Target Date Funds: a bad idea whose time has come

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1 Introduction

As defined benefit (DB) plans disappear, saving for retirement has become the responsibility of individual investors. DB plans have been replaced by defined contribution (DC) plans, whereby an employee and the employer contribute to a tax advantaged account. The employer does not provide any guarantees for this investment account, which has the implication that the entire risk is borne by the employee.

Accumulating assets in DC plans occurs over the entire working career of an investor, which would normally be 30 years or more. This makes retirement savers truly long term investors. A common strategy is to invest a fraction of assets in stocks, with remaining assets in bonds. As time goes on, the actual amounts in stocks and bonds will grow at different rates. The portfolio is then rebalanced back to the desired weights in each asset, typically at quarterly or yearly intervals. A typical strategy would be to choose a weight of 60% stocks and 40% bonds. An easy way to implement this strategy is to use a small set of index Exchange Traded Funds (ETFs). There are now even ETFs which will rebalance to the desired fraction in equities automatically. It is conceivable to implement a constant proportion strategy by owning a single ETF.

A popular alternative to a constant weight strategy is based on the idea of a glide path. This method is implemented many Target Date Funds (TDFs). Denoting time by t, a simple example of a glide path is

$$\text{Fraction invested in equities} = p(t) = \frac{100 - \text{ your age at } t}{100}.$$

The logic behind this idea is that you should take on more risk when you are young (with many years to retirement) and then take on less risk when you are older, with less time to recover from market shocks. This seems quite sensible. The investment portfolio is typically rebalanced at quarterly or yearly intervals, so that the equity fraction is reset back to the glide path value $p(t)$.

This idea is so attractive that TDFs are *Qualified Default Investment Alternatives* (QDIAs) in the US.[1] If an employee has enrolled in an employer-managed DC plan, the assets may be placed in a QDIA as a default option, in the absence of any instructions from the employee.

It is important to note that the glide path $p(t)$ in the age-based example above is only a function of time t. We call this type of strategy a *deterministic* glide path, i.e. this strategy does not adapt

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[1] https://ca.news.yahoo.com/qualified-default-investment-alternative-qdia-214532882.html

to market conditions or the investment goals of the DC plan member, it is predetermined at the initial time.

Of course, the constant proportion strategy is actually a trivial case of a glide path, with $p(t) = \ constant$.

2 What is the point of this article?

At the end of 2020, there was over \$2.8 trillion invested in TDFs in the US.[2] However, recent work has called into question the rationale for these products (Poterba et al., 2009; Basu et al., 2011; Esch and Michaud, 2014; Arnott et al., 2013; Graf, 2017; Forsyth and Vetzal, 2019). In fact, the main conclusion of these cited papers is that glide path TDFs are no better than a simple constant weight strategy.

My objective in writing this article is to summarize this work on TDFs, and I hope to convince you that TDFs are a triumph of marketing over both mathematics and empirical studies. This white paper contains just a bit of mathematics. Perhaps this will interest you to explore these concepts further, by reading some of the papers in the references.

For those who want to skip the mathematics, there is a high level description available.[3]

3 A simple mathematical model

For simplicity we assume that there are only two assets available in the financial market, namely a risky asset and a risk-free asset. In practice, the risky asset would be a broad market index fund, and the risk free asset would a government bond index. We believe that for long-term investors, the major asset allocation decision is the stock-bond split. The choice of which stock index or which bond index to use, is a second order effect.

For those of you who scoff at this simplistic approach, we are only going to use this model as a tool to understanding why TDFs are ineffective. We will test our model analysis on real bootstrapped historical data, where, of course, no assumptions are made.

Now for a little mathematics. We assume that the investment horizon is T. S_t and B_t respectively denote the *amounts* invested in the risky and risk-free assets at time t, $t \in [0, T]$. In general, these amounts will depend on the investor's strategy over time, including contributions, withdrawals, and portfolio rebalances, as well as changes in the unit prices of the assets. The investor can control all of these factors except for the unit prices. To clarify our assumptions regarding asset price dynamics, suppose for the moment that the investor does not take any action with respect to the controllable factors. We refer to this as the absence of control. It implies that all changes in S_t and B_t result from changes in asset prices.

If the absence of control, S_t evolves according to geometric Brownian motion (GBM)

$$\frac{dS_t}{S_t} = \mu dt + \sigma \, d\mathcal{Z} \ , \tag{3.1}$$

where μ is the drift rate, σ is the volatility, $d\mathcal{Z}$ is the increment of a Wiener process.

In the absence of control, we assume that the dynamics of the amount B_t invested in the risk-free asset are

$$dB_t = r B_t \, dt, \tag{3.2}$$

[2] https://www.pionline.com/investing/target-date-funds-see-assets-climb-2020-despite-pandemic-morningstar

[3] P. A. Forsyth, K. R. Vetzal, G. Westmacott, "Target wealth: a better bet for achieving wealth goals." https://cs.uwaterloo.ca/~paforsyt/AAII_Journal_Oct_2017_PortfolioStrategies.pdf

where r is the (constant) risk-free rate.

As well, we define the total wealth W_t at any time as

$$W_t = S_t + B_t \ . \tag{3.3}$$

4 Deterministic glide paths

Let p denote the fraction of total wealth that is invested in the risky asset, i.e.

$$p = \frac{S_t}{S_t + B_t}. \tag{4.1}$$

A *deterministic* glide path restricts the admissible strategies to those with $p = p(t)$, i.e. the optimal strategy cannot take into account the actual value of W_t at any time. Clearly this is a very restrictive assumption, but it is commonly used in TDFs. Although a constant proportion strategy can be seen as a special case of a deterministic glide path where $p(t) = const.$, it is simpler here for expository reasons to reserve the label "deterministic glide path" for cases where $p(t)$ is time-varying.

4.1 Lump sum investment with continuous rebalancing

To gain some intuition about deterministic strategies, we consider first a simple case with a lump sum initial investment and no further cash injections or withdrawals. We also assume here that the portfolio is continuously rebalanced.

Under these assumptions, we can derive the following result

Proposition 4.1 (Inefficiency of glide path strategies for lump sum investments)**.** *Consider a market with two assets following the processes* (3.1) *and* (3.2)*. Suppose we invest a lump sum W_0 at $t = 0$ in a continuously rebalanced portfolio using a deterministic glide path strategy $p = p(t)$, where p is the fraction of total wealth invested in the risky asset. Also consider a strategy with a constant proportion p^* invested in the risky asset, where*

$$p^* = \frac{1}{T} \int_0^T p(s) \, ds. \tag{4.2}$$

Then:

(i) the expected value of the terminal wealth is the same for both strategies; and

(ii) the standard deviation of terminal wealth for the glide path strategy cannot be less than that of the constant proportion strategy.

Proof. See Appendix A. □

This is quite a remarkable result. This says that for *any* glide path, there exists a constant weight strategy (the constant weight is just the time averaged fraction in stocks from the glide path) that has the same expected final wealth, and a smaller standard deviation, compared to the glide path. Let me emphasize that this holds for *any* glide path.

Proposition 4.1 can be generalized to account for non-normal stock returns Forsyth and Vetzal (2019). However, in practical situations, an investor saving in a DC plan account would typically

- Begin with zero initial wealth $W_0 = 0$, and make cash contributions yearly

- Rebalance at quarterly and yearly intervals

In these cases, we cannot obtain nice closed form results as for Proposition 4.1. However, some extensive numerical tests (still assuming simple parametric models for stock and bond returns) indicates that, for practical purposes, the same result holds: for any given glide path, there is a constant weight strategy which gives the same mean and either essentially the same or better standard deviation.

There is also another intriguing result: the glide path mean and standard deviations depend on[4]

$$\int_0^T p(s)\ ds \quad ; \quad \int_0^T (p(s))^2\ ds \tag{4.3}$$

Note that

$$\int_0^T p(s)\,ds = \int_0^T p(T-s)\,ds \quad ; \quad \int_0^T [p(s)]^2\,ds = \int_0^T [p(T-s)]^2\,ds\ , \tag{4.4}$$

so by equations (4.3) the glide path results are the same in this case if we reverse the strategy. In other words, if our glide path starts with a high allocation to stocks and finishes with a low allocation to stocks, we can achieve exactly the same mean-variance result in terms of final wealth by beginning with a low equity allocation and ending with a high equity allocation.

This would suggest that the usual rationale for using a glide path with an initial high allocation to stocks declining to a low allocation is completely misleading, since we can reverse the glide path allocation, and get the same result.

5 Back to reality: empirical tests

Now, we are going carry out some empirical tests on glide paths vs constant weight strategies. Our tests are entirely data driven, i.e. we make no assumptions about the underlying stock and bond processes.

5.1 Data

We will use the Center for Research in Securities (CRSP) capitalization weighted total return index (includes all dividends and distributions).[5] The CRSP data covers the range 1926.00 - 2022.00. We use the monthly data for this series. For the bond index, we consider a short term 30 day US T-bill index.[6] These indexes are also for monthly data. We adjust all the indexes for inflation, by dividing by the CPI (also from CRSP). In other words, all investments are in real dollars.

5.2 Bootstrap resampling

So, in order to evaluate an investment idea, we need to gather some more useful statistics. A popular technique is based on a data driven approach. Here, we make no theoretical assumptions about how markets behave, we simply use the historical data directly. Suppose we are interested in examining

[4]See Appendix A.

[5]More specifically, results presented here were calculated based on data from Historical Indexes, ©2022 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services (WRDS) was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third-party suppliers.

[6]The 30 day T-bill index was obtained from CRSP

a long-term strategy, which would be typical of a person saving for retirement. In this case, for a 35-year old investor, planning to retire at 65, the investment time frame would be 30 years.

The most reliable historical data goes back to about 1926. One popular method for evaluating an investment strategy is based on starting the strategy in 1926, and then seeing how well the portfolio did 30 years later, i.e. in 1956. Then, we do the same thing starting in 1927 and ending in 1957, and so on. The problem with this rolling year approach, is that there is a large overlap between each of the rolling year samples, so we can't use this to generate reliable statistics.

A thirty year investment scenario consists of 360 consecutive one month returns. A single scenario is constructed as follows. We select a month at random from the historical data, and use this as our first month's return. Then, we select another month at random (with replacement) which is the second month's return in our thirty year scenario. We keep doing this until we have a set of 360 returns (one thirty-year path). We then repeat this procedure many times, to produce many 30-year return paths.

However, this bootstrapping approach does not take into account possible serial correlation in the returns. This is just another way of saying that next month's returns may be affected by the returns of the past few months or years.

To take this into account, we select an initial month at random, but use b consecutive monthly returns (starting at the initial random month). We repeat this (360/b) times to generate a single 30 year path. We call b the blocksize.

But we are not done yet. It turns out that a better approach is to not use a fixed blocksize, but to specify an average blocksize b, and randomly vary the blocksize within each thirty year path. This is called the stationary block bootstrap method.

For more details about this method, see (Politis and Romano, 1994; Politis and White, 2004; Patton et al., 2009; Dichtl et al., 2016; Forsyth and Vetzal, 2019).

5.3 Scenario

In order to model a realistic scenario, we consider an investor who has a portfolio of 60% in the equity index, and 40% in a bond index. The investor rebalances the investments in the stock and bond index, back to the 60 : 40 ratio once a year, i.e. at times $t = 0,1,\ldots,29$. The investor starts with 1000, with no injection or withdrawals of cash over the investment horizon, and we examine the statistics of the terminal wealth W_T at $T = 30$ years. We will consider a long term 30 year investment strategy, since this would be typical of an investor saving for retirement.

For a glide path, we will assume that

$$p(t) = 1.0 - t\left(\frac{1.0 - 0.2}{29}\right) \ ; \ t = 0, \ldots, 29 \tag{5.1}$$

i.e., the glide path declines linearly from 100% in stocks to 20% stocks at year 29 (the last rebalancing time). The time averaged allocation is then approximately 60%. Rebalancing to the glide path is also done once per year.

	Constant weight	Glide path
$E[W_T]$	4877	4840
$std[W_T]$	3092	3320
$Med[W_T]$	4136	4021
ES(5%)	1184	981

TABLE 5.1: *Initial stake $W_0 = 1000$, no cash injections or withdrawals, $T = 30$ years. Block bootstrap resampling, expected blocksize 2.0 years. Constant weight: 60% stocks, 40% bonds, rebalanced annually. Glide path: equation (5.1), rebalanced annually. Bond index: constant maturity 30-day US T-bill. Stock index: CRSP capitalization weighted index. Data range 1926.00 - 2022.00. All indexes are deflated by the CPI. $E[W_T]$: mean terminal wealth; $std[W_T]$: standard deviation of W_T; $Med[W_T]$: median of W_T. ES(5%) is the mean of the worst 5% of the outcomes. Parameters for the glide path and constant weight strategies given in Section 5.3.*

5.4 Bootstrap results

In Table 5.1 show the summary statistics for the constant weight strategy, and the glide path, based on bootstrap resampling of the historical data. Since we are using actual historical data here, as well as discrete rebalancing, we cannot expect that Proposition 4.1 will hold precisely. Nevertheless, we can see that a constant weight strategy, with constant weight equal to the time-averaged glide path allocation to stocks, is slightly superior to the glide path strategy. Table 5.1 also shows the expected shortfall at the 5% level (ES(5%)), which is just the mean of the worst 5% of the outcomes. This is a measure of tail risk. Note that since ES is defined in terms of final wealth, a larger value is better than a smaller value.

Rather than look at just a few summary statistics, it is more instructive to examine the entire probability distribution of outcomes. A nice way to do this is to compare the cumulative distribution function (CDF) for both strategies. The CDF in this case is just a plot of $Prob[W_T < W]$ versus W on the x-axis. In other words, for a given value of W, the CDF curve gives the probability that the final wealth W_T will be less than W.

Figure 5.1 shows the CDFs for the glide path and constant weight strategies described in Section 5.3. Remarkably, the CDFs for both strategies are almost coincident for all values of W. Recall that this plot is constructed from bootstrapped historical data, no modelling assumptions are made.

Recall from equation (4.4) for our model stock and bond market, that if we reverse the glide path, then we get the same mean and variance of the final wealth W_T. In other words, instead of equation (5.1), suppose we use

$$p(t) = 0.2 - t\left(\frac{0.2 - 1.0}{29}\right) \ ; \ t = 0, \ldots, 29 \quad , \tag{5.2}$$

so that we start off with a low allocation to stocks (20%), and end up with 100% stocks at year 29. This is, of course, completely contrary to the usual advice. However, this result is based on our simple model of stocks, which follow GBM. Does this still hold with the historical data?

Table 5.2 shows the summary statistics for the forward glide path (5.1) and the reverse glide path (5.2), while Figure 5.2 shows the CDFs for the forward and reverse glide paths. Basically, the two CDFs are indistinguishable. Our simple theory seems to hold up well, even using actual historical data.

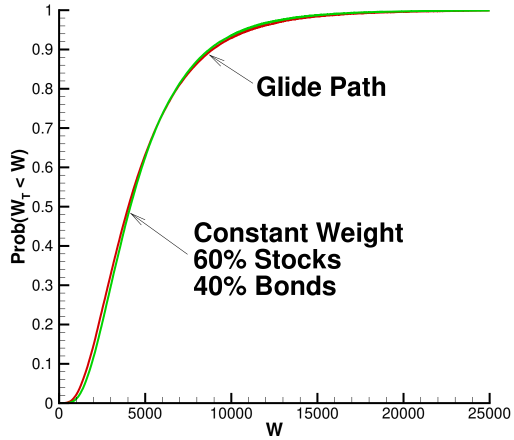

FIGURE 5.1: *Cumulative distribution functions (CDFs) for glide path (5.1) and constant weight ($p = .60$). Initial stake $W_0 = 1000$, no cash injections or withdrawals, $T = 30$ years. Block bootstrap resampling, expected blocksize 2.0 years. Rebalanced annually. Bond index: constant maturity 30 day US T-bill. Stock index: CRSP capitalization weighted index. Data range 1926.00 - 2022.00. All indexes are deflated by the CPI. 10^6 resamples. Parameters for the glide path and constant weight strategies given in Section 5.3.*

5.5 More complex cases

We have focused on simple lump sum investment cases. However, in practice, investors saving for retirement would normally start off with very little wealth, and add to their DC accounts each year. In these more complex cases, we cannot find simple closed form expressions for the variance of the final wealth, for a given glide path. In Forsyth and Vetzal (2019), a numerical optimization technique is used to find the optimal glide path in terms of mean-variance criteria. This is then compared to a constant weight strategy using either Monte Carlo or bootstrap resampling methods. For these more complex cases, we see that the glide path and constant weight strategies give essentially identical CDFs. However, unlike for the lump sum investment case, the glide path does give slightly better standard deviation, compared to the constant weight strategy. But the improvement is almost

	Reverse Glide path	Glide path
$E[W_T]$	4836	4840
$std[W_T]$	3295	3320
$Med[W_T]$	4023	4021
ES(5%)	991	998

TABLE 5.2: *Initial stake $W_0 = 1000$, no cash injections or withdrawals, $T = 30$ years. Block bootstrap resampling, expected blocksize 2.0 years. Forward glide path equation (5.1) rebalanced annually. Reverse glide path: equation (5.2), rebalanced annually. Bond index: constant maturity 30-day US T-bill. Stock index: CRSP capitalization weighted index. Data range 1926.00 - 2022.00. All indexes are deflated by the CPI. $E[W_T]$: mean terminal wealth; $std[W_T]$: standard deviation of W_T; $Med[W_T]$: median of W_T. ES(5%) is the mean of the worst 5% of the outcomes.*

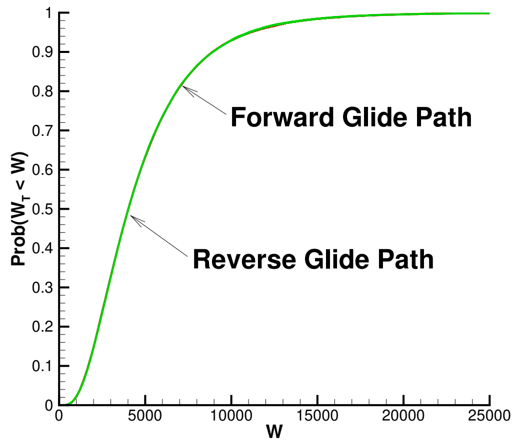

FIGURE 5.2: *Cumulative distribution functions (CDFs) for glide path (5.1) and the reverse glide path (5.2). Initial stake* $W_0 = 1000$*, no cash injections or withdrawals,* $T = 30$ *years. Block bootstrap resampling, expected blocksize 2.0 years. Rebalanced annually. Bond index: constant maturity 30 day US T-bill. Stock index: CRSP capitalization weighted index. Data range 1926.00 - 2022.00. All indexes are deflated by the CPI.* 10^6 *resamples. Forward glide path equation (5.1). Reverse glide path: equation (5.2).*

imperceptible.[7]

We can also replace the 30 day T-bill index by a 10 year US treasury bond index, and we get the same results. Actually, for long term investors, it seems that an equal weight index stochastically dominates a capitalization weighted index.[8] Replacing the cap-weighted CRSP index by the equal weighted CRSP index also gives the same results.

6 Can we do something better than glide paths?

We have seen that, both theoretically and empirically, in terms of the final wealth distributions, a deterministic glide path offers virtually no improvement over a simple constant weight strategy. However, instead of a deterministic strategy, if we allow an adaptive strategy, whereby the fraction in stocks responds to investment experience, then we can obtain a much improved result. In mathematical terms, a deterministic glide path has equity fraction $p = p(t)$, i.e. only a function of time. An adaptive strategy has, at the least, $p = p(W_t,t)$, i.e. the equity fraction is a function of time and the currently observed total wealth. See (Forsyth and Vetzal, 2019) and the references cited therein.

7 Target date funds: the human capital argument

Often, deterministic glide path strategies are justified based on a human capital argument. This says that when investors are young, they have little in the way liquid financial assets, but their

[7] Note that this optimal glide path was obtained by a numerical optimization method. It appears that most TDF glide paths are based on heuristics, or perhaps running a few Monte Carlo simulations. It is unlikely that these glide paths are optimal.

[8] P. A. Forsyth, "Equal Weight vs Capitalization Weight Indexes," (2022) `https://cs.uwaterloo.ca/~paforsyt/equal_vs_cap.pdf`

net worth should also incorporate future employment earnings. Hence, a young investor's net worth is dominated by their bond-like human capital, so that a younger person should have a large commitment to stocks to diversity exposure to work-related earnings.

However, this puts the cart before the horse. The investor's objective is to accumulate enough wealth to finance retirement, and to minimize risk, in terms of the final wealth distribution. From a mathematical point of view, the effect of the human capital is realized by the fact that the investor will make annual contributions to the investment account over her working life. This is a well posed optimal control problem. Rather than assume some sort of hand-wavy diversification argument, why not specify an objective function, and then solve for the optimal control?

In fact, in the case of constant contributions each year, and a mean-variance objective function, the optimal deterministic asset allocation does look like a glide path (Forsyth and Vetzal, 2019). However, the improvement over the equivalent constant weight strategy is virtually non-existent for practical purposes.

Of course this *diversification* argument becomes extremely dubious in the lump sum case, since the results are essentially the same for a forward glide path (large fraction in stocks declining to a small fraction in stocks) and a reverse glide path (small fraction in stocks increasing to a large fraction).

Again, if we allow the strategy to be adaptive, i.e. allow $p = p(W_t,t)$ then this beats the pants off any deterministic strategy. Why handicap yourself with deterministic asset allocations?

8 What if the stock market crashes just before I retire

Given the same wealth just before retirement, then a constant weight strategy will lose more than a glide path, in the event of a market crash, since the glide path will be less exposed to equities (assuming, of course, a standard glide path where the equity fraction decreases with time). However, suppose that stocks have good returns just before retirement. In this case, assuming the same wealth for both the glide path and the constant weight strategy (just before the market uptick), then the constant weight strategy will outperform the glide path.

However, this assumes the same wealth just before retirement. This is an unlikely scenario, since, from the mathematics (and the empirical studies) we know that the final wealth distributions for both strategies are almost the same.[9] This suggests that, on average, possible sudden market crashes near retirement don't affect the final wealth distribution much.

Of course, if we choose a constant weight strategy, then, on any given path, we may be lucky (a market uptick) or unlucky (market crash) compared to a glide path strategy. We cannot guarantee that these two approaches give the same result on any given path, only that, on average, they give the same result. So, you are not going to improve your odds of success by using a TDF compared to a constant weight policy.

However, it is possible to make some sort of behavioural finance argument to prefer a glide path. Even though the investor ends up with the same final wealth distribution, it might be psychologically damaging to experience a large drawdown just before retirement (even if we end up with the same wealth as with a glide path, on average). But if you really worry about drawdown, then you should add a drawdown penalty to the objective function and then optimize the policy. Of course, the more constraints you add to the objective function, the poorer the result in terms of the final wealth distribution.

[9] Note that Proposition 4.1 also holds if we replace the stock GBM process by a jump diffusion process, which models sudden market crashes (Forsyth and Vetzal, 2019).

9 Summary

Theoretical analysis and empirical studies indicate that any glide path strategy, at best, achieves virtually the same final wealth distribution as an appropriately chosen constant weight strategy. A constant weight strategy is simple, transparent, and easy to implement with low-fee ETFs.

If we focus on the final wealth distribution, there is virtually nothing to be gained from a TDF compared to a constant weight strategy. The only argument to be made for a TDF is that perhaps it might provide some psychological comfort to an investor. However, this is not the argument made by purveyors of TDFs.

On the contrary, adaptive strategies (whereby the asset allocation responds to the investment experience) can easily outperform (in terms of final wealth distribution) by a large margin, any constant weight strategy. Since the final wealth distribution of a glide path is indistinguishable from a constant weight strategy, it then follows that an adaptive strategy also outperforms any glide path.

So, why are there no ETFs which use an adaptive strategy, which, mathematically and empirically, is vastly superior to the existing deterministic glide paths used in TDFs?

Appendix

A Proof of Proposition 4.1

Proof. Equations (3.1) and (3.2) imply

$$\begin{aligned} \frac{dW_t}{W_t} &= p(t)\left(\frac{dS_t}{S_t}\right) + (1 - p(t)\ )\left(\frac{dB_t}{B_t}\right) \\ &= [p(t)(\mu - r) + r]\ dt + p(t)\sigma\ d\mathcal{Z}\ . \end{aligned} \tag{A.1}$$

Define

$$p^* = \frac{1}{T}\int_0^T p(s)\ ds\ . \tag{A.2}$$

Then some stochastic calculus (Forsyth and Vetzal, 2019) gives the following

$$\begin{aligned} E[W_T] &= W_0 e^{[p^*(\mu - r) + r]T} \\ std[W_T] &= E[W_T]\left(\exp\left[\sigma^2\int_0^T p(s)^2\ ds\right] - 1\right)^{1/2} \end{aligned} \tag{A.3}$$

where $E[\cdot]$ denotes the expectation, and $std[\,\cdot\,]$ denotes standard deviation.

Now

$$\begin{aligned} \left(\int_0^T p(s)\cdot \mathbf{1}\ ds\right)^2 &\leq \int_0^T p(s)^2\ ds\ \cdot\ \int_0^T \mathbf{1}^2\ ds \quad ; \quad (\text{ Cauchy-Schwartz inequality}) \\ &= \int_0^T p(s)^2\ ds\ \cdot\ T \end{aligned} \tag{A.4}$$

From (A.2), equation (A.4) becomes

$$T^2(p^*)^2 \ \leq \int_0^T p(s)^2\ ds\ \cdot\ T \tag{A.5}$$

or

$$(p^*)^2 T \leq \int_0^T p(s)^2 \; ds \; . \tag{A.6}$$

So, given any glide path $p(t)$, with time averaged allocation p^*, then the expected value of the terminal wealth for both glide path and constant weight strategies is

$$E[W_T] \;\; = \;\; W_0 e^{[p^*(\mu - r) + r]T}] \tag{A.7}$$

and

$$\begin{aligned}
\overbrace{std[W_T]}^{constant\ weight} \quad &= E[W_T]\left(\exp\left[\sigma^2 (p^*)^2 T \right] - 1 \right)^{1/2} \\
&\leq E[W_T]\left(\exp\left[\sigma^2 \int_0^T p(s)^2 \; ds \right] - 1 \right)^{1/2} \quad ; \quad (\text{equation } (A.6)) \\
&= \overbrace{std[W_T]}^{glide\ path}
\end{aligned} \tag{A.8}$$

□

B Another myth: dollar cost averaging

While we are at it, let's explode the myth of dollar cost averaging.

Suppose your rich uncle dies, and leaves you an inheritance of W_0 dollars. Suppose you want to construct a portfolio such that the fraction in stocks is $\hat{p}$.

Common advice is to *dollar cost average* your investments. In other words, you start off putting W_0 into bonds. Then, each month, you buy some stocks. At the end of k months, you achieve your final desired allocation to stocks.

Let's pose this problem mathematically. Suppose that we are dollar cost averaging over the period $[0, \hat{t}]$. From $[\hat{t}, T]$, we will rebalance to a constant weight $\hat{p}$. At $t = T$, we will evaluate our investment performance in terms of the total wealth at that time, W_T.

We can think of dollar cost averaging as a glide path, where

$$p(t) = \min\left[\frac{\hat{p} t}{\hat{t}}, \hat{p} \right] \; . \tag{B.1}$$

Basically, we start off with 100% bonds, then linearly increase the fraction of our total wealth in stocks, until we hit the desired allocation $\hat{p}$, at $t = \hat{t}$. Thereafter, we rebalance to a constant weight $\hat{p}$.

Now, consider the constant weight portfolio with

$$\begin{aligned}
p^* \;\; &= \;\; \frac{1}{T} \int_0^T p(t) \; dt \\
&= \;\; \hat{p}\left(1 - \frac{\hat{t}}{2T} \right) \quad ; \quad (\text{ from } (B.1)) \; .
\end{aligned} \tag{B.2}$$

But from Proposition 4.1 we know that

- $E[W_T]$ is the same for the constant weight strategy (B.2) and the glide path (B.1).
- and

$$\overbrace{std[W_T]}^{const\ weight} \leq \overbrace{std[W_T]}^{glide\ path} \tag{B.3}$$

Hence, in terms of mean-variance criteria, it is better to immediately buy $p^* W_0$ worth of stocks at time zero, rather than dollar cost average.

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