efficient frontiers (multi-period portfolio selection problems) for for allowing bankruptcy ($\mathbb{D} = (-\infty, +\infty)$ and $\mathbb{Q} = (-\infty, +\infty)$), no bankruptcy ($\mathbb{D} = [0, +\infty)$ and $\mathbb{Q} = [0, +\infty)$) and bounded control ($\mathbb{D} = [0, +\infty)$ and $\mathbb{Q} = [0, 1.5]$) cases. Parameters are given in Table 2 except that the contribution rate $\pi = 0$. Values are reported at $(W = 1, t = 0)$.*

7.2 Wealth-to-income Ratio Case

μ_y	0.	ξ_1	0.2
σ_1	0.2	σ_{Y1}	0.05
σ_{Y0}	0.05	π	0.1
T	20 years	λ	0.25
$\mathbb{Q}$	$[0, 1.5]$	$\mathbb{D}$	$[0, +\infty)$

TABLE 6: *Parameters used in the wealth to income ratio pension plan examples.*

In this section, we examine the wealth-to-income ratio case (discussed in Section (4)). Table 6 gives the data used for this example. Table 7 and 8 show a convergence study for the bounded control case. We set $x_{\max}, |x_{\min}| = 1000$ in this case. Increasing $x_{\max}$ had no effect on the solution to six digits. Table 7 reports the value of $E^{q^*}_{t=0,x}[X_T^2]$, and Table 8 reports the values of $E^{q^*}_{t=0,x}[X_T]$ and $\text{Std}^{q^*}_{t=0,x}[X_T]$. The results show that the numerical solutions of $E^{q^*}_{t=0,x}[X_T^2]$ and $E^{q^*}_{t=0,x}[X_T]$ converge at a first order rate as mesh and timestep size tends to zero. No analytic solutions are available in this case.

Nodes ($X \times Q$)	Timesteps	Normalized CPU Time	$E^{q^*}_{t=0,x}[X_T^2]$	Ratio
90 × 16	40	1.	15.1154	
179 × 31	80	17.	15.2894	
357 × 61	160	104.	15.3453	3.113
713 × 121	320	794.50	15.3696	2.300
1425 × 241	640	6430.01	15.3814	2.059
2849 × 481	1280	52513.05	15.3871	2.070

TABLE 7: *Convergence study of the wealth-to-income ratio case, bounded control. The proportion of the total wealth invested in the risky asset is used as the control ($q = p$). We set $q = p \in [0, 1.5]$. Fully implicit timestepping is applied, using constant timesteps. Parameters are given in Table 6, with $\lambda = 0.25$. Values of $E^{q^*}_{t=0,x}[X_T^2]$ are reported at ($X = 0.5, t = 0$). Ratio is the ratio of successive changes in the computed values for decreasing values of the discretization parameter h. CPU time is normalized. We take the CPU time used for the first test in this table as one unit of CPU time, which uses 90×16 nodes for $W \times Q$ grid and 40 timesteps.*

Efficient frontiers for the wealth case are shown in Figure 3, with parameters given in Table 6. The curve for bankruptcy case is determined by using monetary amount invested in the risky asset as the control. As for the wealth case, the curve for the case of no bankruptcy is also two overlapping curves. One is from the solutions using the monetary amount invested in the risky asset as the control, and the other is from the solutions using the proportion as the control. Again, the strategy given by the allowing bankruptcy case is the most efficient, and the strategy given by the bounded control case is the least efficient.

Remark 7.1 *As we discussed in Remark 6.2, in the case of bankruptcy prohibition, we have to have $\lim_{z\to 0}(p^* z) = 0$ so that negative wealth is not admissible, where $z = w$ or x. Our numerical tests show that as z goes to zero, $p^* z = O(z^\beta)$. For a reasonable range of parameters, we have*

Nodes ($W \times Q$)	Timesteps	$\text{Std}^{q^*}_{t=0,x}[X_T]$	$E^{q^*}_{t=0,x}[X_T]$	Ratio for $\text{Std}^{q^*}_{t=0,x}[X_T]$	Ratio for $E[X_T]$
90 × 16	40	1.37474	3.63669		
179 × 31	80	1.35197	3.66900		
357 × 62	160	1.33799	3.68172	1.629	2.540
713 × 121	320	1.33060	3.68770	1.892	2.127
1425 × 241	640	1.32688	3.69063	1.987	2.041
2849 × 481	1280	1.32500	3.69208	1.979	2.021

TABLE 8: *Convergence study of the wealth-to-income ratio case, bounded control. The proportion of the total wealth invested in the risky asset is used as the control ($q = p$). We set $q \in [0, 1.5]$. Fully implicit timestepping is applied, using constant timesteps. Parameters are given in Table 6, with $\lambda = 0.25$. Values of $Std^{q^*}_{t=0,x}[X_T]$ and $E^{q^*}_{t=0,x}[X_T]$ are reported at ($X = 0.5, t = 0$). Ratio is the ratio of successive changes in the computed values for decreasing values of the discretization parameter h.*

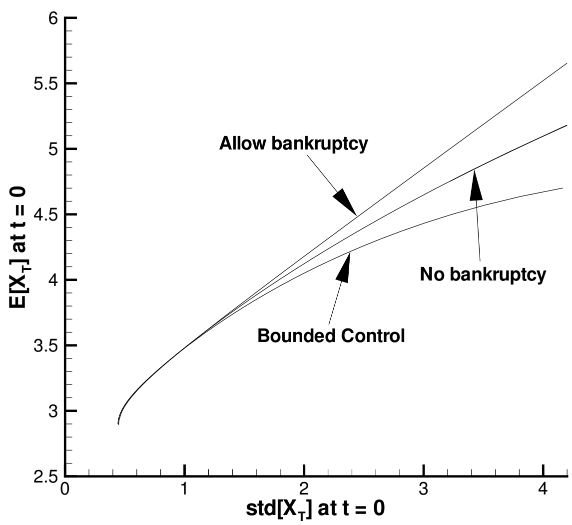


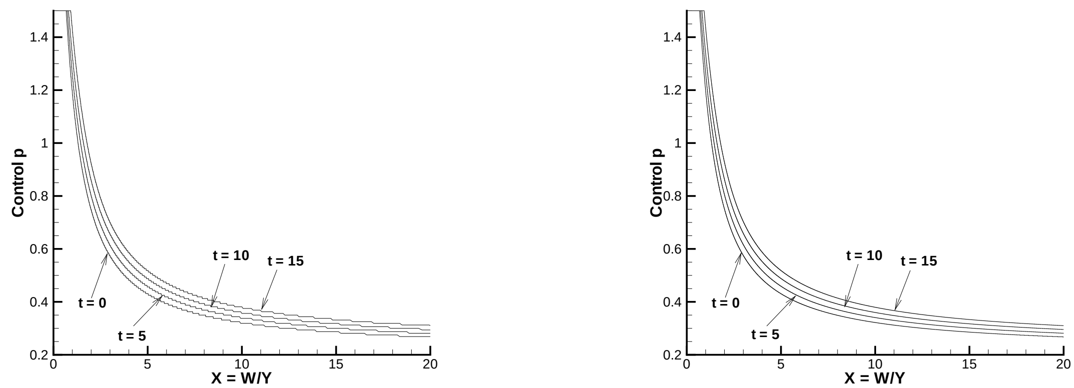
FIGURE 3: *Time-consistent efficient frontiers (wealth-to-income ratio case) for allowing bankruptcy ($\mathbb{D} = (-\infty, +\infty)$ and $\mathbb{Q} = (-\infty, +\infty)$), no bankruptcy ($\mathbb{D} = [0, +\infty)$ and $\mathbb{Q} = [0, +\infty)$) and bounded control ($\mathbb{D} = [0, +\infty)$ and $\mathbb{Q} = [0, 1.5]$) cases. Parameters are given in Table 6. Values are reported at ($X = 0.5, t = 0$).*

$0.9 < \beta < 1$. *As discussed in [11], zero is an unattainable boundary for the stochastic process (2.2) if* $\beta > 0.5$. *Hence, this verifies that the boundary conditions (6.6) ensure that negative wealth is not admissible under the optimal strategy.*

Figure 4 shows the values of the optimal control (the investment strategies) at different times t for a fixed $T = 20$ and $\lambda = 0.25$ for the bounded control case. Under these inputs, if $X(t = 0) = 0.5$, $(\text{Std}^{q^*}_{t=0,x}[X_T], E^{q^*}_{t=0,x}[X_T]) = (1.32500, 3.69208)$ from the finite difference solution. From this figure, we can see that the control q is a increasing function of time t for a fixed X. This behavior of the optimal strategy is also seen in the analytic solution for the wealth case with bankruptcy allowed (Equation 6.3). This result is also the same as for the pre-commitment case [19]. In other words, if time goes on, and wealth remains constant, then the investor's optimal strategy is to invest more in the risky asset. Note that curves for the control is not very smooth in Figure 4 (a). This is due to the fact that we have discretized the control in each interval $[\tau_n, \tau_{n+1}]$. Recall equation (5.5),

$$
\begin{aligned}
\hat{\mathbb{Q}} = [p_0, p_1, ..., q_m] \quad &, \quad \text{with } p_0 = q_{\min} \; ; \; q_m = q_{\max} \; , \\
\max_{0 \le j \le m-1} (q_{j+1} - q_j) \quad &= \quad C_1 h \; .
\end{aligned}
\tag{7.1}
$$

The curves for the control in Figure 4 converge to smooth curves as $h \to \infty$. Figure 4 (b) is computed by using a finer grid and more timesteps.



(a) (b)

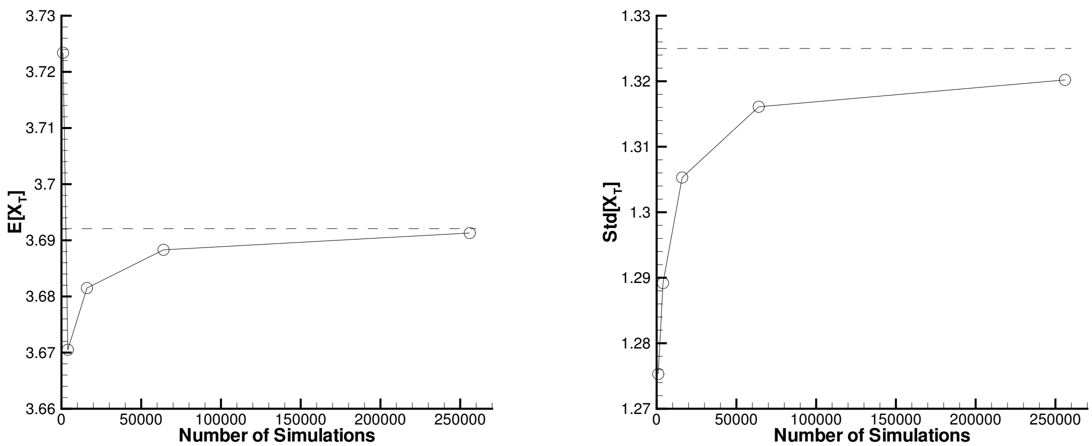
FIGURE 4: *Optimal control as a function of* (X, t)*. Parameters are given in Table 6 with* $\lambda = 0.25$*. Under these inputs, if* $X(t = 0) = 0.5$, $(Std_{q^*}^{t=0}[X_T], E_{q^*}^{t=0}[X_T]) = (1.32500, 3.69208)$ *from the finite difference solution. Figure (a) uses 4560 nodes for X grid, 433 nodes for the control grid, and 640 timesteps. Figure (b) uses 9120 nodes for X grid, 865 nodes for the control grid, and 1280 timesteps.*

7.3 Monte-Carlo Simulation

In this section, we carry out Monte-Carlo simulation. We use the wealth-to-income ratio case with a bounded control as an example. Using the parameters in Table 6, we solve the stochastic optimal control problem (equation (4.5)) and store the optimal strategies for each $(X = x, t)$. We then carry out Monte-Carlo simulations based on the stored strategies for $X(t = 0) = 0.5$ initially. The value for $(\text{Std}_{t=0,x}^{q^*}[X_T], E_{t=0,x}^{q^*}[X_T])$ is $(1.32500, 3.69208)$ (from the finite difference solution). Table 9 shows a convergence study of Monte-Carlo simulations, and Figure 5 shows a plot of the convergence study. As the number of simulations increases and the timestep size decreases, the results given by Monte-Carlo simulation converge to the values given by solving the finite difference solution.

# of Simulations	MC Timestep	$E^{q^*}_{t=0,x}[X_T]$	$\text{Std}^{q^*}_{t=0,x}[X_T]$
1000	0.25	3.7234	1.2753
4000	0.125	3.6705	1.2892
16000	0.0625	3.6815	1.3053
64000	0.03125	3.6883	1.3161
256000	0.015625	3.6913	1.3202

TABLE 9: *Convergence study for the Monte-Carlo Simulations (bounded control). Parameters are given in Table 6. Values for* $E^{q^*}_{t=0,x}[X_T]$ *and* $Std^{q^*}_{t=0,x}[X_T]$ *are reported at* $(X = 0.5, t = 0)$. *The finite difference values are:* $E^{q^*}_{t=0,x}[X_T] = 3.69208$ *and* $Std^{q^*}_{t=0,x}[X_T] = 1.32500$.

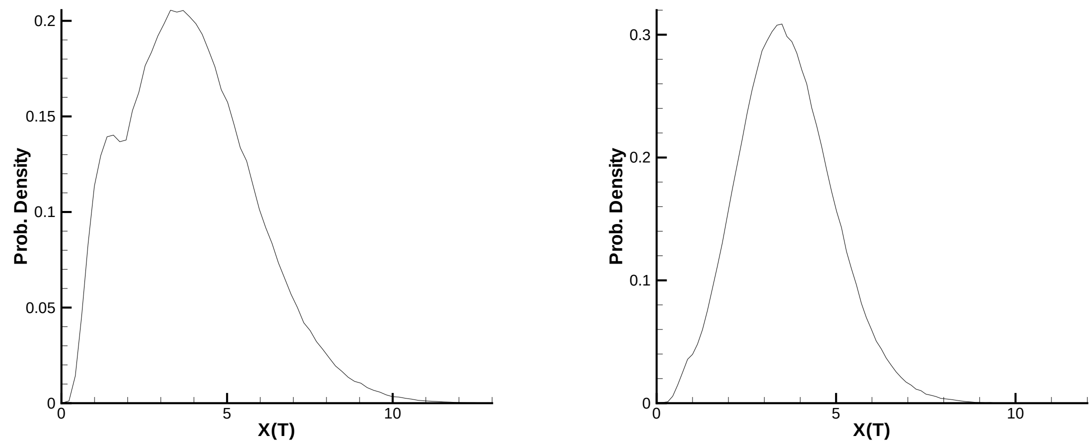


(a) **(b)**

FIGURE 5: *Convergence study for Monte-Carlo Simulation (bounded control). Parameters are given in Table 6. Figure (a) shows the plot of* $E^{q^*}_{t=0,x}[X_T]$. *Figure (b) shows the plot for* $Std^{q^*}_{t=0,x}[X_T]$. $E^{q^*}_{t=0,x}[X_T]$ *and* $Std^{q^*}_{t=0,x}[X_T]$ *are written as* $E[X_T]$ *and* $Std[X_T]$ *in the figure. Values for* $E^{q^*}_{t=0,x}[X_T]$ *and* $Std^{q^*}_{t=0,x}[X_T]$ *are reported in Table 9. The finite difference values are* $(Std^{q^*}_{t=0,x}[X_T], E^{q^*}_{t=0,x}[X_T]) = (1.32500, 3.69208)$.

Figure 6 shows the probability density function of Monte-Carlo simulations (500000 simulations). Figure 6 (a) uses $\lambda = 0.15$ $((\text{Std}^{q^*}_{t=0,x}[X_T], E^{q^*}_{t=0,x}[X_T]) = (1.91306, 4.01011))$, while Figure 6 (b) uses $\lambda = 0.25$ $((\text{Std}^{q^*}_{t=0,x}[X_T], E^{q^*}_{t=0,x}[X_T]) = (1.32500, 3.69208))$. The shape of the probability density function depends on input parameters (λ in this example). The double peak in Figure 6 (a) is due to the same effect as described in [19].

Figure 7 shows the mean and standard deviation for the strategy $q(t, x) = p(t, x)$ as time changes. Figure 7 (a) uses $\lambda = 0.15$ $((\text{Std}^{p^*}_{t=0,x}[X_T], E^{p^*}_{t=0,x}[X_T]) = (1.91306, 4.01011))$, while Figure 7 (b) uses $\lambda = 0.25$ $((\text{Std}^{p^*}_{t=0,x}[X_T], E^{p^*}_{t=0,x}[X_T]) = (1.32500, 3.69208))$. Both of these Figures show that the mean of $p(t, x)$ is a decreasing function of time t, i.e., as time goes on, the investor switches into the less risky strategy (on average). Since the value of $E^{p^*}_{t=0,x}[X_T]$ in Figure 7 (b) is higher



(a) $\lambda = 0.15$ **(b)** $\lambda = 0.25$

FIGURE 6: *Probability density function for Monte-Carlo Simulation, bounded control,* 500,000 *simulations and 1280 simulation timesteps. Parameters are given in Table 6. Values for* $(E^{q^*}_{t=0,x}[X_T], Std^{q^*}_{t=0,x}[X_T])$ *are reported at* $(X = 0.5, t = 0)$. *Figure (a) uses* $\lambda = 0.15$, *while Figure (b) uses* $\lambda = 0.25$. *For Figure (a),* $(Std^{q^*}_{t=0,x}[X_T], E^{q^*}_{t=0,x}[X_T]) = (1.91306, 4.01011)$ *from the finite difference solution; For Figure (b),* $(Std^{q^*}_{t=0,x}[X_T], E^{q^*}_{t=0,x}[X_T]) = (1.32500, 3.69208)$ *from the finite difference solution.*

than the one in Figure 7 (a), the mean strategy in Figure 7 (b) is more risky compared to Figure 7 (a).

7.4 Comparison with the Pre-commitment Strategy

In this section, we briefly compare the time-consistent strategy with the pre-commitment strategy.

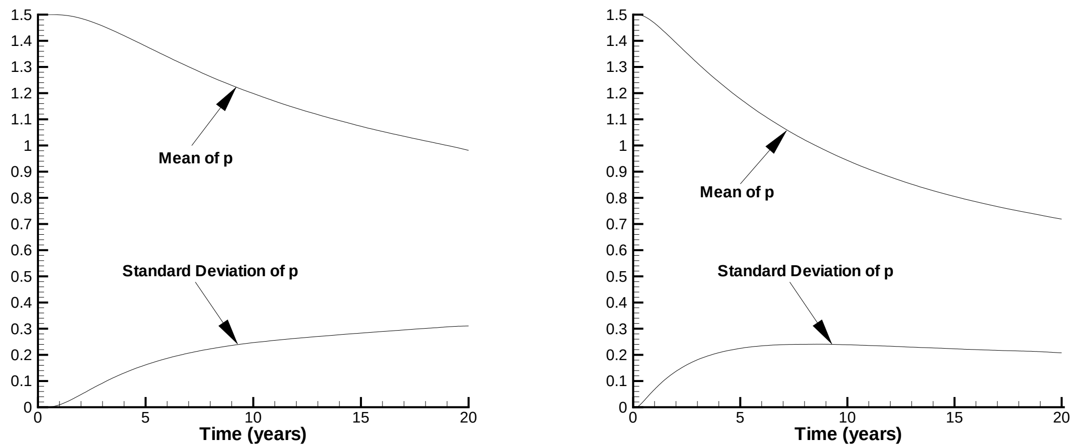
We first study the wealth case. Figure 8 shows the efficient frontiers for the case of allowing bankruptcy for the two strategies. The analytic solution for the pre-commitment strategy is given in [10],

$$\begin{cases} Var^{t=0}[W_T] = \frac{e^{\xi_1^2 T}-1}{4\lambda^2} \\ E^{t=0}[W_T] = \hat{w}_0 e^{rT} + \pi\frac{e^{rT}-1}{r} + \sqrt{e^{\xi_1^2 T} - 1}\text{Std}(W_T)\ , \end{cases} \tag{7.2}$$

and the optimal control $(q = p)$ at any time $t \in [0, T]$ is

$$q^*(t, w) = -\frac{\xi_1}{\sigma_1 w}[w - (\hat{w}_0 e^{rt} + \frac{\pi}{r}(e^{rt} - 1)) - \frac{e^{-r(T-t)+\xi_1^2 T}}{2\lambda}]\ . \tag{7.3}$$

The figure clearly shows that the pre-commitment strategy is more efficient than the time-consistent strategy, since the pre-commitment strategy is a globally optimal strategy in terms of an efficient frontier. The two efficient frontiers are both straight lines, and pass through the same point at $(\text{Std}(W_T), E(W_T)) = (0, \hat{w}_0 e^{rT} + \pi\frac{e^{rT}-1}{r})$. At that point, the plan holder simply invests all her



(a) $\lambda = 0.15$ **(b)** $\lambda = 0.25$

FIGURE 7: *Mean and standard deviation for the control $q(t, x) = p(t, x)$. There are 64000 simulations and 1280 simulation timesteps. Parameters are given in Table 6. Figure (a) uses $\lambda = 0.15$, while Figure (b) uses $\lambda = 0.25$. For Figure (a), $(Std^{p^*}_{t=0,x}[X_T], E^{p^*}_{t=0,x}[X_T]) = (1.91306, 4.01011)$ from the finite difference solution; For Figure (b), $(Std^{p^*}_{t=0,x}[X_T], E^{p^*}_{t=0,x}[X_T]) = (1.32500, 3.69208)$ from the finite difference solution.*

wealth in the risk free bond all the time, so the standard deviation is zero. The slope $(= \sqrt{e^{\xi_1^2 T} - 1})$ of the pre-commitment strategy is larger than the slope $(= \xi_1 \sqrt{T})$ of the time-consistent strategy. But note that $\sqrt{e^{\xi_1^2 T} - 1} \to \xi_1 \sqrt{T}$ as $T \to 0$, so the two strategies are the same as $T \to 0$. This is easy to understand, since as $T \to 0$, finding the global optimal strategy (pre-commitment case) is the same as finding the local optimal strategy (time-consistent case).

Figure 9 (a) shows a comparison for the two strategies of the no bankruptcy case, and Figure 9 (b) is for the bounded control case. Similar to the allowing bankruptcy case, the pre-commitment strategy is more efficient. For the bounded control case, the two efficient frontiers have the same end points. The lower end corresponds to the most conservative strategy, i.e. the total wealth is invested in the risk free bond at any time. The higher end corresponds to the most aggressive strategy, i.e. choose the control p to be the upper bound $p_{max} (= 1.5)$ at any time. Figure 8 and 9 show that the difference between the efficient frontier solutions of the pre-commitment and time-consistent strategies become smaller after adding constraints.

It is not surprising that the pre-commitment strategy is more efficient than the time-consistent strategy, since the pre-commitment policy is the strategy which optimizes the objective function at the initial time ($t = 0$). However, in practice, investors may be more likely to choose the time-consistent strategy. This is because investors often prefer to choose the "optimal" strategy based on the current state, without regard to investment targets specified at the initial time. This strategy is also more natural to follow if the parameters of the problem (volatility, Sharpe ratio) change with time. However, if we can repeat a similar trading strategy many times (e.g. optimal execution in [7]), we should choose the pre-commitment strategy, since the average outcome will then end up on the globally optimal pre-commitment efficient frontier.

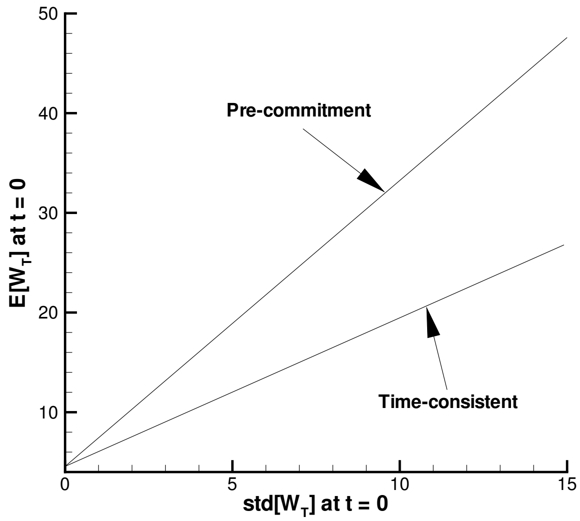
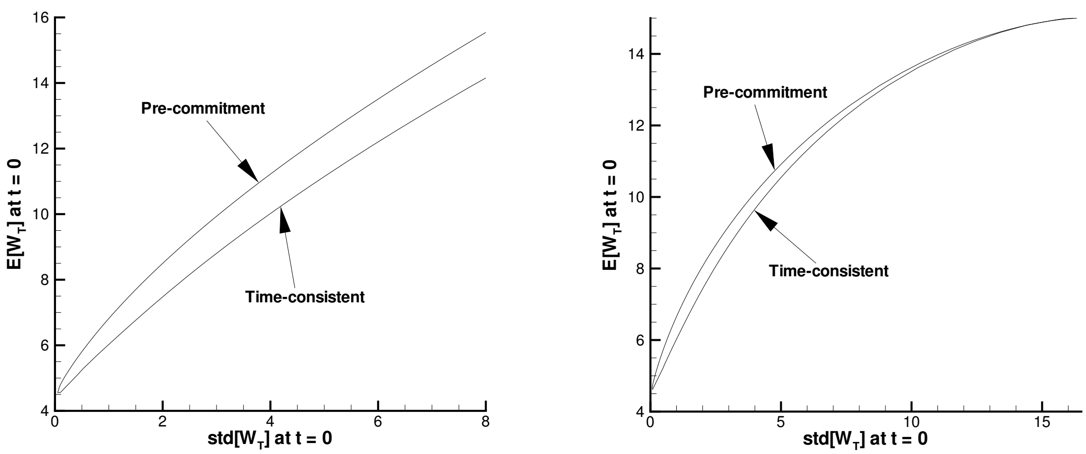


FIGURE 8: *Time-consistent vs Pre-commitment: Wealth case, allowing Bankruptcy. Parameters are given in Table 2.*

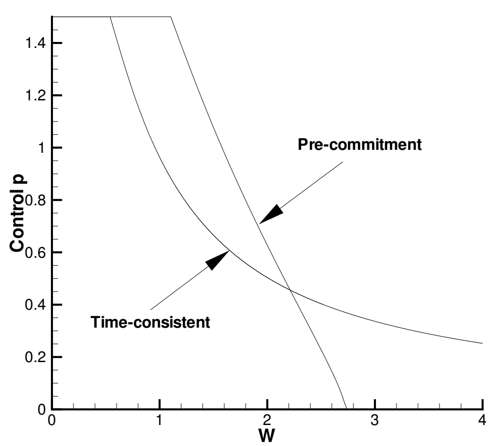


(a) No Bankruptcy **(b) Bounded Control**

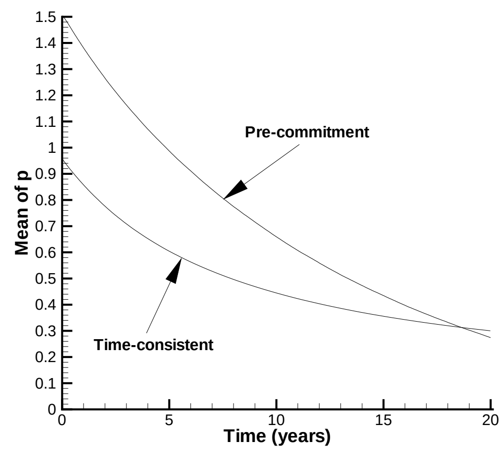
FIGURE 9: *Time-consistent vs Pre-commitment: Wealth case. (a): no bankruptcy case; (b): bounded control case. Parameters are given in Table 2.*

In Figure 10, we compare the control policies of the time-consistent and pre-commitment strategies. Parameters are given in Table 2, and we use the wealth case with bounded control ($q = p \in [0, 1.5]$). We fix $\text{Std}^{p^*}_{t=0,w}[W_T] \simeq 1.24$ for this test. Figure 10 shows that the control policies given by the two strategies are significantly different. This is true even for the bounded control case, where the expected value for pre-commitment and time-consistent policies is similar for a fixed standard deviation. Figure 10 (a) shows the control policies at $t = 0$. We can see that once the wealth W is large enough, the control policy for the pre-commitment strategy is to invest all wealth in the risk free asset. The reason for this is that for the pre-commitment strategy, there is an effective investment target given at $t = 0$, which depends on the value of λ. Once the target is reached, the investor will not take anymore risk and switch all wealth into bonds. However, there is no investment target for the time-consistent case, so the control never reaches zero. Figure 10 (b) shows the mean of the control policies versus time $t \in [0, T]$. The mean of both policies is a decreasing function of time, i.e. both strategies are less risky as we near maturity.

Similar to Figure 10, Figure 11 shows a comparison of the control policies of the time-consistent and pre-commitment strategies. Parameters are given in Table 6, and we use wealth case with bounded control ($q = p \in [0, 1.5]$). Figure 11 uses a more risky strategy. We fix $\text{Std}^{p^*}_{t=0,w}[W_T] \simeq 8.17$ for this test. The comparison results are similar to the results from Figure 10. Although the pre-commitment and the time-consistent strategies have a similar pair of expected value and standard deviation, the control policies are significantly different.



(a)



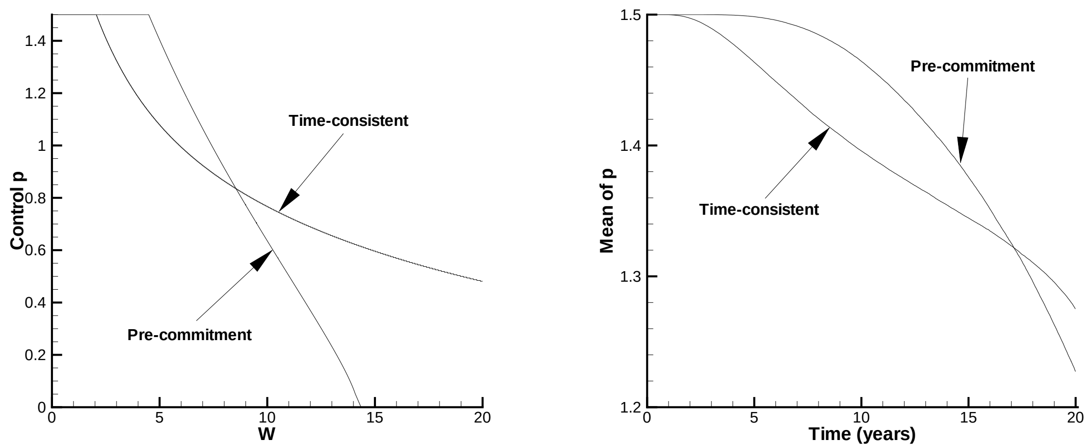
(b)

FIGURE 10: *Comparison of the control policies: wealth case with bounded control ($q = p \in [0, 1.5]$). Parameters are given in Table 2. We fix $std^{p^*}_{t=0,w}[W_T] \simeq 1.24$ for this test. More precisely, from our finite difference solutions, $(Std^{p^*}_{t=0,w}[W_T], E^{p^*}_{t=0,w}[W_T]) = (1.23975, 6.39296)$ for the time-consistent strategy; and $(Std^{p^*}_{t=0,w}[W_T], E^{p^*}_{t=0,w}[W_T]) = (1.23805, 7.03097)$ for the pre-commitment strategy. Figure (a) shows the control policies at $t = 0$; Figure (b) shows the mean of the control policies versus time $t \in [0, T]$.*

For the wealth-to-income ratio case, the comparison is similar to the wealth case.

8 Conclusions

In this article, we have developed a numerical technique for determining the optimal time-consistent mean variance investment strategy. We discuss two cases for the pension plan problem: the wealth



(a) (b)

FIGURE 11: *Comparison of the control policies: wealth case with bounded control ($q = p \in [0, 1.5]$). Parameters are given in Table 2. We fix $std^{p^*}_{t=0,w}[W_T] \simeq 8.17$ for this test. More precisely, from our finite difference solutions, $(Std^{p^*}_{t=0,w}[W_T], E^{p^*}_{t=0,w}[W_T]) = (8.17494, 12.6612)$ for the time-consistent strategy; and $(Std^{p^*}_{t=0,w}[W_T], E^{p^*}_{t=0,w}[W_T]) = (8.17453, 12.8326)$ for the pre-commitment strategy. Figure (a) shows the control policies at $t = 0$; Figure (b) shows the mean of the control policies versus time $t \in [0, T]$.*

case and the wealth-to-income ratio case. Our method can handle various constraints on the control policy. We study three types of control: allowing bankruptcy case, no bankruptcy case, and bounded control case.

We use the piecewise constant policy technique [12] to solve the optimization problem for the efficient frontier solutions. In [12], the authors applied the piecewise constant timestepping to a scalar HJB equation, and proved that the solution converges to the viscosity solution. However, in our case, since the time-consistent problem can be formulated as a system of HJB differential algebraic equations, this falls outside the viscosity solution theory. Hence we have no formal proof of convergence of our method. Nevertheless, our technique does converge to analytic solutions where available.

We compare the solutions for time-consistent policy with the solutions for the pre-commitment policy. Since we can view the time-consistent mean variance strategy as the pre-commitment strategy with a constraint forcing time consistency, the efficient frontier for the time-consistent strategy can never be above the efficient frontier for the pre-commitment strategy. When realistic constraints are applied to the investment policy, the efficient frontiers for the time-consistent strategy and the pre-commitment strategy are quite similar, and they have the same end points. However, the investment policies (not at the end points of the efficient frontier) are substantially different. This suggests that the choice between a pre-commitment or time-consistent strategy cannot be made by examining the efficient frontier, but rather should be based on the qualitative behavior of the optimal policies.

A Discrete Equation Coefficients

Let q_i^n denote the optimal control q^* at node i, time level n and set

$$a_i^{n+1} = a(z_i, q_i^n),\ b_i^{n+1} = b(z_i, q_i^n)\ , \tag{A.1}$$

where $a(z,q)$ and $b(z,q)$ are defined in equation (5.26). Then, we can use central, forward or backward differencing at any node.

Central Differencing:

$$\begin{aligned}
\alpha_{i,central}^n &= \left[\frac{2a_i^n}{(z_i - z_{i-1})(z_{i+1} - z_{i-1})} - \frac{b_i^n}{z_{i+1} - z_{i-1}} \right] \\
\beta_{i,central}^n &= \left[\frac{2a_i^n}{(z_{i+1} - z_i)(z_{i+1} - z_{i-1})} + \frac{b_i^n}{z_{i+1} - z_{i-1}} \right]\ .
\end{aligned} \tag{A.2}$$

Forward/backward Differencing: $(b_i^n > 0/\ b_i^n < 0)$

$$\begin{aligned}
\alpha_{i,forward/backward}^n &= \left[\frac{2a_i^n}{(z_i - z_{i-1})(z_{i+1} - z_{i-1})} + \max(0, \frac{-b_i^n}{z_i - z_{i-1}}) \right] \\
\beta_{i,forward/backward}^n &= \left[\frac{2a_i^n}{(z_{i+1} - z_i)(z_{i+1} - z_{i-1})} + \max(0, \frac{b_i^n}{z_{i+1} - z_i}) \right]\ .
\end{aligned} \tag{A.3}$$

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